Ray Shooting from Convex Ranges¶

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Abstract

We consider geometric and graph-theoretic aspects of the well-known paradigm "killing two birds with one stone". Consider that we have a set X of n-points in space and a compact plane convex set S. We define a graph $G(X,S)$, called the *ray shooting graph*, on X as follows: The points of X are the vertices and $\{p_i, p_j\}$, $p_i, p_j \in X$, is an edge if and only if the line joining p_i to p_j intersects S. Ray shooting graphs are shown to be comparability graphs, but the converse is shown not to be true. We provide a characterization of such graphs in terms of geometric containment orders and show that the recognition problem when S is a triangle is NP-complete.

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1 Introduction

Suppose that an archer is hunting birds flying over hunting grounds described as a bounded region possibly with holes formed by obstacles such as mountains, lakes, dense forests, etc. In an attempt to minimize the number of arrows used, the archer tries to identify pairs of birds that can be pierced by a single arrow; this is possible if the positions of two birds line up with some point on the hunting grounds. This corresponds to the well-known paradigm "killing two birds with one stone".

The archer problem can be modeled as follows. Let $X = \{p_1, ..., p_n\}$ be a collection of points in $\mathbb{E}^3$ (in general position) such that the z-coordinate of each element of X is strictly greater than 0 and S a compact plane set of $\mathbb{E}^3$ contained in the hyperplane $H_0 = \{p \in \mathbb{E}^3\colon$ the z$-$coordinate of p is $0\}$. Given X and S construct a graph $G(X,S)$ with vertex set X such that two vertices p_i, p_j of G are adjacent if the line through p_i and p_j intersects S. $G(X,S)$ will be called as the 3-*dimensional ray shooting graph* of X and S.

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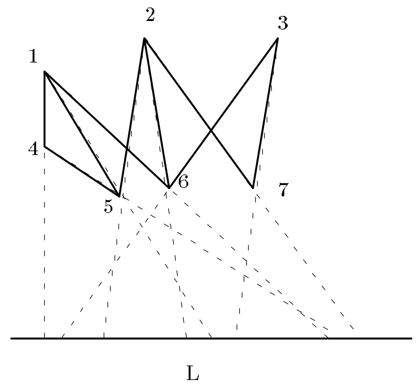

Figure 1: Representation of the graph with vertices $1, 2, 3, 4, 5, 6, 7$ and edges $\{1,4\}, \{1,5\}, \{1,6\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,7\}, \{4,5\}$.

In [2] the planar case of the above paradigm was studied. Consider a line segment L, called the stage, contained in the x-axis of the plane and a set of points X with positive y-coordinates. We define a graph $G(X, L)$ with vertex set X and two vertices as adjacent if the (infinite) line connecting them intersects L (see Figure 1). In [2] a characterization for this class of graphs, called 2-dimensional ray shooting graphs, is presented. They show that G is a 2-dimensional ray shooting graph if and only if G is the comparability graph of an order of dimension 2. Comparability graphs are undirected graphs G which can be oriented in such a way that if we have $u \to v$ and $v \to w$ in G then $u \to w$ is also an edge of G. Using this characterization theorem, they provide efficient algorithms for several problems, including maximum matching and optimal scheduling in permutation graphs, reporting dominances for a planar point set, recognition algorithm for this class of graphs, etc.

In this paper we study the 3-dimensional generalization of 2-shooting graphs. In the rest of the paper we refer to "3-dimensional ray shooting graphs" as "shooting graphs".

We show that all shooting graphs are comparability graphs. We also provide a characterization of shooting graphs in terms of geometric containment orders, that is, ordered sets arising from containment relations among the elements of families of convex sets on the plane.

