On Systems with Sense of Direction *

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Abstract

For a system with Sense of Direction, there are several possible consistent coding functions c and corresponding decoding functions d. Thus, it is desirable for a given system to identify, among all the couples (c, d) which are Sense of Direction, those who have additional properties. In fact, by using such properties, the solution of problems in a system could be further improved without any modification in the labeling.

We first consider two such properties: a strong form of decoding consistency called *everywhere consistency*, and the coding property called *everywhere associativity*. We prove that for any system with Sense of Direction there always exists at least a couple (c, d) where c is everywhere associative and d is everywhere consistent. The proof is constructive.

We then focus on systems with *edge symmetry*. For these systems we study three important properties in presence of Sense of Direction: *backward consistency*, *name symmetry*, and *homonymity*. We establish a number of results for each of these properties, as well as some relationships among them. Some of these results have immediate practical applications. For example, our results imply that all proper colorings can be tested in (low) polynomial time.

1 Introduction

A *distributed system* is a collection of autonomous entities communicating by the exchange of messages. The communication topology of the system can be represented as a graph $G(V, E)$ where nodes correspond to the system entities and edges represent pairs of neighboring entities (i.e., entities which can communicate directly). Each entity has a local (partial) view of the system. In particular, a node has *local orientation*; that is, it has a distinct label, sometimes called *port number*, associated to each of its incident edges. Let $\lambda_x(\langle x, y\rangle)$ be the label associated by $x \in V$ to the edge $\langle x, y\rangle \in E$. Note that, since labels are assigned locally, two neighbors may associate different labels to their incident edge; i.e., possibly $\lambda_x(\langle x, y\rangle) \neq \lambda_y(\langle y, x\rangle)$. The entire system can thus be denote by (G, λ), where $\lambda = \{\lambda_x : x \in V\}$.

The solution to many problems in distributed computing can be greatly simplified by the appropriate choice of the labeling λ. In other words, the communication complexity is in general sensitive to the properties of the labeling. In particular, it is well known that, if λ

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satisfies the set of consistency constraints called *Sense of Direction* [3, 16], the communication complexity of several distributed problem is drastically reduced (e.g., see [2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18]). Because of its impact on complexity, a great deal of research has been devoted to the analysis of properties of $\mathcal{SD}$ (e.g., see [1, 4, 5, 17]).

Informally, a system (G, λ) has Sense of Direction if there exists both a *consistent coding* function c for λ and a *consistent decoding* function d for c; should this be the case, we shall say that (c, d) is a $\mathcal{SD}$ in (G, λ). The existence of these functions allow the nodes to solve global problems while working solely an truly in a local mode [3].

For a system (G, λ) with Sense of Direction, there are several possible consistent coding functions and corresponding decoding functions. Thus, it is desirable for a given system to identify, among all the couples (c, d) which are $\mathcal{SD}$ in (G, λ), those who have important or useful additional properties. In fact, by using such additional properties, the solution of problems in a system could be further improved without any modification in the labeling.

We first consider two such properties: a strong form of decoding consistency called *everywhere consistency*, and the coding property called *everywhere associativity*.

We prove that for any system (G, λ) with Sense of Direction there always exists at least a couple (c, d) where c is everywhere associative and d is everywhere consistent. The proof is constructive.

We then focus on systems with *edge symmetry*; that is, systems where the labels assigned by neighbors to their incident edge are related by some function ψ. Practically all systems with $\mathcal{SD}$ studied in the literature have edge symmetry (e.g., in a torus with "compass" $\mathcal{SD}$, a "north" edge is labeled "south" at the other incident node). A particular form of edge symmetry is when the function ψ is the identity function; in this case, the labeling is said to be a *matching* labeling or a *coloring*.

For these systems we study three important properties in presence of Sense of Direction: *backward consistency* ($\mathcal{BC}$), *name symmetry* ($\mathcal{NS}$), and *homonymity* ($\mathcal{H}$).

