Sense of Direction:
Definitions, Properties and Classes

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Abstract

An extensive body of evidence exists of the impact that specific edge labelings have on the communication complexity of distributed problems. It has been long suspected that these very different labelings share a common property, named a long time ago *Sense of Direction*. In spite of the large amount of investigations, and of the obvious practical importance, a formal characterization of this property did not exist.

In this paper, we finally provide a formal definition of *Sense of Direction*. We show that in Sense of Direction there is a very specific link between three factors: the labeling, the topological structure, and the local view that an entity has of the system. In a way, Sense of Direction is the capacity of a node in the system to use the labeling to translate the local view of its neighbors into its own. Using the formal definition as an observational platform, we describe several properties which allow the translation process to be possible beyond the immediate neighborhood. Finally, we identify four general classes of labelings and analyzed their properties; these classes include all the labelings used in the literature.

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1 Introduction

The ultimate goal of the research in *Distributed Computing* is to understand the nature, the properties and the limits of computing in a system of *autonomous communicating agents*. To this end, it is crucial to identify those factors which are significant for the computability and the communication complexity of problems.

A *distributed system* is a collection of processing entities (e.g., processors) connected by a communication network, where each entity has a local non-shared memory and can communicate by sending messages to and receiving messages from its neighbors. Every entity has a distinct label (e.g., port number) associated to each of its incident links. Thus, the entire system can be viewed as a graph where each node corresponds to a system entity, and each edge corresponds to a direct communication link between two entities; furthermore, every edge has two labels, one for each of its incident nodes. A classical example is a ring network where each edge is labeled "right" at an incident node and "left" at the other.

A very important fact is that some assignments of labels (or, labelings) have a dramatic effect on the communication complexity of distributed problems.

This fact has been made explicit by the surprising result by [23]: the "distance" labeling of a complete graph allows the message complexity of the Election process to be reduced from $\Omega(n \log n)$ (for the unlabeled case [17]) to $O(n)$ messages, where n is the number of entities in the system (see also [28, 41]).

Since this first result, the evidence of the impact of specific labelings in particular graphs has been accumulating in the recent years. For example, in Chordal Rings without labeling, the Election process requires $\Omega(n \log n)$ messages; with a "distance" labeling there exist algorithms whose complexity depends on the chord structure and can be linear [3, 16, 26, 34].

Similarly, $O(n)$ election algorithms exist for an Hypercube with the traditional "dimensional" labeling [9, 36, 43, 46]; without labeling, the best known complexity is $O(n \log n)$ which is probably optimal. An even simpler $O(n)$ technique has been found if the hypercube has the "distance" labeling [9].

In systems of unknown topology (the so-called *arbitrary graph* case), the availability of the "neighboring" labeling reduces the complexity of the Election problem from $\Omega(e + n \log n)$ (for the unlabeled case [1, 37]) to $O(e)$ messages; with the same labeling, the message complexity of Depth First Traversal drops from $\Omega(e)$ to $O(n)$ [40]. The same reductions for both the Election and Depth First Traversal problems can be obtained also with the simpler "distance" labeling [10, 24].

The properties of some of these labelings have been intensely studied and applied. For example, the "distance" labeling in chordal rings and complete graphs has been used for the Weak Unison problem [15] and for Fault-Tolerant Election [25, 29, 30, 31]; the "dimensional" labeling in hypercubes has been investigated for its impact on computability when the system is anonymous and possibly faulty [18, 21]; the complexity of constructing the traditional "left-right" labeling of a ring has been studied [2, 14, 42], and lower bounds for the Election problem in presence of such a labeling have been established [6]. The construction of the traditional labelings of the well-known topologies (hypercubes, tori, etc.) has been the object of the extensive study of [44].

Incomplete labelings of specific topologies have also been the object of investigations (e.g., [3, 13, 39]); it has been shown that even in this case there is an impact on complexity. For example, it is possible to elect a leader in a complete graph with $O(n)$ messages (instead of $\Omega(n \log n)$) even if each node has the "distance" labeling on only $O(\log n)$ appropriate incident arcs [3].

All these labelings differ greatly from each other: the "distance" labeling in chordal rings, the "dimensional" labeling in hypercubes, the "neighboring" labeling in arbitrary graphs, etc. Still, the way they impact on the complexity (i.e., the manner in which the solution algorithms exploit the labelings) is similar, hinting the presence of a *common* extremely useful property. This property has been named, a long time ago, *Sense of Direction* [38], and is generally described as the presence of some "global consistency".

The knowledge acquired by most investigations concentrates on specific problems in particular topologies with particular labeling; thus, it provides information on *instances* of Sense of Direction. Other information on Sense of Direction is given implicitly by the related investigations on the impact of network structure in anonymous systems [2, 4, 20, 33, 47], on the difference between labeled and unlabeled anonymous systems [19], on the relationship between graph symmetricity and labelings [47], and, to some extent, by the investigations on *implicit routing* (see [45] for a survey).

Unfortunately, in spite of the large amount of investigations, of the extensive body of knowledge, and of the evident practical importance, a formal definition of Sense of Direction did not exist; actually, it has been missing even an understanding of what "global consistency" is and why it works in a context larger that the single instances.

In this paper, we finally provide a formal definition of Sense of Direction. In particular, we define the properties whose presence in a labeling make possible the reduction in communication complexity uncovered by the previous investigations. This is achieved by identifying the mechanisms which operate in the reduction, and determining the conditions for the existence of those mechanisms.

From the definition, it emerges that in Sense of Direction there is a very specific link between three factors: the *labeling*, the *topological structure*, and the *local view* that an entity has of the system. In a way, Sense of Direction is the capacity of a node in the system to use the labeling to *translate* the local view of its neighbors into its own.

Using the formal definition as an observational platform, we derive several previously unknown properties of Sense of Direction as well as properties implied by having Sense of Direction in a system.

Based on the formal definition, four general classes of labelings are identified and defined. These classes include all the labelings used in the field.

The paper is organized as follows. In the next Section we give an informal description of Sense of Direction. In Section 3 we introduce the notion of local edge and node labelings and, on the basis of these notions, we formally define Sense of Direction.

In Section 4, we discuss properties of Sense of Direction which allow a node to derive information about the local view of other nodes in the system. In Section 5, we introduce several instances of Sense of Direction, we group them in four general classes, and we analyze their properties. Finally, in Section 6, we discuss some open problems.

2 An Informal Description

In this Section, we will provide an intuitive description of the three fundamental properties which characterize the notion of sense of direction; these properties are (sometimes obscurely) implicit in the previous (topology-dependent) results.

First of all, in Sense of Direction, there is a link between labeling and capability of distinguishing among paths. Each node x has a unique label associated to each of its incident edges; let $\lambda_x(\langle x, z\rangle)$ be the label associated by x to the edge $\langle x, z\rangle$.

Intuitively, when the labeling is a Sense of Direction it is possible to understand, from the labels associated to the edges, whether different paths from any given node x end in the same node or in different nodes.

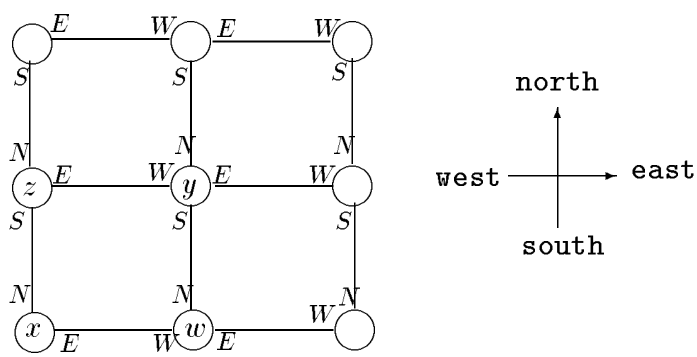

Figure 1: Labeling which is a $\mathcal{SD}$.

For example, consider the system depicted in Figure 1: the communication topology is a 2-dimensional mesh where the edge labels are from the set $\{north,\ south,\ east,\ west\}$ and are assigned in the natural globally consistent way. This labeling is a Sense of Direction (for an appropriate choice of the node names, as discussed later). Consider for instance the three paths, starting from x, whose associated labels are $c_1 = [north,\ north,\ east,\ south]$, $c_2 = [\ east,\ east,\ north,\ west]$, and $c_3 = [east, east]$. Using the rules of the globally consistent labeling, it is trivial to deduce that the two paths corresponding to c_1 and c_2 will end in the same node, while the one corresponding to c_3 will end in a different node.