Given a convex set S on the plane, and a family $F = \{S_1, \ldots, S_n\}$ of n convex sets, homothetic to S, we define a partial order on $F = \{S_1, \ldots, S_n\}$ in which $S_i < S_j$ iff S_i is contained in S_j. The partial order thus obtained is called an S-order. We prove that the set of shooting graphs $G(X, S)$ obtained from a convex set S contained in the hyperplane $H_0 = \{p \in \mathbb{E}^3$: the z-coordinate of p is $0\}$ and collections $X = \{p_1, \ldots, p_n\}$ of points in a vector space $\mathbb{E}^3$ such that the z-coordinate of each element of X is strictly greater than 0, is exactly the set of comparability graphs of S-orders. In particular it follows that the set of shooting graphs obtained when S is a triangle is exactly the set of comparability graphs of orders of dimension at most 3, and thus already for 3 dimensions, the recognition problem is NP-complete. Moreover when S is a circle, we obtain comparability graphs of circle orders

[8]. Finally, we prove that shooting graphs are comparability graphs of orders with crossing number at most 2 (as defined in [8]), and thus not all comparability graphs are shooting graphs.

2 Preliminaries

2.1 Terminology and Definitions

A binary relation $<$ over a set X defines a *partial order* $P(X,<)$ on X if it satisfies:

1. $x<y, y<z$ implies $x<z$ (transitivity), and
2. $x<x$ (antisymmetry).

The partially ordered set $P(X,<)$ is a *linear order* if it also satisfies $x<y$ or $y<x$, for all distinct $x,y \in X$.

Let $P(X,<)$ be a poset. A realizer of P of size $k+1$ is a collection of linear orders $\{L_0(X,<_0), L_1(X,<_1), \ldots, L_k(X,<_k)\}$ such that

$$L_0(X,<_0) \cap L_1(X,<_1) \cap \cdots \cap L_k(X,<_k) = P(X,<),$$

where the intersection is defined by $x<y \Leftrightarrow x <_i y$, for all i. It can be proved easily that every poset can be obtained as the intersection of a number of linear orders. Dushnik and Miller [4] define the dimension of P, denoted $\dim P$, to be the smallest possible size of a realizer of P. Such a realizer is called a minimum realizer of P.

2.2 Function diagrams and crossing numbers

Let $\xi = \{f_1, \ldots, f_m\}$ be a family of continuous functions $f_i\colon [0,1] \to \mathbb{R}$, $i=1,\ldots,m$. The family ξ is called *regular* if the following conditions are satisfied.

1. For any pair of elements $f_i, f_j \in \xi, i \neq j$, the set of values $S(i,j) = \{x \in [0,1]\colon f_i(x) = f_j(x)\}$ is finite.
2. $f_i(0) \neq f_j(0), f_i(1) \neq f_j(1);\ i \neq j$.
3. Each time the graphs of two different functions intersect, they cross each other; that is if $f_i(x_0) = f_j(x_0)$ there exists an $\epsilon > 0$ such that $x_0 - \epsilon < x < x_0 < y < x_0 + \epsilon$ implies that $f_i(x) < f_j(x)$ and $f_i(y) > f_j(y)$ or $f_i(x) > f_j(x)$ and $f_i(y) < f_j(y)$.

Informally speaking, a set of functions ξ as above is regular if the graphs of any two elements of ξ intersect a finite number of times and each time they intersect, they cross each other.

Let $X = \{x_1, \ldots, x_m\}$ be a set, and $P(X,<)$ a partial order on X. $P(X,<)$ is called a *function order* (f-order for short) if there exists a regular set of functions $\xi = \{f_1, \ldots, f_m\}$ such that $x_i < x_j$ if $f_i(x) < f_j(x)$, for all $x \in [0,1]$. The set ξ will be called an *f-diagram* for $P(X,<)$. We will also say that $P(X,<)$ represents ξ. It is easy to prove that every poset is an f-order [10].

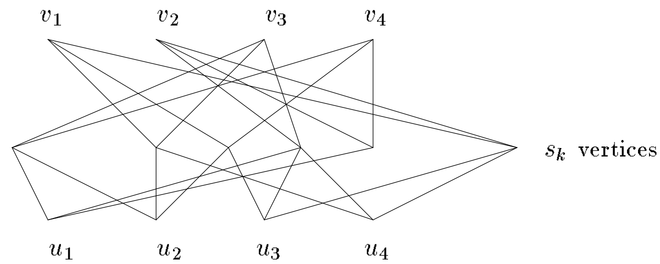

Figure 2: The order set Ψ_n.