We establish a number of results for each of these properties, as well as some relationships among them. For instance, we completely characterize the relationship between edge symmetry and homonymity, as well as the one between name symmetry and edge symmetry. A partial characterization is provided for backward consistency.

Some of these results have immediate practical applications. For example, since the testing algorithm of [1] requires homonymity, our results imply that all proper colorings can be tested in (low) polynomial time.

2 Preliminaries

2.1 Labelings

Let $G(V, E)$ be a graph where nodes correspond to entities and edges correspond to direct bidirectional communication links between entities. Let $E(x)$ denote the set of edges incident to node x.

Given a graph $G = (V, E)$ and a set $\mathcal{L}$ of labels, a *local edge-labeling* (or labeling) function for $x \in V$ is any function $\lambda_x : E(x) \to \mathcal{L}$ which associates a label $l \in \mathcal{L}$ to each of its incident edges $e \in E(x)$.

The *labeling* λ of G is the set of local labeling functions, that is $\lambda = \{\lambda_x : x \in V\}$. By (G, λ) we shall denote a *labeled graph*, that is a graph G on which it is defined a labeling λ.

A labeling λ is a *Local Orientation* when each node can distinguish among its incident edges; that is, $\forall x \in V$, $\forall e_1, e_2 \in E(x)$, $\lambda_x(e_1) = \lambda_x(e_2)$ iff $e_1 = e_2$.

We now extend the definition of the labeling function from edges to paths. A *path* π in G is a sequence of edges in which the endpoint of one edge is the starting point of the next edge; the reverse path will be denoted by $\overline{\pi}$. A path is a *cycle* if the starting point coincides with the ending point; a path is *simple* if it does not contain any cycle. Let $P[x]$ denote the set of all the paths with $x \in V$ as a starting point, and let $P[x, y]$ denote the set of paths starting from node $x \in V$ and ending in node $y \in V$.

Given a labeling λ and a node $x \in V$, let $\Lambda_x : P[x] \to \mathcal{L}^+$ be the *path-labeling function* defined as follows: for every path $\pi \in P[x_1]$ starting from x_1,

$$\Lambda_{x_1}(\pi) = [\lambda_{x_1}(\langle x_1, x_2\rangle), \ldots, \lambda_{x_m}(\langle x_m, x_{m+1}\rangle)]$$

where $\pi = [\langle x_1, x_2\rangle, \ldots, \langle x_m, x_{m+1}\rangle]$.

2.2 Consistency

Intuitively, in a labeled graph (G, λ), the labeling is consistent if it is possible to understand, from the labels associated to the edges, whether different paths from any given node x end in the same node or in different nodes.

Definition 1 *A consistent coding function c is any function with domain $\mathcal{L}^+$, such that $\forall x, y, z \in V$, $\forall \pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$ $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_2)) \Leftrightarrow y = z$. The codomain of c will be denoted by $\mathcal{N}_c$.*

In other words, in a consistent coding function, paths originating from the same node are mapped to the same value (called local name) if and only if they end in the same node.

Definition 2 *Given a consistent coding function c, a consistent decoding function d for c is any function $d : \mathcal{L} \times \mathcal{N}_c \to \mathcal{N}_c$ such that $\forall \langle x, y\rangle \in E(x)$, $\pi \in P[y, z]$*

$$d(\lambda_x(\langle x, y\rangle), c(\Lambda_y(\pi))) = c(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))$$

where $\cdot$ is the concatenation operator.

Definition 3 *A system (G, λ) has* Sense of Direction *($\mathcal{SD}$) iff the following conditions hold:*
1) λ is a Local Orientation,
2) there exists a consistent coding function c for λ,
3) there exists a consistent decoding function d for c.

We shall also say that (c, d) *is a $\mathcal{SD}$ in* (G, λ); in the following, when no ambiguity arises, we shall omit the reference to (G, λ).