In other words, when there is Sense of Direction, the global consistency of the labeling allows to distinguish for each pair of nodes, x and y, in the set of paths starting from x, the ones which terminate in y.

The second property of Sense of Direction is the existence of a link between edge labelings and the local node names. Each node x refers to the other nodes using local names. Let us stress that these local names are *not* necessarily identities (i.e., unique global identifiers); in fact, the system could be anonymous. The set of local names of x will be called the *local view* of x. Let $\beta_x(y)$ be the local name by which y is known at x; obviously, this name is the one which will be used whenever x wants to refer to y.

Intuitively, in a Sense of Direction, there is a function which maps the sequences of labels associated to the paths from x to y to the local name $\beta_x(y)$ used by x to refer to y. This means that, in presence of Sense of Direction, x must be able to associate names to paths in such a way that different names are associated to path ending in different nodes, and the same name to paths ending in the same node.

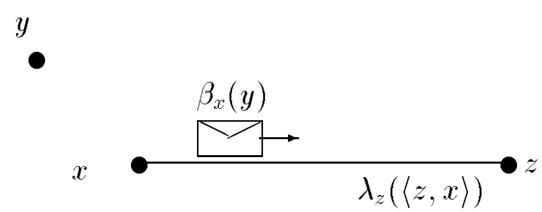

Figure 2: Communication of information about node y from node x to z.

The third property is, perhaps, the most important property of Sense of Direction and refers to the "translation" capability of a node. This property can be described by an example.

Consider the situation of node x sending to its neighbor z information about a node y (see Figure 2). Node y is known at x as $\beta_x(y)$; thus, the message sent by x will contain information about a node called "$\beta_x(y)$". Suppose that this information is received by z along the edge locally labeled with $\lambda_z(\langle z, x\rangle)$.

Informally, if there is sense of direction, node z, based on the label $\lambda_z(\langle z, x\rangle)$ and on the name $\beta_x(y)$, can deduce that the received information is about the node locally called $\beta_z(y)$.

In other words, when a labeling is a *Sense of Direction*, each node can consistently *translate the local views* of its neighbors.

Together these three properties indicate that, in a Sense of Direction, there is a very specific link between edge labels, local names and paths. The nature of this link, and thus, the definition of Sense of Direction, will be formally described in the next Sections.

3 Sense of Direction

Let $G(V, E)$ be a graph where nodes correspond to entities and edges correspond to direct bidirectional communication links between entities. Let $E(x)$ denote the set of edges incident to node x.

A *path* in G is a sequence of edges $[\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m\rangle]$, $\langle x_i, x_{i+1}\rangle \in E(x_i)$, in which the endpoint of one edge is the starting point of the next edge. A path is a *cycle* if the starting point x_0 coincides with the ending point x_m; a path is *simple* if it does not contain any cycle.

Let $P[x]$ denote the set of all the paths with $x \in V$ as a starting point, and let $P[x, y]$ denote the set of paths starting from node $x \in V$ and ending in node $y \in V$.

3.1 Local Edge-Labelings

Each node $x \in V$ has a local function which is used by x to assign labels from a set $\mathcal{L}$ to its incident edges $\langle x, y\rangle \in E(x)$.

Definition 1 *Given a graph $G = (V, E)$ and a set $\mathcal{L}$ of labels, a* local edge-labeling *(or labeling) function for $x \in V$ is any function $\lambda_x : E(x) \to \mathcal{L}$ which associates a label $l \in \mathcal{L}$ to each edge $e \in E(x)$.*

The *labeling* λ of G is the set of local labeling functions, that is $\lambda = \{\lambda_x : x \in V\}$. By (G, λ) we shall denote a *labeled graph*, that is a graph G on which it is defined a labeling λ.

Definition 2 $\mathcal{LO}$ - Local Orientation
A labeling λ is a Local Orientation *iff $\forall x \in V$, $\forall e_1, e_2 \in E(x)$,*

$$\lambda_x(e_1) = \lambda_x(e_2) \quad \text{iff} \quad e_1 = e_2$$

That is, a labeling is a *local orientation* when each node can distinguish among its incident edges.

Definition 3 $\mathcal{ES}$ - Edge Symmetry
A labeling λ has edge symmetry *if there exists a function $\psi : \mathcal{L} \to \mathcal{L}$, such that, $\forall\langle x, y\rangle \in E$, $\lambda_x(\langle x, y\rangle) = \psi(\lambda_y(\langle y, x\rangle))$.*

Definition 4 $\mathcal{LSO}$ - Locally Symmetric Orientation
A labeling is a Locally Symmetric Orientation *when it is a Local Orientation with Edge Symmetry.*

We now extend the definition of the labeling function from edges to paths. Given a labeling λ and a node $x \in V$, let $\Lambda_x : P[x] \to \mathcal{L}^*$ be the *path-labeling function* defined as follows: for every path $\pi \in P[x]$ starting from x,

$$\Lambda_x(\pi) = (\lambda_x(\langle x, x_1\rangle), \lambda_{x_1}(\langle x_1, x_2\rangle), \ldots, \lambda_{x_{m-1}}(\langle x_{m-1}, x_m\rangle))$$

where $\pi = (\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{m-1}, x_m \rangle)$.

Let $\alpha = [\alpha_1, \dots, \alpha_m]$ be a sequence of labels corresponding to a path in G, and let $\Psi : \mathcal{L}^* \to \mathcal{L}^*$ be defined as follows: $\Psi(\alpha) = [\psi(\alpha_1), \dots, \psi(\alpha_m)]$.

Property 1 $\forall \pi \in P[x,y]$, $\Lambda_x(\pi) = \Psi(\Lambda_y(\pi))$

Proof By definition of Ψ. □

3.2 Local Node-Labeling and Local View

Each node x refers to the other nodes using local names. Let us stress that these local names are *not* necessarily identities (i.e., unique global identifiers); in fact, the system could be anonymous.

Definition 5 Local Naming Function
A local node-labeling (or naming) function *for $x \in V$ is a function $\beta_x : V \to \mathcal{N}$ such that $\forall y, z \in V$*

$$\beta_x(y) = \beta_x(z) \text{ iff } y = z$$

The *naming* function β for (G, λ) is the set of local naming functions, that is $\beta = \{\beta_x : x \in V\}$. We shall denote by (G, λ, β) the labeled graph with naming β.

Each node $x \in V$ has a *local view* of the system. A local view consists of the set of names $\{\beta_x(y) : y \in V\}$ used by x.

Definition 6 Local View
Given (G, λ, β), the local view $W(x)$ of node x is the set $W(x) = \{\beta_x(y) : y \in V\}$ of names.

A naming function is said to have Name Symmetry when, for any two nodes x and y, there exists a relationship between the name that x associates to y and the name that y associates to x.

Definition 7 $\mathcal{NS}$ - Name Symmetry
A naming function β has Name Symmetry *if there exists a function $\mu : \mathcal{N} \to \mathcal{N}$, such that, $\forall x, y \in V$, $\beta_x(y) = \mu(\beta_y(x))$.*

3.3 Sense of Direction

We will now introduce the formal definition of sense of direction.

A *coding function* f of a graph (G, λ, β) is a function that associates names to sequences of labels of paths in G.

Definition 8 Coding Function
Given a labeling λ, *a* coding function *for* λ *is any function* $f : \mathcal{L}^* \rightarrow \mathcal{N} \cup \{\star\}$, *where* $\star \notin \mathcal{N}$ *is a distinguished element called the* null name, *such that:*

$$f(\alpha) \in \mathcal{N} \text{ iff } \exists x \in V,\ \pi \in P[x] :\ \alpha = \Lambda_x(\pi)$$

Definition 9 Consistent Coding Function
A coding function f *is* consistent *in* (G, λ, β) *iff* $\forall x, y \in V,\ \pi \in P[x, y]$,

$$f(\Lambda_x(\pi)) = \beta_x(y)$$

Intuitively, the labeling is consistent if it is possible to understand, from the labels associated to the edges, whether different paths from any given node x end in the same node or in different nodes.
By the above definition,

Property 2 *If* f *is consistent, then:* $\forall x, y, z \in V$, $\forall \pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$

$$f(\Lambda_x(\pi_1)) = f(\Lambda_x(\pi_2)) \quad \text{iff} \quad y = z$$

In other words, if a coding function is consistent, then paths originating from the same node are mapped to the same name if and only if they end in the same node.

A *decoding function* h for f in a graph (G, λ, β) is a function which associates a name to a given name and a label.