Given an f-diagram $\xi = \{f_1, \ldots, f_m\}$, the *crossing number* $\chi(\xi)$ is defined as the maximum over the set $\{|S(i,j)|: f_i, f_j \in \xi, i \neq j\}$; that is the maximum number of times two elements of ξ intersect. The crossing number $\chi(P(X,<))$ of a poset $P(X,<)$ is now defined as $\min\{\chi(\xi)$: ξ is an f-diagram for $P(X,<)\}$.

Informally speaking, every partial order can be represented in many ways using a regular set $\xi = \{f_1, \ldots, f_m\}$ of continuous real functions with domain $[0,1]$. In each such representation, the graphs of some elements of ξ intersect a number of times. The crossing number of a poset $P(X,<)$ is k if in any f-diagram $\xi = \{f_1, \ldots, f_m\}$ representing $P(X,<)$ there are at least two elements of ξ that intersects at least k times. Notice that if $\chi(P(X,<)) = 0$, then $P(X,<)$ has an f-diagram ξ in which no pair of functions of ξ intersect, thus $P(X,<)$ is a linear order. It is also easy to prove that if $\chi(P(X,<)) = 1$, then $\dim P(X,<) = 0$ and that in general $\chi(P(X,<)) \leq \dim P(X,<) - 1$ [8].

Let $H_n(X,<)$ be the ordered set with elements $X = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ such that $u_i < v_j, i \neq j$, and all other pairs of elements in $H_n(X,<)$ are not comparable.

Let Ψ_n be the ordered set obtained from $H_n(X,<)$ as follows. For each subset S_k of $\{1, \ldots, n\}$ with either $\lfloor n/2 \rfloor$ or $\lfloor (n+1)/2 \rfloor$ elements, insert in $H_n(X,<)$ a new element s_k such that $s_k > u_j, j \in S_k, s_k < v_i, i \notin S_k; s_i < s_j$ if $S_i \subset S_j, i \neq j$ (see Figure 2). The next result which will prove useful was proved in [8]:

Theorem 2.1 *The ordered set Ψ_n has crossing number $n-1$ and dimension n.* ■

2.3 Geometric Containment Orders

Let $H = \{T_1, \ldots, T_n\}$ be a family of plane convex sets. A *containment order* $P(H,<)$ on H can be defined in which $S_i < S_j$ if and only if S_i is a subset of S_j. Geometric containment orders have been studied intensely in the literature; see [1, 5, 6, 7, 8, 9]. In the case that all the elements of H are circles, i.e., circles together with their interiors, $P(H,<)$ is called a *circle order*; when the elements of H are polygons with n vertices, $P(H,<)$ is called an *n-gon order* [8]. When, in addition, the elements of H are *regular n-gons* with the same orientation, $P(H,<)$ is called a regular n-gon order [7]. Given a convex set S and a family $F = \{S_1, \ldots, S_n\}$ of convex sets homothetic to S, the containment order $P(F,<)$ arising from F will be called an S-order, e.g. if S is a circle, $P(F,<)$ is a circle order. A family

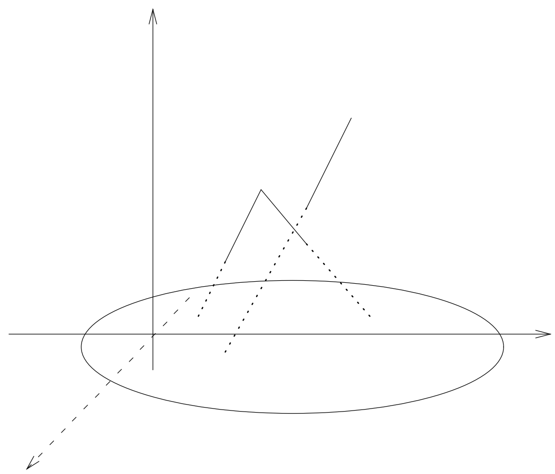

Figure 3: A circle representation of a graph.

$H = \{T_1, \ldots, T_n\}$ of convex sets is called *normal* if the intersection of all the elements of H is non-empty.