3 Everywhere Consistency and Associativity

3.1 Everywhere Consistent Decoding

Definition 4 Everywhere Consistent Decoding
Given a consistent coding function c, a decoding function d for c is everywhere consistent *iff* $\forall a \in \mathcal{L}, \alpha \in \mathcal{L}^+$*:* $d(a, c(\alpha)) = c(a \cdot \alpha)$.

Clearly, everywhere consistency implies consistency. The converse is not necessarily true.

Theorem 1 *If (c, d) is a $\mathcal{SD}$, then there exist a consistent coding function $\overline{c}$ such that $(\overline{c}, d)$ is a $\mathcal{SD}$ and d is everywhere consistent for $\overline{c}$.*

Proof Define the coding function $\overline{c}$ as follows: for each $a \in \mathcal{L}$ $\overline{c}(a) = c(a)$ and for each $w \in \mathcal{L}^+$:

$$\overline{c}(a \cdot w) = \begin{cases} c(a \cdot w) & \text{if } \exists x \in V, \pi \in P[x] \text{ s.t. } a \cdot w = \Lambda_x(\pi) \\ d(a, \overline{c}(w)) & \text{otherwise} \end{cases}$$

We will first prove that $\overline{c}(w)$ is well defined by induction on the length of w. When $|w| = 1$ $\overline{c}(w) = c(w)$ which is defined. Let it hold for each w such that $|w| \leq n$ Consider the string $a \cdot w \in \mathcal{L}^+$ where $a \in \mathcal{L}^+$. If there exist $x \in V$ and $\pi \in P[x]$ s.t. $\Lambda_x(\pi) = a \cdot w$ then $\overline{c}(a \cdot w) = c(a \cdot w)$ which is defined. Otherwise, since $\overline{c}(w)$ is defined and $\overline{c}(a \cdot w) = h(a, \overline{c}(w))$ is defined.
Note that c and $\overline{c}$ agree on paths: $\forall x \in V, \pi \in P[x]$ $c(\Lambda_x(\pi)) = \overline{c}(\Lambda_x(\pi))$.
We will now prove that $\overline{c}$ is consistent. Since c and $\overline{c}$ agree on paths, for each $x, y, z \in V$ and each $\pi_1 \in P[x, y], \pi_2 \in P[x, z]$, $\overline{c}(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_2))$ iff $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_1))$ because π_1 and π_2 are paths and c and $\overline{c}$ agree on paths. Furthermore by consistency of c $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_1))$ iff $y = z$.
We will now prove that d is a consistent decoding for $\overline{c}$. Since $\overline{c}$ and c agree on paths for each x, y, z, each $\langle x, y\rangle \in E(x)$ and each $\pi \in P[y, z]$, $d(\lambda_x(\langle x, y\rangle), \overline{c}(\Lambda_y(\pi))) = d(\lambda_x(\langle x, y\rangle), c(\Lambda_y(\pi)))$. On the other hand,by consistency of d, $d(\lambda_x(\langle x, y\rangle), c(\Lambda_y(\pi))) = c(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))$. By the existence of path $\langle x, y\rangle\pi$ and by definition of $\overline{c}$, we have that $\overline{c}(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi)) = c(\lambda_x(\langle x, y\rangle) \cdot \Lambda_y(\pi))$. Thus d is consistent and $(\overline{c}, d)$ is a $\mathcal{SD}$.
Now we prove that d is everywhere consistent, that is $d(a, \overline{c}(w)) = \overline{c}(a \cdot w)$ for each $w \in \mathcal{L}^+$. If there exist $x, y, z \in V$ and $\langle x, y\rangle \in E(x)$ and $\pi \in P[y, z]$ s.t. $a = \lambda_x(\langle x, y\rangle)$ and $w = \Lambda_x(\pi)$ then, by consistency of d, $d(a, c(w)) = c(a \cdot w)$. Furthermore the existence of the paths $\langle x, y\rangle\pi$ and π implies $\overline{c}(a \cdot w) = c(a \cdot w)$ and $\overline{c}(w) = c(w)$. Thus $d(a, \overline{c}(w)) = d(a, c(w)) = c(a \cdot w) = \overline{c}(a \cdot w)$. If no x, y, z and π path exist $\overline{c}(a \cdot w) = d(a, \overline{c}(w))$ by definition of $\overline{c}$. □