Definition 10 Decoding Function
Given a coding function f, *a* decoding function h *for* f *is any function* $h : \mathcal{L} \times \mathcal{N} \rightarrow \mathcal{N} \cup \{\star\}$, *where* $\star \notin \mathcal{N}$ *is a distinguished element called the* null name, *such that:*

$$h(l, q) \in \mathcal{N} \text{ iff } \exists \langle x, y \rangle \in E(x),\ \pi \in P[y] :\ l = \lambda_x(\langle x, y \rangle) \text{ and } q = f(\Lambda_y(\pi))$$

To guarantee a consistent "translation" mechanism, a decoding function requires an additional property called *Consistent Local Decoding*.

Definition 11 Consistent Local Decoding
Given a consistent coding function f, *a decoding function* h *for* f *is* consistent *iff* $\forall \langle x, y \rangle \in E(x), \pi \in P[y, z]$

$$h(\lambda_x(\langle x, y \rangle), f(\Lambda_y(\pi))) = \beta_x(z)$$

The existence of a consistent decoding function is clearly a crucial property since it would allow the nodes to solve global problems while working solely and truly in a local mode.

Definition 12 $\mathcal{SD}$ - Sense of Direction
Given (G, λ, β), λ *is a* Sense of Direction *(*$\mathcal{SD}$*) iff the following conditions hold:*
1) λ *is a Local Orientation,*
2) there exists a consistent coding function f,
3) there exists a consistent decoding function h *for* f.

Example 1 - Labeling which is a $\mathcal{SD}$
Consider a system (G, λ, β) where

G is a 2-dimensional mesh;

λ is the natural "compass" assignment of the labels $\mathcal{L} = \{north,\ south,\ east,\ west\}$ (see Figure 1); the labeling has clearly edge symmetry, e.g. $north = \psi(south)$.

β is the following function: $\forall x, y \in V$, $\beta_x(y)$ is the (lexicographically ordered) sequence of labels corresponding to the shortest path between x and y. For example, in Figure 1, $\beta_x(y) = [east, north]$

Note that, in this system, $\mathcal{N} \subset \mathcal{L}^*$. We will now show that this labeling λ is a $\mathcal{SD}$. Given a sequence α of labels, let $\bar{\alpha}$ be the sequence obtained from α by deleting every pair of labels l, l' such that $l = \psi(l')$ and lexicographically sorting the resulting sequence. To show that λ is a $\mathcal{SD}$ we show that there exists a consistent coding function f in (G, λ, β). Consider, for example, the function $f : \mathcal{L}^* \to \mathcal{N} \cup \{\star\}$ such that

$$f(\alpha) = \begin{cases} \bar{\alpha} & \text{if } \exists x \in V,\ \pi \in P[x] :\ \alpha = \Lambda_x(\pi) \\ \star & \text{otherwise} \end{cases}$$

It is easy to verify that function f, applied to any path between x and y, coincides with $\beta_x(y)$. In the example of Figure 1, we have $f([north,\ east,\ north,\ south]) = [east, north] = \beta_x(y)$ and $f([west, east, north, east]) = [east, north] = \beta_x(y)$. It is easy to see that the function $h(l, n) = f([l \circ n]$, where $\circ$ is the concatenation operator, is a consistent decoding function for f.
This labeling is an instance of *Contracted Sense of Direction*, which will be discussed in Section 5.3.

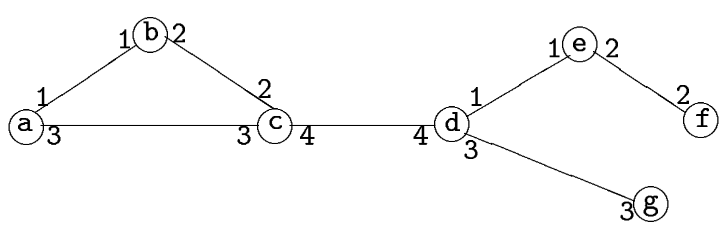

Figure 3: Labeling which is not a $\mathcal{SD}$.

Example 2 - Labeling which is not a $\mathcal{SD}$
Consider the system (G, λ) of Figure 3. To see that λ cannot be a $\mathcal{SD}$ for any choice of β, consider the four paths $\pi_1 = [\langle a, b\rangle,\ \langle b, c\rangle]$, $\pi_2 = [\langle a, c\rangle]$, $\pi_3 = [\langle d, e\rangle,\ \langle e, f\rangle]$ and

$\pi_4 = [\langle d, g\rangle]$. For these paths we should have $f(\Lambda_a(\pi_1)) = f(\Lambda_a(\pi_2)) = \beta_a(c)$ and $f(\Lambda_d(\pi_3)) \neq f(\Lambda_d(\pi_4))$. On the other hand, we have $\Lambda_a(\pi_1) = [1,2] = \Lambda_d(\pi_3)$, and $\Lambda_a(\pi_2) = [3] = \Lambda_d(\pi_4)$.

3.4 Weak Sense of Direction and Associativity

A weaker form of Sense of Direction in a system (G, λ, β) is represented by a labeling λ such that there exists a consistent coding function f, but not necessarily a consistent decoding function h for f.

Definition 13 $\mathcal{WSD}$ - Weak Sense of Direction
Given (G, λ, β), λ is a Weak Sense of Direction *($\mathcal{WSD}$) iff the following conditions hold:*
1) λ is a Local Orientation, and
2) there exists a consistent coding function f.

Definition 14 *Let $\mathcal{N} \subseteq \mathcal{L}^*$. A coding function f is* associative *iff: $\forall\langle x, y\rangle \in E(x)$, $\forall \pi \in P[y]$*

$$f([\lambda_x(\langle x,y\rangle) \circ f(\Lambda_y(\pi))]) = f([\lambda_x(\langle x,y\rangle) \circ \Lambda_y(\pi)])$$

where $\circ$ is the concatenation operator.

Theorem 1 *Let $\mathcal{N} \subseteq \mathcal{L}^*$ and λ be a Weak Sense of Direction. If the corresponding coding function f is associative, λ is a Sense of Direction.*

Proof Consider the following decoding function $h(l, n) = f([l \circ n])$, where $l \in \mathcal{L}$, $n \in \mathcal{N}$ and $\circ$ is the concatenation operator. Note that, $\mathcal{N} \subseteq \mathcal{L}^*$, thus $[l \circ n] \in \mathcal{L}^*$. We have that $\forall\langle x, y\rangle$, $\forall \pi \in P[y, z]$,

$$h(\lambda_x(\langle x,y\rangle), f(\Lambda_y(\pi))) = f([\lambda_x(\langle x,y\rangle) \circ f(\Lambda_y(\pi)])$$

By definition of associative coding function, we have that:

$$f([\lambda_x(\langle x,y\rangle)) \circ f(\Lambda_y(\pi)]) = f([\lambda_x(\langle x,y\rangle) \circ \Lambda_y(\pi)])$$

But

$$f([\lambda_x(\langle x,y\rangle) \circ \Lambda_y(\pi)]) = \beta_x(z)$$

Thus, $h(\lambda_x(\langle x,y\rangle), f(\Lambda_y(\pi))) = \beta_x(z)$, and thus, the decoding function $h(l,n) = f([l \circ n])$ is consistent for f in (G, λ, β). $\square$

4 Translation of Local Views

In the previous Section, we have seen that the availability of Sense of Direction in (G, λ, β) implies the existence of a decoding function h which allows any node x to translate the Local View of its neighbors into its own Local View.

In this Section, we consider a system (G, λ, β), where λ is a $\mathcal{SD}$, and focus on what other knowledge of the system can be derived from the viewpoint of a node x. We discuss properties of $\mathcal{SD}$ which allow x to derive information about the Local Views of other nodes. In particular, we show that, under certain conditions, the availability of $\mathcal{SD}$ allows the translation process to be possible beyond the immediate neighborhood.

4.1 Translation of Incident Paths

The following properties of Sense of Direction express what knowledge can be derived by x from the sequence of labels corresponding to a path in $P[x]$.

Given a system (G, λ, β), where λ is a $\mathcal{SD}$, let $\pi = [\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m\rangle]$ and let $\alpha = \Lambda_{x_0}(\pi)$ be the corresponding sequence of labels. If node x_0 knows the sequence α, then the following properties hold.

Property 3 *Node x_0 can derive the local names of all the nodes on the path; that is,* $\{\beta_{x_0}(x_i) : i = 1 \ldots m\}$.