3 Ray Shooting From Convex Ranges

In this section we study ray shooting graphs in the three-dimensional vector space $\mathbb{E}^3$. Let $X = \{p_1, \ldots, p_n\}$ be a collection of points in $\mathbb{E}^3$ such that the z-coordinate of each element of X is strictly greater than 0 and S a compact plane-convex set of $\mathbb{E}^3$ contained in the hyperplane $H_0 = \{p \in \mathbb{E}^3\colon \text{the } z\text{-coordinate of } p \text{ is } 0\}$. Given X and S we can now construct a graph $G(X, S)$ with vertex set X such that two vertices p_i, p_j of $G(X, S)$ are adjacent if the line through p_i and p_j intersects S (see Figure 3).

In the rest of this section, unless otherwise specified, S will always refer to a two-dimensional convex set of $\mathbb{R}^3$ contained in the plane $z = 0$ such that S contains the origin. $F = \{S_1, \ldots, S_n\}$ will be a family of convex sets homothetic to S and contained in the plane $z = 0$. X will always refer to a collection of points in $\mathbb{R}^3$ such that the z-coordinate of all the elements of X is strictly greater than 0. We say that a graph G is a shooting graph if there is a two-dimensional convex set S (contained in the plane $z = 0$ and containing the origin) and a collection of points X (with z-coordinate greater than 0) such that G is isomorphic to $G(S, X)$.

3.1 Characterization

In this subsection we first show that shooting graphs are comparability graphs and then provide characterizations in terms of geometric containment orders.

Lemma 3.1 *Let S be a plane convex set of $\mathbb{R}^3$ and X a collection of points in $\mathbb{R}^3$ with z coordinates greater than 0. Then $G(S, X)$ is a comparability graph.*

PROOF For every point $p_i \in X$ let $C(p_i, S)$ be the truncated cone formed by the set of all line segments joining p_i to points in S. It is easy to see that if the z coordinate of p_i is smaller than the z coordinate of p_j then p_i and p_j are adjacent in $G(S, X)$ if and only if $C(p_i, S)$ is contained in $C(p_j, S)$. In this case, orient the edge $\{p_i, p_j\}$ of $G(S, X)$ as $p_i \rightarrow p_j$. To verify the transitivity of this orientation, we simply observe that if $p_i \rightarrow p_j$ and $p_j \rightarrow p_k$ then $C(p_i, S)$ is contained in $C(p_j, S)$, which in turn is contained in $C(p_k, S)$. It follows that $C(p_i, S)$ is contained in $C(p_k, S)$ and thus $p_i \rightarrow p_k$. ∎

The orientation induced on $G(S, X)$ induces a partial order $P(X, <)$ on X. $P(X, <)$ will be called a *ray shooting order*. If, in addition, we want to specify that a ray shooting order $P(X, <)$ arises from a specific convex set S, we will call $P(X, <)$ an *S-shooting order*. Natural questions arise:

> *Is it true that for every ordered set $P(Y, <)$ there is a convex set S and a point set X such that $P(Y, <)$ is isomorphic to the ray shooting order generated by X and S? Can we characterize ray shooting orders?*

Our main objective is to show that there are ordered sets that are not ray shooting orders regardless of the choice of S. We will also give a partial characterization of ray shooting orders in terms of "geometric containment" orders.

Lemma 3.2 *Let $P(H, <)$ be a containment order of a normal family $H = \{T_1, \ldots, T_n\}$ of convex sets such that the boundaries of every two elements T_i and T_j intersects at most k times. Then the crossing number of $P(H, <)$ is at most k.*

PROOF To prove our result all we need to do is to produce a function diagram for $P(H, <)$ in which every pair of functions intersect at most k times. Since H is normal there is a point p in the interior of all T_i, $i = 1, \ldots, n$. Using what in topology is known as surgery, cut the plane along a ray emanating from P (that does not go through any point in the intersection of the boundaries of any two S_i, S_j) and map the two sides of the cut line to the lines $x = 0$ and $x = 1$ of the plane; the upper cut to $x = 1$ and the lower to $x = 0$. It is easy to see (e.g., using polar coordinates) that this can be done in such a way that the boundary of each S_i is mapped to a continuous function from $[0, 1]$ to the reals and that intersection points of boundaries are mapped to intersection points of corresponding functions. The result now follows (see Figure 4). ∎