3.2 Everywhere Associative Coding

Definition 5 Everywhere Associative Coding
Let $\mathcal{N} \subseteq \mathcal{L}^+$. A coding function c is everywhere associative *iff:* $\forall a \in \mathcal{L}$, $\forall w \in \mathcal{L}^+$

$$c(a \cdot w) = c(a \cdot c(w))$$

Definition 6 *Given a set S, a function $s : 2^S \rightarrow S$ is a* selection function *iff:*

$$\forall A \subseteq S \;\; s(A) \in A.$$

Definition 7 *Given a coding function* $c : \mathcal{L} \to \mathcal{N}$ *and a selection function* $s : 2^{\mathcal{N}} \to \mathcal{N}$*, the* selection inverse *of* c *is the function* $c_s^{-1} = s \circ c^{-1}$*, where* $c^{-1} : \mathcal{N} \to 2^{\mathcal{N}}$ *is the inverse function s.t.* $c^{-1}(n) = \{\Lambda_x(\pi) : \forall x, \pi \in P[x] \cap c(\Lambda_x(\pi)) = n\}$.

Property 1 *For each* c *and each* s*:* $c \circ c_s^{-1} \circ c = c$

Proof $c_s^{-1} \circ c(\alpha) = \alpha'$ s.t. $c(\alpha) = c(\alpha')$. Thus $c(c_s^{-1}(c(\alpha))) = c(\alpha') = c(\alpha)$. □

Property 2 *If* $c_s^{-1} \circ c(\alpha) = c_s^{-1} \circ c(\beta)$*, then* $c(\alpha) = c(\beta)$.

Proof Suppose that $c_s^{-1}(c(\alpha)) = c_s^{-1}(c(\beta))$. Let $c^{-1}(c(\alpha)) = N$ and $c^{-1}(c(\beta)) = M$. $s(N) = s(M)$ implies $s(N) \in N \cap M$.
Then there exists $\gamma = s(N)$ s.t $c(\alpha) = c(\gamma)$ and $c(\beta) = c(\gamma)$. Thus, $c(\alpha) = c(\beta)$. □

Theorem 2 *If* (c, d) *is a Sense of Direction, then there exists a Sense of Direction* (c', d') *with* c' *everywhere associative and* d' *everywhere consistent.*

Proof By Theorem 1 we can suppose that d is everywhere consistent for c. Let $c' = c_s^{-1} \circ c$. c' is consistent because, by Property 2, for each $x, y, z \in V$ and each $\pi_1 \in P[x, y]$ and $\pi_2 \in P[x, z]$, $c_s^{-1} \circ c(\Lambda_x(\pi_1)) = c_s^{-1} \circ c(\Lambda_x(\pi_2))$ iff $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_1))$; furthermore, by consistency of c, $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_1))$ iff $y = z$.

Now we prove that $c_s^{-1} \circ c$ is everywhere associative. Since c is everywhere consistent, for each $a \in \mathcal{L}$ and each $w \in \mathcal{L}^+$ $c'(a \cdot c'(w)) = c_s^{-1}(c(a \cdot c_s^{-1}(c(w)))) = c_s^{-1}(d(a, c(c_s^{-1}(c(w)))))$. By Property 1 and by everywhere consistency of d, it follows that $c_s^{-1}(d(a, c(c_s^{-1}(c(w))))) = c_s^{-1}(d(a, c(w))) = c_s^{-1}(c(a \cdot w)) = c'(a \cdot w)$.