Proof By definition of consistent coding function, to derive $\beta_{x_0}(x_i)$, it suffices to compute $f(\alpha_i)$ where $\alpha_i = \Lambda_{x_0}([\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{i-1}, x_i\rangle])$. □

That is, x_0 can translate into its local view the names of all the nodes on the paths. It can actually do more, as expressed by the following:

Property 4 *For any x_i, x_j in the path, $i < j$, node x_0 can derive* $\{\beta_{x_i}(x_j) : j = 1 \ldots m,\ i = 1, \ldots j\}$.

Proof To derive the name that x_i associates to x_j, with $i < j$, it suffices to use the coding function. In fact, by definition of consistent coding function, $\beta_{x_i}(x_j) = f(\alpha_{i,j})$, where $\alpha_{i,j} = \Lambda_{x_i}([\langle x_i, x_{i+1}\rangle, \langle x_{i+1}, x_{i+2}\rangle, \ldots, \langle x_{j-1}, x_j\rangle])$ □

In other words, x_0 can translate into the local view of x_i the names of the nodes following x_i in the path.
If there is also Edge Symmetry, the translation capabilities of x_0 increase; as shown by the following.

Property 5 *If λ has Edge Symmetry, node x_0 can derive $\{\beta_{x_i}(x_j) : i, j = 1 \ldots m\}$ for any x_i, x_j in the path.*

Proof Let λ have Edge Symmetry, and let ψ be the edge symmetry function. Consider now any x_i, x_j in the path. If $i \leq j$, the Property holds by Property 4. Let $i > j$. By definition of consistent coding function, $\beta_{x_i}(x_j) = f(\alpha_{i,j})$, where $\alpha_{i,j} = \Lambda_{x_i}([\ \langle x_i, x_{i-1}\rangle, \ldots, \langle x_{j+1}, x_j\rangle\ ]) = [\lambda_{x_i}(\langle x_i, x_{i-1}\rangle), \ldots, \lambda_{x_{j+1}}(\ \langle x_{j+1}, x_j\rangle)]$. The sequence $\alpha_{i,j}$ is clearly derivable since, by definition of edge symmetry function, $\lambda_{x_l}(x_{l-1}) = \psi(\lambda_{x_{l-1}}(x_l))$, for any $x_l \in \pi$. □

4.2 Translation of Remote Paths

The Properties of the previous Section refer to the knowledge which can be derived by a node x from the labels of a path incident on x. We consider now what x can derive from the labels of a pathwhich might not contain x.

Given a system (G, λ, β), where λ is a $\mathcal{SD}$, let $\pi = [\langle y_0, y_1\rangle, \langle y_1, y_2\rangle, \ldots, \langle y_{m-1}, y_m\rangle]$, and $\alpha = \Lambda_{y_0}(\pi)$. If node x knows the sequences α, then the following properties hold.

Property 6 *For any $y_j \in \pi$, node x can derive $\{\beta_{y_0}(y_j) : \ j = 1 \ldots m\}$.*

Proof By definition of consistent coding function, $\beta_{y_0}(y_j) = f(\Lambda_{y_0}(\alpha_j))$, where $\alpha_j = \Lambda_{y_0}([\ \langle y_0, y_1\rangle, \ldots, \langle y_{j-1}, y_j\rangle])$. □

That is, x can derive the name of any node in the path in the local view of the origin of the path. It can actually do more, as expressed by the following:

Property 7 *For any $y_i, y_j \in \pi$, $i < j$, node x can derive $\{\beta_{y_i}(y_j)\}$.*

Proof By definition of consistent coding function, the names can be computed by using the coding function, in fact $\beta_{y_i}(y_j) = f(\alpha_{i,j})$, where $\alpha_{i,j} = \Lambda_{y_i}([\langle y_i, y_{i+1}\rangle, \ldots, \langle y_{j-1}, y_j\rangle])$. □

In other words, x can derive how a node y_i refers to the nodes following it in the path. An even stronger translation capability exists in presence of Edge Symmetry.

Property 8 *If λ has Edge Symmetry, node x can derive $\{\beta_{y_i}(y_j)\}$, for any $y_i, y_j \in \pi$.*

Proof Let λ have Edge Symmetry, and let ψ be the edge symmetry function. Consider now any $y_i, y_j \in \pi$. If $i \leq j$, the Property holds by Property 7. Let $i > j$. By definition of consistent coding function, $\beta_{y_i}(y_j) = f(\alpha_{i,j})$, where $\alpha_{i,j} = \Lambda_{y_i}([\ \langle y_i, y_{i-1}\rangle, \ldots, \langle y_{j-1}, y_j\rangle])$. The sequence $\alpha_{i,j}$ is easilyclearly derivable since, by definition of edge symmetry function, $\lambda_{y_l}(y_{l-1}) = \psi(\lambda_{y_{l-1}}(y_l))$, for $l = 1 \ldots i-1$. □

Note that all the translation of remote paths described above hold without requiring x to know the local name of any node in the path.

Further note that, unless x is in the path (which is the case covered by Section 4.1), it cannot in general translate those names in *its* own local view.

Knowledge of the local name of the origin of the path does not seem to make any difference for the systems considered here. In the next Section we will consider the impact of such a knowledge in stronger systems.

4.3 Translation with Symmetric Sense of Direction

In this Section we will show that, in systems with both Edge and Name Symmetry, knowledge of the origin of a path has an impact on the translation capacities of a node outside the path.

Definition 15 $\mathcal{SSD}$ - Symmetric Sense of Direction
Given (G, λ, β)*,* λ *is a* Symmetric Sense of Direction *if* λ *is a* $\mathcal{SD}$ *with Edge Symmetry and* β *has Name Symmetry.*

Example 4 Consider the system (G, λ, β) of Example 1. Recall that $\beta_x(y)$ is the (lexicographically ordered) sequence of labels corresponding to the shortest path between x and y. We can easily see that β has name symmetry and, thus, λ is a Symmetric Sense of Direction. Let $\beta_x(y) = [l_0, l_1, \ldots, l_k]$, and let ψ be the edge symmetry function; then the function $\mu : \mathcal{L}^\star \to \mathcal{L}^\star$, such that, $\forall x, y \in V$

$$\mu(\beta_x(y)) = [\psi(l_k), \psi(l_{k-1}), \ldots, \psi(l_0)] = \beta_y(x)$$

is the needed name symmetry function.

We will now show the impact of $\mathcal{SSD}$ on the translation of local views. Given a system (G, λ, β), where λ is a $\mathcal{SSD}$, let $\pi = [\langle y_0, y_1\rangle, \langle y_1, y_2\rangle, \ldots, \langle y_{m-1}, y_m\rangle]$, and $\alpha = \Lambda_{y_0}(\pi)$. Let node x know the sequence α and that the name $\beta_x(y_0)$ corresponds to the origin of π.

Property 9 *Node* x *can derive the local name* $\beta_x(y_1)$ *of* y_1*.*

Proof Let ψ and μ be the edge and the name symmetry function, respectively. By definition of name symmetry function,

$$\beta_{y_0}(x) = \mu(\beta_x(y_0)) \text{ and } \beta_x(y_1) = \mu(\beta_{y_1}(x))$$

By definition of consistent decoding function,

$$\beta_{y_1}(x) = h(\lambda_{y_1}(\langle y_1, y_0\rangle), \beta_{y_0}(x))$$

By definition of edge symmetry function,

$$\lambda_{y_1}(\langle y_1, y_0\rangle) = \psi(\lambda_{y_0}(\langle y_0, y_1\rangle))$$

It follows that

$$\beta_x(y_1) = \mu(\beta_{y_1}(x)) = \mu(h(\psi(\lambda_{y_0}(\langle y_0, y_1\rangle)), \mu(\beta_x(y_0)))$$

□

The previous property shows that, when there is a Symmetric Sense of Direction, x can translate into its local view the name of the immediate neighbor y_1 of y_0. We can now obtain the following:

Property 10 *Node x can derive the local names $\beta_x(y_i)$ of all $y_i \in \pi$.*

Proof Let ψ be the edge symmetry function and let μ be the name symmetry function. By Property 9, x can derive the name $\beta_x(y_1)$. By induction, assume that x can derive $\beta_x(y_i)$, and consider $\beta_x(y_{i+1})$. By definition of consistent decoding function,

$$\beta_x(y_{i+1}) = h(\lambda_{y_{i+1}}(\langle y_{i+1}, y_i\rangle), \beta_x(y_i))$$

By definition of edge symmetry function,

$$\lambda_{y_{i+1}}(\langle y_{i+1}, y_i\rangle) = \psi(\lambda_{y_i}(\langle y_i, y_{i+1}\rangle))$$

and by the induction hypothesis $\beta_x(y_i)$ is derivable. Thus the Property follows. □

5 Classes of Sense of Direction

In this Section, several instances of $\mathcal{SD}$ are introduced. These instances include all the labelings used in the literature on Senses of Direction, and are grouped in four general classes: *Cartographic*, *Chordal*, *Contracted*, and *Neighboring* $\mathcal{SD}$s.