We now prove the following result:

Theorem 3.1 *Let S be a convex set and $P(X, <)$ an S-order. Then the crossing number of $P(X, <)$ is at most 2.*

Some terminology and basic geometric results will be needed before we prove our result. Let S be a plane convex set of $\mathbb{R}^3$ and $F = \{S_1, \ldots, S_n\}$ be a family of convex sets homothetic to S. Let $C(S)$ be the cone containing all the rays joining the point $(0, 0, 2)$ to points in S. The point $(0, 0, 2)$ will be called the apex of $C(S)$. It is easy to see that every $S_i \in F$ is the intersection of the plane $z = 0$ with a translate $C_i(S) = \{q = p + v_i; p \in C(S)\}$ by a vector $v_i, i = 1, \ldots, n$. The point $a_i = (0, 0, 2) + v_i$ will be called the apex of $C_i(S), i = 1, ..., n$.

Let H_m be the plane $z = -m$ of $\mathbb{R}^3$, m a positive integer. Let us now define $F_m = \{S_{i,m} = C_i(S) \cap H_m,\ i = 1, ..., n\}$. The following lemma is straightforward.

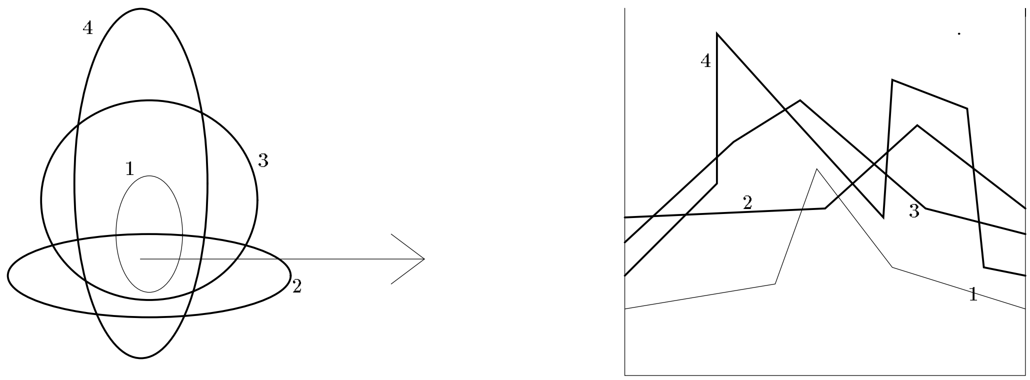

Figure 4:

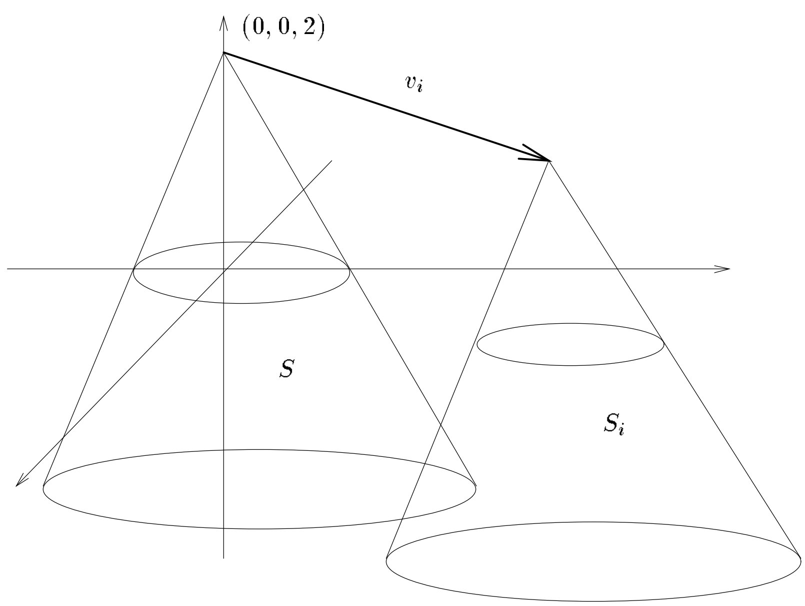

Figure 5: S, S_i are homothetic and contained in the plane $z = 0$,

Lemma 3.3 *The containment order induced by F_m is the same as that of F. Moreover if m is large enough, then F_m is a normal family of sets, i.e. the intersection of all sets of F_m is non-empty.* ■