Let $d'(a, c'(w)) = c'(a \cdot c'(w))$ By everywhere associativity of c' we have that $c'(a \cdot c'(w)) = c'(a \cdot w)$, thus d' is an everywhere decoding of c'. □

4 Symmetries and Backward Consistency

In this section we consider systems with Edge symmetry; that is systems in which there exists a precise relation between the labels at the two sides of an edge. For those systems, we focus on three properties: *Backward Consistency*, *Homonymity* and *Name Symmetry*. We study under what conditions it is possible to construct a couple (c, d) with such properties.

4.1 Symmetry

Definition 8 $\mathcal{ES}$ - Edge Symmetry
A labeling λ *has* edge symmetry *if there exists a function* $\psi : \mathcal{L} \to \mathcal{L}$*, such that,* $\forall \langle x, y\rangle \in E$*,* $\lambda_x(\langle x, y\rangle) = \psi(\lambda_y(\langle y, x\rangle))$.

Property 3 $\psi \circ \psi \circ \lambda_x = \lambda_x$

Proof We prove that $\forall \langle x, y\rangle \in E(x)$, $\psi(\psi(e))) = \lambda_x(e)$, where $e = \langle x, y\rangle$. $\psi(\lambda_x(\langle x, y\rangle)) = \lambda_y(\langle y, x\rangle)$ by definition of ψ. $\psi(\lambda_y(\langle y, x\rangle)) = \lambda_x(\langle x, y\rangle) = \lambda_x(e)$ by definition of ψ. □

A particular form of edge symmetry is the one where the edge symmetry function is the identity function; any such a labeling is said to be *matching labeling*, or *coloring*.

Let $\alpha = [\alpha_1, \ldots, \alpha_m]$ be a sequence of labels corresponding to a path in G, and let $\Psi : \mathcal{L}^* \rightarrow \mathcal{L}^*$ be defined as follows: $\Psi(\alpha) = [\psi(\alpha_m), \ldots, \psi(\alpha_1)]$.

Property 4 $\forall \pi \in P[x,y]$, $\Lambda_x(\pi) = \Psi(\Lambda_y(\overline{\pi}))$

Proof By definition of Ψ. □

Property 5 *(1)* $\Psi \circ \Psi \circ \Lambda_x = \Lambda_x$ *and (2)* $\Psi \circ \Psi = Id_\Lambda$, *where* $\Lambda = \{\Lambda_x(\pi) : \forall x \in V, \pi \in P[x]\}$.

Proof (1) Consider an arbitrary path $\pi = [\langle x_1, x_2\rangle, \ldots, \langle x_n, x_{n+1}\rangle]$. By definition of Ψ and Λ_{x_1}, we have $\Psi(\Lambda_{x_1}(\pi)) = [\psi(\lambda_{x_n}(\langle x_n, x_{n+1}\rangle), \ldots, \psi(\lambda_{x_1}(\langle x_1, x_2\rangle))]$. Thus, $\Psi([\psi(\lambda_{x_n}(\langle x_n, x_{n+1}\rangle), \ldots, \psi(\lambda_{x_1}(\langle$ $= [\psi(\psi(\lambda_{x_1}(\langle x_1, x_2\rangle))), \ldots, \psi(\psi(\lambda_{x_n}(\langle x_n, x_{n+1}\rangle)))] = [\lambda_{x_1}(\langle x_1, x_2\rangle) \ldots \lambda_{x_n}(\langle x_n x_{n+1}\rangle)]$ $= \Lambda_{x_1}(\pi)$ by definition of Ψ, by Property 3 and by definition of Λ_{x_1}.
(2) Trivial.

□

Definition 9 $\mathcal{H}$ - Homonymity
A consistent coding function c is homonymous, *iff* $\forall x, y \in V$, $\pi_1 \in P[x,x]$,$\pi_2 \in P[y,y]$: $c(\Lambda_x(\pi_1)) = c(\Lambda_y(\pi_2))$

In other words, when c is homonymous, every node associates to itself the same local name.