5.1 Cartographic Sense of Direction

A *Cartographic Sense of Direction* is any $\mathcal{SD}$ which uses properties of an embedding of $G = (V, E)$ in the plane. Instances of Cartographic $\mathcal{SD}$s are the following.

5.1.1 Coordinate $\mathcal{SD}$

A Coordinate labeling is one which labels the edge $\langle u, v\rangle$ at u by the relative coordinates of v, (See Figure 4).

Definition 16 Coordinate labeling
Given an embedding of G in the plane, λ is a Coordinate labeling *iff:*

$$\forall \langle u, v\rangle \in E[u] \quad \lambda_u(\langle u, v\rangle) = (x_1 - x_0, y_1 - y_0)$$

where (x_0, y_0) and (x_1, y_1) are the coordinates of u and v, respectively, in the embedding.

Note that the labels are elements of $\mathbf{R}^2$. When the local names of the nodes are the appropriate relative coordinates, the Coordinate labeling is a $\mathcal{SD}$.

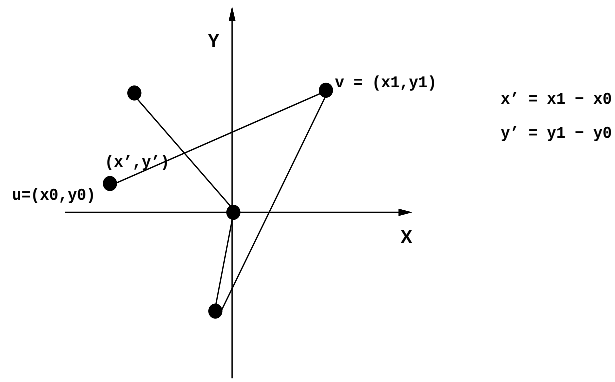

Figure 4: Example of Coordinate $\mathcal{SD}$.

Theorem 2 *Let λ be a Coordinate labeling and $\forall u, v \in V$, let $\beta_u(v) = (x_1 - x_0, y_1 - y_0)$, where $u = (x_0, y_0)$ and $v = (x_1, y_1)$. Then λ is a $\mathcal{SD}$.*

Proof To verify that λ is a $\mathcal{SD}$, consider the coding function f defined as follows: $\forall \pi = [\langle u_0, u_1\rangle, \ldots, \langle u_{m-1}, u_m\rangle] \in P[u_0, u_m]$, where $u_i = (x_i, y_i)$

$$f(\Lambda_{u_0}(\pi)) = f([(x_1 - x_0, y_1 - y_0), \ldots, (x_m - x_{m-1}, y_m - y_{m-1})]) =$$

$$\left(\sum_{i=1}^{m} x_i - x_{i-1}, \sum_{i=1}^{m} y_i - y_{i-1}\right)$$

It follows that

$$f(\Lambda_{u_0}(\pi)) = (x_m - x_0, y_m - y_0) = \beta_{u_0}(u_m)$$

thus, f is consistent in (G, λ, β).
Consider now the following decoding function h:
$\forall \langle u_0, u_1\rangle \in E(u_0)$, $\forall \pi \in P[u_1]$, $\pi = [\langle u_1, u_2\rangle, \ldots, \langle u_{m-1}, u_m\rangle]$ where $u_i = (x_i, y_i)$

$$h(\lambda_{u_0}(\langle u_0, u_1\rangle), f(\Lambda_{u_1}(\pi))) = (x_m - x_0, y_m - y_0)$$

The decoding function is consistent; in fact,

$$(x_m - x_0, y_m - y_0) = \beta_{u_0}(u_m)$$

Thus, λ is a $\mathcal{SD}$. □

We shall call this labeling a *Coordinate* $\mathcal{SD}$.

Theorem 3 *Coordinate Sense of Direction is Symmetric.*

Proof Let λ be a Coordinate $\mathcal{SD}$ in (G, λ, β). To prove the Theorem, we must show that λ and β have edge and name symmetry, respectively. Consider the function $\psi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, $\psi((x, y)) = -(x, y)$. Function ψ is an edge symmetry function: let $u = (x_0, y_0)$ and $v = (x_1, y_1)$ be neighbors; then,

$$\psi(\lambda_u(\langle u, v\rangle)) = \psi(x_1 - x_0, y_1 - y_0) = -(x_1 - x_0, y_1 - y_0)$$

It follows that

$$\psi(\lambda_u(\langle u, v\rangle)) = (x_0 - x_1, y_0 - y_1) = \lambda_v(\langle v, u\rangle)$$

Thus, λ has edge symmetry. Since $\beta_u(v) = -\beta_v(u)$, it follows that ψ is also a name symmetry function; thus, β has name symmetry and λ is a Symmetric Sense of Direction. $\square$

5.1.2 Polar Sense of Direction

A particular class of embeddings of G is obtained by placing the nodes on the unit circle centered in the origin, and by connecting each pair of incident nodes by a straight line. (See Figure 5). Any embedding of this type will be called a *polar representation* of G.

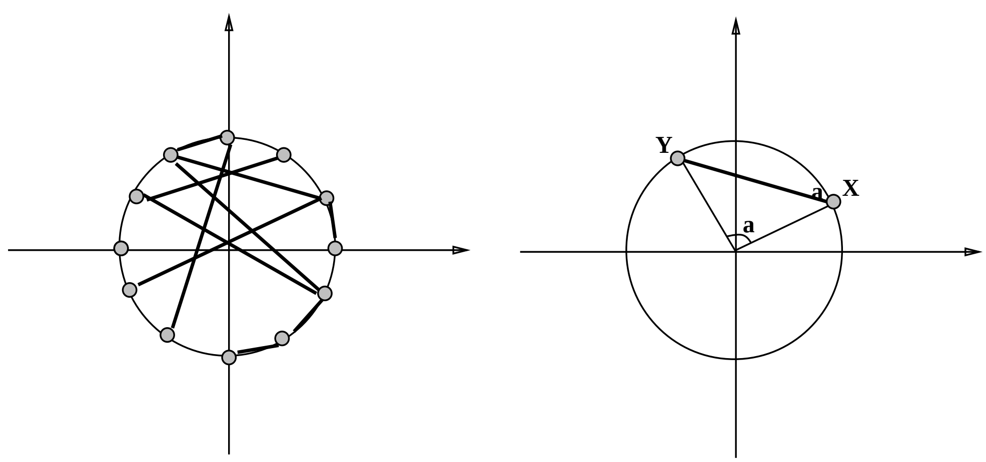

Figure 5: Polar Representation of a Graph and Polar $\mathcal{SD}$.

Definition 17 Polar labeling
Given a graph (G, λ) *in polar representation,* λ *is a* Polar labeling *iff:*

$$\forall \langle x, y\rangle \in E[x] \quad \lambda_x(\langle x, y\rangle) = \alpha_{xy}$$

where α_{xy} *is the angle under the arc* $\langle x, y\rangle$.

When the local names of the nodes are the appropriate angles, the Polar labeling is a $\mathcal{SD}$.

Theorem 4 *Let λ be a Polar labeling and $\forall x, y \in V$ let $\beta_x(y) = \alpha_{xy}$. Then λ is a $\mathcal{SD}$.*

Proof To verify that it is a $\mathcal{SD}$, consider the following coding function f:
$\forall \pi \in P[x_0]$, $\pi = [\langle x_0, x_1 \rangle, \ldots, \langle x_{m-1}, x_m \rangle]$

$$f(\Lambda_x(\pi)) = f(\alpha_{x_0 x_1}, \alpha_{x_1 x_2}, \ldots, \alpha_{x_{m-1} x_m}) = \sum_{i=0}^{m-1} \alpha_{x_i x_{i+1}} \bmod 2\pi$$

That is, $f(\Lambda_x(\pi)) = \alpha_{x_0 x_m} = \beta_{x_0}(x_m)$; thus, f is consistent in (G, λ, β).
Consider now the following decoding function h:
$\forall \langle x_0, y_0 \rangle \in E(x_0)$, $\forall \pi \in P[y_0]$, $\pi = [\langle y_0, y_1 \rangle, \ldots, \langle y_{m-1}, y_m \rangle]$

$$h(\lambda_{x_0}(\langle x_0, y_0 \rangle), f(\Lambda_{y_0}(\pi))) = \alpha_{x_0 y_0} + f(\Lambda_y(\pi)) = \alpha_{x_0 y_0} + \alpha_{y_0 y_m} = \alpha_{x_0 y_m}$$

The decoding function is consistent; in fact,

$$\alpha_{x_0 y_m} = \beta_{x_0}(y_m)$$

□

We shall call the above labeling a *Polar* $\mathcal{SD}$.