PROOF of Theorem 3.1. Let $F = \{S_1, \ldots, S_n\}$ be a family of convex sets on the plane homothetic to a convex set S and let $P(F, <)$ be the containment order set generated by F. By Lemma 3.3, there is an m such that $F_m = \{S_{1,m}, \ldots, S_{n,m}\}$ is a normal family of convex sets homothetic to S such that F_m has the same containment order as F.

But by Lemma 3.2 and the fact that the boundaries of any two of the homothetic convex sets intersect at most twice, the crossing number of $P(F, <)$ is at most 2. This completes the proof of Theorem 3.1. ■

Theorem 3.2 *Not every comparability graph is a shooting graph.*

PROOF It is proved in [8] that for every n there are orders of dimension n and crossing number $n-1$. Since the crossing number of ray shooting orders is at most two, our result now follows. ■

Theorem 3.3 *An ordered set $P(Y, <)$ is a S-shooting order if and only if $P(Y, <)$ is an S-ordered set.*

Some extra results will be needed to prove Theorem 3.3. Given $C(S)$, let $D(S)$ be the set of directions of the rays contained in $C(S)$ emanating from the apex $(0, 0, 2)$ of $C(S)$, that is the set of unit vectors u_i of $\mathbb{R}^3$ such that $(0, 0, 2) + u_i$ belongs to $C(S)$. $D(S)$ is a compact subset of the unit sphere in $\mathbb{R}^3$.

Consider a convex set S' contained in the plane $z = 0$ homothetic to S and containing all the elements of F. Let $C'(S)$ be a translate of $C(S)$ such that $C'(S) \cap H_0 = S'$. Since S' contains all the elements of F, $S'_m = H_m \cap C'(S)$ contains all the elements of F_m. The next lemma is easy to prove.

Lemma 3.4 *For every $\epsilon > 0$, there is an m_0 such that if $m > m_0$ and $q \in S'_m$ then the unit vector u determined by the ray from the apex a_i of $C_i(S)$ to q belongs to the set of directions $D(S)$ of $C(S)$ or is at distance at most ϵ from $D(S)$, $i = 1, \ldots, n$.* ■

PROOF of Theorem 3.3. We prove first that S-shooting orders are S-containment orders. Let X be a collection of points with z coordinate greater than 0 and let $P(X, <)$ be the S-shooting order generated by S and X. Assume without loss of generality that all the elements of X have z-coordinate greater than 2. Each point p_i of X together with S defines a truncated cone $C(p_i, S)$ containing all line segments joining p_i to points in S. Consider the plane $z = 1$. Each truncated cone $C(p_i, S)$ intersects the plane $z = 1$ in a convex set Q_i homothetic to S. Moreover if an element p_i is smaller than p_j in $P(X, <)$ then $C(p_i, S)$ is contained in $C(p_j, S)$ and then Q_i is contained in Q_j. It now follows that $P(X, <)$ is an S-order.

Conversely, let $P(F, <)$ be an S-containment order and $F = \{S_1, \ldots, S_n\}$ a family of n convex sets homothetic to S that generates $P(F, <)$. For every S_i of F let $C_i(S)$ be the cone defined in the proof of Theorem 3.2. As in Theorem 3.2 let $S_{i,m} = C_i(S) \cap H_m$ and let $S'_m = H_m \cap C'(S)$.