Definition 10 $\mathcal{NS}$ - Name Symmetry
A consistent coding function c has name symmetry *iff there exists a function* $\mu : \mathcal{N}_c \rightarrow \mathcal{N}_c$ *s.t.* $\forall \pi \in P[x,y]$: $\mu(c(\Lambda_x(\pi))) = c(\Lambda_y(\overline{\pi}))$.

Theorem 3 *A consistent coding function c has name symmetry iff* $\forall \pi_1 \in P[s,t], \pi_2 \in P[w,z]$ $c(\Lambda_s(\pi_1)) = c(\Lambda_w(\pi_2)) \Rightarrow c(\Psi(\Lambda_s(\pi_1))) = c(\Psi(\Lambda_w(\pi_2)))$.

Proof ($\Rightarrow$) By contradiction, suppose that μ exists and that exist $\pi_1 \in P[s,t], \pi_2 \in P[w,z]$ s.t. $c(\Lambda_s(\pi_1)) = c(\Lambda_w(\pi_2))$ while $c(\Psi(\Lambda_s(\pi_1))) \neq c(\Psi(\Lambda_w(\pi_2)))$.

Since $\overline{\pi_1} \in P[t,s]$ and $\overline{\pi_2} \in P[z,w]$, by Property 4, $\Lambda_t(\overline{\pi_1}) = \Psi(\Lambda_s(\pi_1))$ and $\Lambda_z(\overline{\pi_2}) = \Psi(\Lambda_w(\pi_2))$. By name symmetry it must be $\mu(c(\Lambda_s(\pi_1))) = c(\Lambda_t(\overline{\pi_1}))) = c(\Psi(\Lambda_s(\pi_1))))$ and $\mu(c(\Lambda_w(\pi_2))) = c(\Lambda_z(\overline{\pi_2}))) = c(\Psi(\Lambda_w(\pi_2))))$. But $c(\Lambda_s(\pi_1)) = c(\Lambda_w(\pi_2))$ implies $\mu(c(\Lambda_s(\pi_1)))$ $= \mu(c(\Lambda_w(\pi_2)))$, which implies $c(\Psi(\Lambda_s(\pi_1))) = c(\Psi(\Lambda_w(\pi_2)))$. Contradiction.

($\Leftarrow$) We first prove that $c \circ \Psi \circ c_s^{-1} \circ c = c \circ \Psi$. By Property 1, $c \circ c_s^{-1} \circ c = c$. By hypothesis, $f \circ \Psi \circ f_s^{-1} \circ c = c \circ \Psi$. Let $\mu = c \circ \Psi \circ c_s^{-1}$; then $\mu(c(\Lambda_x(\pi))) = c(\Psi(c_s^{-1}(c(\Lambda_x(\pi)))) =$ $c(\Psi(\Lambda_x(\pi))) = c(\Lambda_y(\overline{\pi}))$. □

4.2 Backward Consistency

Definition 11 $\mathcal{BC}$ - Backward Consistency
A coding function c has Backward Consistency, *iff* $\forall x, y, z \in V$, $\pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$: $c(\Lambda_x(\pi_1)) = c(\Lambda_y(\pi_2)) \Leftrightarrow x = y$

Definition 12 Backward Decoding Function
Given a consistent coding function c, a backward consistent decoding function *for c is any function* $d_b : \mathcal{N}_c \times \mathcal{L} \to \mathcal{N}_c$ *s.t.* $\forall \pi \in P[x, y]$, $\langle y, z\rangle \in E(y)$,

$$d_b(c(\Lambda_x(\pi)), \lambda_y(\langle y, z\rangle)) = c(\Lambda_x(\pi) \cdot \lambda_y(\langle y, z\rangle))$$

If $\forall \alpha \in \mathcal{L}^+$, $a \in \mathcal{L}$, $d_b(c(\alpha), a) = c(\alpha \cdot a)$, then d_b is said to be *everywhere backward consistent.*