Theorem 5 *Polar Sense of Direction is Symmetric.*

Proof Let λ be a Polar $\mathcal{SD}$ in (G, λ, β). To show that it is a Symmetric Sense of Direction we have to show that λ and β have edge and name symmetry, respectively. Consider the following function $\psi : \mathcal{L} \to \mathcal{L}$, $\psi(\alpha) = 2 \cdot \pi - \alpha$. Function ψ is an edge symmetry function, in fact, let x, y be two neighbors;

$$\psi(\lambda_x(\langle x, y \rangle)) = \psi(\alpha_{xy}) = 2 \cdot \pi - \alpha_{xy}$$

where α_{xy} is the angle under the arc $\langle x, y \rangle$. It follows that

$$\psi(\lambda_x(\langle x, y \rangle)) = \lambda_y(\langle y, x \rangle)$$

Thus, λ has edge symmetry. It is immediate to see that the name symmetry function μ coincides with the edge symmetry function. Thus, λ is a Symmetric Sense of Direction.
□

5.2 Chordal Sense of Direction

A *Chordal* labeling of a graph $G = (V, E)$, with $|V| = n$, is defined by fixing a cyclic ordering of the nodes and labeling each incident link by the distance in the above cycle.

Definition 18 *Let $\gamma : V \to V$ be a successor function defining a cyclic ordering of the nodes of (G, λ); and let $\gamma^k(x) = \gamma^{k-1}(\gamma(x))$ for $k > 0$. Let $\delta : V \times V \to \{0, \ldots, n-1\}$ be the corresponding distance function; i.e., $\delta(x, y)$ is the smallest k such that $\gamma^k(x) = y$. The labeling λ is a* Chordal *labeling iff, $\forall \langle x, y \rangle \in E(x)$:*

$$\lambda_x(\langle x, y \rangle) = \delta(x, y)$$

Note that γ is the function defining the cyclic ordering of the nodes; thus, different *Chordal* labelings arise from different γs. Further note that, if the link between p and q is labeled by d at node p, it is labeled by $n - d$ at node q (see Figure 6).

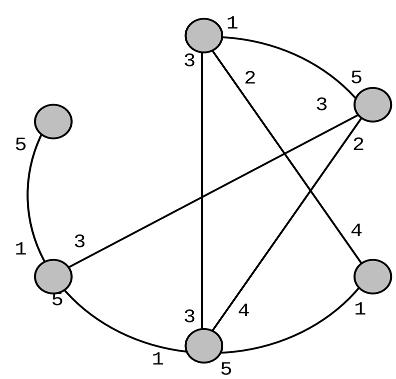

Figure 6: A Graph with Chordal Sense of Direction.

When also the local names of the nodes are relative distances in the cyclic ordering, the Chordal labeling is a $\mathcal{SD}$.

Theorem 6 *Let λ be a Chordal labeling and $\forall x, y$ let $\beta_x(y) = \delta(x, y)$. Then λ is a $\mathcal{SD}$.*

Proof To verify that it is a $\mathcal{SD}$, consider the coding function f defined as follows:
$\forall \pi \in P[x_0]$, $\pi = (\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \ldots, \langle x_{m-1}, x_m \rangle)$

$$f(\Lambda_{x_0}(\pi)) = f(\lambda_{x_0}(\langle x_0, x_1 \rangle), \lambda_{x_1}(\langle x_1, x_2 \rangle), \ldots, \lambda_{x_{m-1}}(\langle x_{m-1}, x_m \rangle)) =$$

$$\sum_{i=0}^{m-1} \lambda_{x_i}(\langle x_i, x_{i+1} \rangle) \bmod n$$

It follows that

$$f(\Lambda_{x_0}(\pi)) = \delta(x_0, x_m) = \beta_{x_0}(x_m)$$

Consider now the following decoding function h:
$\forall\langle x_0, y_0\rangle \in E(x_0),\ \forall\pi \in P[y_0]$

$$h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = \lambda_{x_0}(\langle x_0, y_0\rangle) + f(\Lambda_{y_0}(\pi))$$

h is clearly consistent; in fact,

$$h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = \lambda_{x_0}(\langle x_0, y_0\rangle) + \sum_{i=1}^{m-1} \lambda_{y_i}(\langle y_i, y_{i+1}\rangle) \bmod n = \beta_{x_0}(x_m)$$

thus, λ is a $\mathcal{SD}$. □

Note that the set of names and the set of labels coincide: $\mathcal{L} = \mathcal{N} = \mathbf{Z}_n^+$. We shall call this labeling *Chordal* $\mathcal{SD}$.

Theorem 7 *Chordal Sense of Direction is Symmetric.*

Proof Let λ be a Chordal $\mathcal{SD}$ in (G, λ, β). To prove that it is a Symmetric Sense of Direction we have to show that λ and β have edge and name symmetry, respectively. Consider the function $\psi : \mathbf{Z}_n^+ \to \mathbf{Z}_n^+$, $\psi(d) = n - d$, where $n = |V|$ and $d \in \mathbf{Z}_n^+$. Function ψ is an edge symmetry function, in fact, let x, y be neighbors;

$$\psi(\lambda_x(\langle x, y\rangle)) = n - \lambda_x(\langle x, y\rangle)$$

It follows that

$$\psi(\lambda_x(\langle x, y\rangle)) = \lambda_y(\langle y, x\rangle)$$

Thus, λ has edge symmetry. Since $\beta_u(v) = n - \beta_v(u)$, it follows that ψ is also a name symmetry function; thus, β has name symmetry and λ is a Symmetric Sense of Direction. □

The Chordal labeling is the natural labeling for *chordal rings* (also called *circulant graphs* [5], or *loop networks* [7]), from which it takes the name. It can obviously be defined for any graph. In the literature, the Chordal $\mathcal{SD}$ has been extensively investigated in specific topologies. Sometimes called *Distance* $\mathcal{SD}$, it has been studied in complete graphs [15, 23, 28, 29, 30, 31, 41] and chordal rings [3, 16, 34]. Its impact has been also investigated in hypercubes [9], as well as in systems of unknown topology (the *arbitrary graph* case) [24].

5.3 Contracted Sense of Direction

In this Section, we will analyze a rather general class of $\mathcal{SD}$ based on labelings with Locally Symmetric Orientation (i.e., with both Local Orientation and Edge Symmetry). As we will see, this class contains the traditional labelings for meshes, tori, and hypercubes, among others.

Let λ be a labeling with locally symmetric orientation, and let ψ be the corresponding edge symmetry function.

Definition 19 Contraction
Given a sequence $\alpha \in \mathcal{L}^*$, the contraction of α is the sequence $\overline{\alpha}$ of labels obtained from α by deleting every pair of labels l and l' such that $l = \psi(l')$, and lexicographically sorting the resulting sequence.

Definition 20 Contracted Labeling
A labeling λ with edge symmetry is contracted *iff $\forall x, y \in V,\ \forall \pi_1, \pi_2 \in P[x, y]$*

$$\overline{\Lambda_x(\pi_1)} = \overline{\Lambda_x(\pi_2)}$$

That is, if λ is *contracted*, then all the sequences of all the paths from x to y have the same contraction, which we shall denote by $\overline{\Lambda_{x,y}}$.

When the local names of the nodes are the appropriate contractions, the Contracted labeling is a $\mathcal{SD}$.

Theorem 8 *Let λ be a Contracted labeling and $\forall x, y \in V$ let $\beta_x(y) = \overline{\Lambda_{x,y}}$. Then λ is a $\mathcal{SD}$.*

Proof To verify that it is a $\mathcal{SD}$, consider the coding function f defined as follows: $\forall \pi \in P[x_0]$, $\pi = [\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m\rangle]$

$$f(\Lambda_{x_0}(\pi)) = \overline{\Lambda_{x_0,x_m}}$$

It follows that

$$f(\Lambda_{x_0}(\pi)) = \beta_{x_0}(x_m)$$

thus, f is consistent.