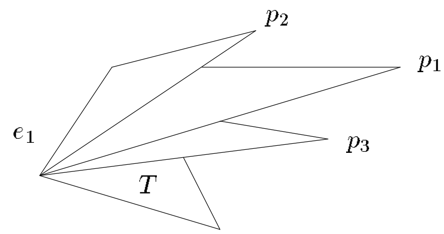

Figure 6: L_1 is the order $p_3 < p_1 < p_2$

Notice that if two sets S_i and S_j of F are such that S_i is contained in S_j then the line joining the apex a_i of $C_i(S)$ to the apex a_j of $C_j(S)$ intersects S'_m. Suppose then that S_i is not contained in S_j and S_j is not contained in S_i. Assume without loss of generality that the z-coordinate of a_i is greater than that of a_j. We proceed now to prove that if m is large enough, then the line joining a_i to a_j does not intersect $S_{i,m}$. To prove this is equivalent to proving that the ray emanating at a_i through a_j does not intersect $S_{i,m}$. To prove this, we notice that since S_i is not contained in S_j and S_j is not contained in S_i, the direction $d_{i,j}$ determined by the ray from a_i to a_j does not belong to $D(S)$. Let $e_{i,j} > 0$ be the distance from $d_{i,j}$ to $D(S)$ and let $\epsilon = \min\{\epsilon_{i,j}\colon S_i \not\subseteq S_j \ \&\ S_j \not\subseteq S_i\}$. Then, by Lemma 3.4, there is an m_0 such that if $m > m_0$ then the direction joining any a_i to any point of $S_{i,m}$ is at distance at most ϵ from $D(S)$. If S_i is not contained in S_j and S_j is not contained in S_i, the line joining a_i to a_j cannot intersect $S_{i,m}$, and the direction of its distance to $D(S)$ is greater than or equal to ϵ which is a contradiction. ■

Corollary 3.1 *If S is a circle, n-gon, regular n-gon, etc., then $P(Y,<)$ is a circle-order, n-gon order, regular n-gon order, etc., respectively.*

3.2 Recognition of triangle-shooting graphs

In this subsection we show that recognizing triangle-shooting graphs is equivalent to recognizing orders of dimension 3, which is NP-hard [11].

Theorem 3.4 *An ordered set $P(X,<)$ is a triangle-shooting order if and only if the dimension of $P(X,<)$ is at most* 3.

Proof Let T be a triangle contained in the plane $z = 0$ of $\mathbb{E}^3$ with edges e_1, e_2, e_3, and as usual, X a set of n points in $\mathbb{E}^3$ with z-coordinate greater than 0. Let $P(T,<)$ be the ray shooting order generated by T and X. For every point p_i of X consider the triangles $T_{i,j}$ defined by e_j and p_i, $i = 1,\ldots,n$, $j = 1,2,3$. For each $k = 1,2,3$ we can define a linear order L_k on X in which a point p_i is smaller than p_j if, when we rotate T along e_k in the upward direction, we meet p_i before we meet p_j. It is easy to see that $P(X,<) = L_1 \cap L_2 \cap L_3$ (see Figure 6).

Conversely, let $P(Y,<)$ be an ordered set of dimension at most 3 and three linear extensions L_1, L_2, L_3 of $P(Y,<)$ such that $P(Y,<) = L_1 \cap L_2 \cap L_3$. Consider a point q in the interior of T and the perpendicular L to T through q. Choose n points q_i in general position "around" L with z-coordinate equal to i, $i = 1, \ldots, n$. For $k = 1, 2, 3$ let $H_{k,i}$ be the plane containing e_k and q_i, $k = 1, 2, 3$, $i = 1, \ldots, n$. For every $y \in Y$ let $r(k, y)$ be the rank or position of y in L_k, $k = 1, 2, 3$. It is now easy to see that the set of points $X = \{H_{k,r(1,y)} \cap H_{k,r(2,y)} \cap H_{k,r(3,y)}: y \in Y\}$ is such that the ray shooting order induced by T and X is isomorphic to $P(Y, <)$. ■

4 Conclusion

In this paper we provide a characterization of shooting graphs; they are exactly the set of comparability graphs of containment orders arising from families of plane homothetic convex sets. We prove that, unlike in the planar case, the recognition problem of shooting graphs is NP-hard. We also prove that the crossing number of these ordered sets is at most 2, and that not all comparability graphs are 3-dimensional ray shooting graphs. A natural question arises now: Is it true that the comparability graph of every ordered set with crossing number 2 is a 3-ray shooting graph?

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