Consider now the following decoding function h:
$\forall \langle x_0, y_0\rangle \in E(x_0)$, $\forall \pi \in P[y_0]$, $\pi = [\langle y_0, y_1\rangle, \ldots, \langle y_{m-1}, y_m\rangle]$:

$$h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = \overline{\lambda_{x_0}(\langle x_0, y_0\rangle) \circ f(\Lambda_{y_0}(\pi))}$$

where $\circ$ is the concatenation operator. It follows that

$$h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = \overline{\lambda_{x_0}(\langle x_0, y_0\rangle) \circ \overline{\Lambda_{y_0 y_m}}} =$$

$$\overline{\Lambda_{x_0 y_m}} = \beta_{x_0}(y_m)$$

thus, h is consistent and λ is a $\mathcal{SD}$. □

We shall call this labeling *Contracted* $\mathcal{SD}$.

Theorem 9 *Contracted Sense of Direction is Symmetric.*

Proof Let λ be a Contracted $\mathcal{SD}$ in (G, λ, β). To prove that it is a Symmetric Sense of Direction, we have to show that λ and β have edge and name symmetry, respectively. The labeling λ has edge symmetry by definition; let ψ be the edge symmetry function, and let Ψ be the corresponding path symmetry function. We will now show that Ψ is also a name symmetry function. By Property 1, $\forall \pi \in P[x, y]$, $\Lambda_x(\pi) = \Psi(\Lambda_y(\pi))$. By definition of β and since λ is a contracted labeling, it follows that

$$\beta_x(y) = \overline{\Lambda_x(\pi)} = \overline{\Psi(\Lambda_y(\pi))} = \Psi(\overline{\Lambda_y(\pi)}) = \Psi(\beta_y(x))$$

thus, Ψ is a name symmetry function and β has name symmetry. It follows that λ is a Symmetric Sense of Direction. $\square$

Example - Contraction in Hypercubes: Dimensional $\mathcal{SD}$

The traditional labeling of a $d - dimensional$ hypercube, shown in Figure 7 for $d = 3$, is an instance of Contracted $\mathcal{SD}$ where the local name $\beta_x(y)$ is the (sorted) sequence of labels (dimensions) on the shortest path between x and y.

In fact, it is a locally symmetric orientation where the edge symmetry function ψ is the identity function. It is easy to verify that, in the hypercube, this labeling is a contracted labeling. Consider, for example, the two paths π_1 and π_2 from x to y, in Figure 7 with $\Lambda_x(\pi_1) = [3, 2, 3, 1, 3]$ and $\Lambda_x(\pi_2) = [1, 2, 3]$; in this case we have $\overline{\Lambda_x(\pi_1)} = [1, 2, 3] = \overline{\Lambda_x(\pi_2)}$, and $\beta_x(y) = \beta_y(x) = [1, 2, 3]$.

The impact of this labeling, also called *Dimensional* $\mathcal{SD}$, in hypercubes has been extensively studied in the literature (e.g., [9, 36, 43]).

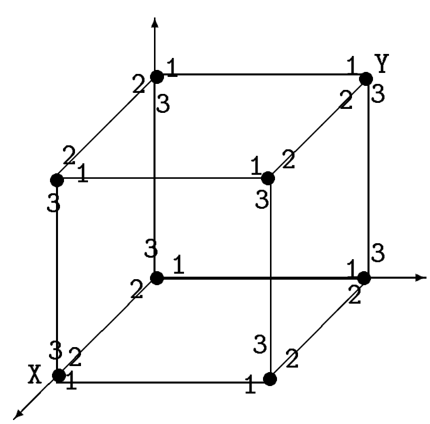

Figure 7: Hypercube with Dimensional Sense of Direction.

Example - Contraction in Meshes: Compass $\mathcal{SD}$.

Each type of d-dimensional mesh (e.g., quadrilateral, hexagonal, etc.) has a natural labeling which forms a particular case of Contracted $\mathcal{SD}$ where the local name $\beta_x(y)$ is the (sorted) sequence of labels on the shortest path between x and y.

Consider, for example, the traditional labeling of a d-dimensional quadrilateral mesh, shown in Figure 1 for $d = 2$. This labeling is a locally symmetric orientation where the edge symmetry function ψ is such that $\psi(north) = south$, $\psi(east) = west$, and so on. It is easy to verify that this labeling is contracted. Consider, for example, the two paths π_1 and π_2 from x to y, in Figure 1 with $\Lambda_x(\pi_1) = [north,\ east,\ north,\ south]$ and $\Lambda_x(\pi_2) = [east,\ north,\ east,\ west]$; we have $\overline{\Lambda_x(\pi_1)} = [east, north] = \overline{\Lambda_x(\pi_2)}$. In this case, $\beta_x(y) = [east, north]$, while $\beta_y(x) = \mu(\beta_x(y)) = [south, west]$, where μ is the name symmetry function.

In the literature, the impact of this type of labelings, sometimes called Compass $\mathcal{SD}$, has been studied only for the cases of quadrilateral meshes, considered above, and of hexagonal meshes [27, 35, 44]).

5.3.1 Contraction with Wraparound

An immediate generalization of the Contracted $\mathcal{SD}$ is the one which applies to topologies with *wraparound* (e.g., rings, tori, etc.). In this case, the sequences associated to paths are transformed so to use only a subset of the labels (termed "allowed directions") and to take into account the structure of the wraparound .

Let λ be a labeling with locally symmetric orientation, and let ψ be the corresponding edge symmetry function.

Definition 21 Contraction with Wraparound
Let $L = \{l_1, \ldots, l_m\} \subset \mathcal{L}$ where $l_i \neq \psi(l_j)$ for $i \neq j$, and let $W = \{w_1, \ldots, w_m\} \subseteq \mathbf{Z}^m$. Given a sequence of labels $\alpha \in \mathcal{L}^*$, the contraction with Wraparound W and Allowed Directions L *(shortly, LW-*contraction*) of α is the sequence $\widehat{\alpha}$ of labels obtained from the contraction $\overline{\alpha}$ by*
1. replacing any subsequence of k l_is with a subsequence of $w_i - k$ $\psi(l_i)$, where $k \geq 0$, $l_i \in L$, $w_i \in W$; and
2. lexicographically sorting the resulting sequence.

Using this operation, the notions of contracted labeling and contracted $\mathcal{SD}$ are extended as follows.

Definition 22 LW-Contracted Labeling
*A labeling λ with edge symmetry is LW-*contracted *iff $\forall x, y,\ \forall \pi_1, \pi_2 \in P[x, y]$*

$$\widehat{\Lambda_x(\pi_1)} = \widehat{\Lambda_x(\pi_2)}$$

That is, if λ is LW-contracted, then all the sequences of all the paths from x to y have the same LW-contraction, which we shall denote by $\widehat{\Lambda_{x,y}}$.

Theorem 10 *Let λ be a LW-Contracted labeling and $\forall x, y$ let $\beta_x(y) = \widehat{\Lambda}_{x,y}$. Then λ is a $\mathcal{SD}$.*

The proof follows the same lines as the one of Theorem 8. Similarly, we can prove the following.

Theorem 11 *LW-Contracted Sense of Direction is Symmetric.*

Each d-dimensional torus has a natural labeling which forms a particular case of LW-Contracted $\mathcal{SD}$ where the local name $\beta_x(y)$ is the (sorted) sequence of labels on the shortest path between x and y using only the allowed directions.

Example - Contraction in Rings
Consider a ring (i.e., a 1-dimensional torus) of size n with the traditional labeling with $\mathcal{L} = \{left,\ right\}$ and with edge symmetry function ψ: $right = \psi(left)$. This labeling is LW contracted where the wrap around is $W = \{n\}$ and the direction is, for example, $L = \{left\}$. Consider, for example, the two paths π_1 and π_2 from x to y, in a ring of size $n = 7$, with $\alpha_1 = \Lambda_x(\pi_1) = [left, left, left, left]$ and $\alpha_2 = \Lambda_x(\pi_2) = [right, right, left, right, right]$. The corresponding LW-contractions are $\widehat{\alpha_1} = [right, right, right] = \widehat{\alpha_2}$. In this case $\beta_x(y) = [right, right, right]$.

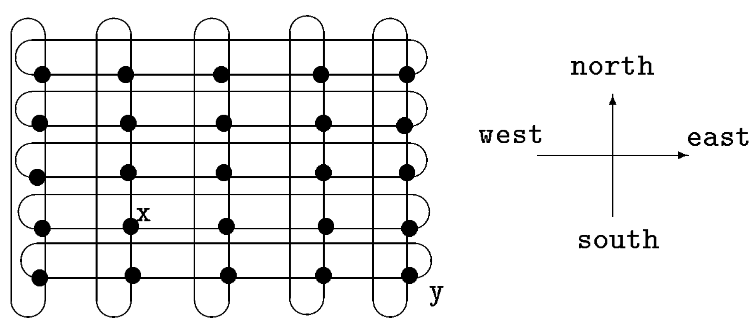

Figure 8: A Torus with LW-Contracted Sense of Direction.

Example - Contraction in Tori: Compass $\mathcal{SD}$
Consider the 2-dimensional torus of size $n_1 \times n_2$ with the traditional "compass" assignment of the labels $\mathcal{L} = \{north,\ south,\ east,\ west\}$ (see Figure 9) and edge symmetry function ψ: $north = \psi(south)$, $east = \psi(west)$, and so on. Clearly, the set of wraparounds is $W = \{n_1, n_2\}$; the corresponding set of allowed directions is, for example, $L = \{south, west\}$. The labeling λ is a LW-contracted labeling. Consider, for example, the two paths π_1 and π_2 from x to y, in Figure 9 with $\alpha_1 = \Lambda_x(\pi_1) =$

[*east*, *south*, *west*, *west*, *west*] and $\alpha_2 = \Lambda_x(\pi_2) =$ [*north*, *north*, *east*, *north*, *north*, *east*, *east*];

The contractions of α_1 and α_2 are $\overline{\alpha_1}$ = [*south*, *west*, *west*] and $\overline{\alpha_2}$ = [*east*, *east*, *east*, *north*, *north*, *north*, *north*]. The corresponding LW-contractions are $\widehat{\alpha_1}$ = [*east*, *east*, *east*, *south*] $= \widehat{\alpha_2}$. In this case, $\beta_x(y)$ = [*east*, *east*, *east*, *south*].

5.4 Neighboring Sense of Direction

In this Section, we describe a class of labelings, which we will show are very powerfull ones.

Definition 23 *Given a graph* (G, λ), λ *is a* Neighboring labeling *iff:* $\forall \langle x, y\rangle \in E[x]$, $\langle z, w\rangle \in E[z]$,

$$\lambda_x(\langle x, y\rangle) = \lambda_z(\langle z, w\rangle) \text{ iff } y = w$$

That is, in a neighboring labeling, all the links ending in the same node x are labeled with the same label, which we shall denote by $l_{(x)}$ (see Figure 9).

Theorem 12 *Let* λ *be a Neighboring labeling, and,* $\forall x, y \in V$, *let* $\beta_x(y) = l_{(y)}$. *Then* λ *is a* $\mathcal{SD}$.

Proof To verify that it is a $\mathcal{SD}$, consider the coding function f with $\mathcal{N} = \mathcal{L}$, defined as follows: $\forall \pi \in P[x_0],\ \pi = (\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \ldots, \langle x_{m-1}, x_m\rangle)$

$$f(\Lambda_x(\pi)) = \lambda_{x_{m-1}}(\langle x_{m-1}, x_m\rangle)$$

Since, by definition of Neighboring labeling, $\lambda_{x_{m-1}}(\langle x_{m-1}, x_m\rangle) = l_{(x_m)}$ it follows that

$$f(\Lambda_x(\pi)) = \beta_{x_0}(x_m)$$

thus, f is consistent.
Consider now the following decoding function:
$\forall \langle x_0, y_0\rangle \in E(x_0),\ \forall \pi \in P[y_0],\ \pi = (\langle y_0, y_1\rangle, \langle y_1, y_2\rangle, \ldots, \langle y_{m-1}, y_m\rangle)$

$$h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = f(\Lambda_{y_0}(\pi))$$

h is consistent, in fact:

$$h(\lambda_{x_0}(\langle x_0, y_0\rangle), f(\Lambda_{y_0}(\pi))) = \lambda_{y_{m-1}}(\langle y_{m-1}, y_m\rangle) = l_{(y_m)} = \beta_{x_0}(y_m)$$

thus, λ is a $\mathcal{SD}$. □

We shall call this labeling a *Neighboring* $\mathcal{SD}$.

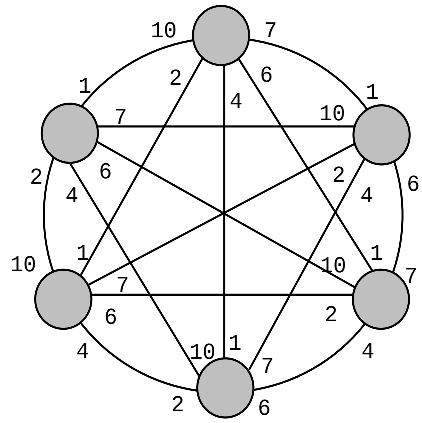

Figure 9: A Complete Network with Neighboring $\mathcal{SD}$.

Let us observe that, unlike all the previous classes of $\mathcal{SD}$s, the Neighboring Sense of Direction is *not* Symmetric. This implies that we cannot apply the properties of Section 4.3; recall that, if a labeling is symmetric, knowledge of the origin of a path has an impact on the translation capacities of a node outside the path.

However, we will now show that the Neighboring $\mathcal{SD}$ has actually a very strong property; in fact, with such a labeling, the translation capacities of a node outside the path are the same as for a symmetric labeling, *even* without knowledge of the origin of the path.

Given a system (G, λ, β), where λ is a Neighboring $\mathcal{SD}$, let $\pi = [\langle y_0, y_1\rangle, \langle y_1, y_2\rangle, \ldots, \langle y_{m-1}, y_m\rangle]$, and let $\alpha = \Lambda_{y_0}(\pi)$. Let node x know the sequence α.

Property 11 *Node x can derive the local names $\beta_x(y_i)$ of all $y_i \in \pi$.*

Proof It trivially follows since $\beta_x(y_i) = l_{(y_i)} = \lambda_{y_{i-1}}(\langle y_{i-1}, y_i\rangle)$. □

The above Property shows an aspect of the *strength* of the Neighboring $\mathcal{SD}$, which sets it apart from the other classes of $\mathcal{SD}$s. Another, even more startling proof of this *strength* is given by the following:

Property 12 *Given an anonymous system (G, λ, β), if λ is a Neighboring $\mathcal{SD}$, then the* Election Problem *is solvable in G.*

Proof Let λ be a Neighboring $\mathcal{SD}$. Then each node x can acquire a unique global identifier; e.g., by asking an arbitrary neighbor for the label of the link connecting them, and assuming such a label as its identifier. In presence of a unique global identifier for each node, the Election Problem can be solved using any of the existing algorithms. □

To fully appreciate this result, recall that the Election Problem is unsolvable in an anonymous unlabeled G, and that similar results do not exist for the other classes of $\mathcal{SD}$s described above.

Note that this strenght of the Neighbouring $\mathcal{SD}$ is also its weakness. In fact, exactly because the Election problem is unsolvable in anonymous graphs, it follows that the Neighbouring $\mathcal{SD}$ cannot be constructed in anonymous systems.

In the literature, the Neighboring $\mathcal{SD}$ has been studied solely in systems of unknown topology [22, 40].

6 Conclusions and Open Problems

In this paper, we have provided a formal definition of Sense of Direction. In particular, we have identified the properties whose presence in a labeling make possible the reduction in communication complexity uncovered by the previous investigations.

Using the formal definition as an observational platform, we have derived previously unknown properties of Sense of Direction as well as properties implied by having Sense of Direction in a system. Based on the formal definition, we have identified and defined four general classes of labelings which include all the labelings used in the field.

A major contribution of this paper is to provide researchers with a firm starting point as well as a powerful formal tool. From this point and using this tool, many intriguing questions are now open, all of them of immediate practical relevance; here are just a few:

What is the nature of the relationship between graph topology and Sense of Direction?

Which topological properties guarantee the existence of a "minimal" Sense of Direction (i.e., with smallest number of labels)?

What is the complexity of deciding if a labeling is a Sense of Direction?

Investigations in these directions have already started, providing the first partial results [8, 12, 32].

Another important research area is the application of Sense of Direction to distributed problems. In a companion paper [11], we have shown that all the existing results for general graphs follow as simple applications of the definition or of the derived properties. That is, in arbitrary graphs, the complexity improvements obtained with very specific labelings can be obtained with *any* Sense of Direction. As for specific topologies, no such a result exists. The open problem is thus to understand how *topology-dependent* distributed algorithms (e.g., election protocols for hypercubes) can be constructed which would be efficient with *any* Sense of Direction.

Finally, a very interesting research direction now open is the study of the interplay between *Implicit Routing* (e.g, [45]) and Sense of Direction.

Acknowledgement. This work has been supported in part by N.S.E.R.C, under grant #A2415, and by a N.A.T.O. Advanced Reasearch Fellowship.

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