Let (c, d) be a $\mathcal{SD}$ in (G, λ). We can easily extend the domain of a consistent decoding function d to $\mathcal{L}^+ \times \mathcal{N}_c$ in the following recursive way: $d(a \cdot \omega, c(\beta)) = d(a, d(\omega, c(\beta)))$, where $a \in \mathcal{L}$, $\omega, \beta \in \mathcal{L}^+$. The extended decoding function is consistent for c, that is $\forall \pi_1 \in P(x, y), \pi_2 \in P[y, z]$, $d(\Lambda_x(\pi_1), c(\Lambda_y(\pi_1))) = c(\Lambda_x(\pi_1) \cdot \Lambda_y(\pi_2))$. Analogously we can extend the backward decoding function d_b.

Theorem 4 $\mathcal{ES} \not\Rightarrow \mathcal{BC}$: *Edge symmetry is not sufficient for backward consistency.*

Proof Consider the graph (G, λ) of Figure 1 where the labeling is a coloring; thus, it has edge symmetry. It is easy to see that there exist a Sense of Direction (c, d) in (G, λ). By contradiction, assume that there is backward consistency. By consistency of c, we have that $c(2) = c(4)$; it follows that $d(1, c(2)) = d(1, c(4))$, which implies $c(1 \cdot 2) = c(1 \cdot 4)$. We also have, by consistency of c, that $c(1 \cdot 4) = c(3)$; thus, we also have that $c(1 \cdot 2) = c(3)$ which contradicts the assumption of backward consistency. □

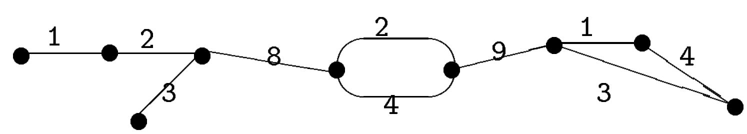

Figure 1: A labeling which has $\mathcal{ES}$ but does not have $\mathcal{BC}$ and $\mathcal{NS}$.

Lemma 1 *Let* (c, d) *be a Sense of Direction in* (G, λ), *and let* d *be everywhere consistent, then* $\forall \alpha, \omega \in \mathcal{L}^+ : c(\alpha) = d(\alpha, c(\omega \cdot \Psi(\omega)))$

Definition 13 *A consistent coding function c has* suffix reduction everywhere *iff* $\forall \alpha, \beta, \omega \in \mathcal{L}^+$: $c(\alpha \cdot \omega) = c(\beta \cdot \omega) \Rightarrow c(\alpha) = c(\beta)$

Lemma 2 *Let* (c, d) *be a Sense of Direction in* (G, λ) *with d everywhere consistent. If there exists a everywhere consistent backward decoding function* d_b *for c, then c has suffix reduction everywhere.*

Proof Let (c, d) be such Sense of Direction in (G, λ). Let $\alpha, \beta, \omega \in \mathcal{L}^+$, and let $c(\alpha \cdot \omega) = c(\beta \cdot \omega)$.
By Lemma 1, $c(\alpha) = d(\alpha, c(\omega \cdot \Psi(\omega))) = c(\alpha \cdot \omega \cdot \Psi(\omega))$, by definition of backward decoding $c(\alpha \cdot \omega \cdot \Psi(\omega)) = d_b(c(\alpha \cdot \omega), \Psi(\omega)) = d_b(c(\beta \cdot \omega), \Psi(\omega)) = c(\beta \cdot \omega \cdot \Psi(\omega))$. But, again by Lemma 1, $c(\beta \cdot \omega \cdot \Psi(\omega)) = c(\beta)$; thus, $c(\alpha) = c(\beta)$. □

Theorem 5 *Let (c,d) be a Sense of Direction in (G,λ). with d everywhere consistent. If there exists a backward decoding function d_b for c everywhere consistent, then c has backward consistency.*

Proof Let (c,d) be a Sense of Direction in (G,λ) and let d and d_b be everywhere consistent. By contradiction, let $\exists x,y,z \in V$, $\pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$: $c(\Lambda_x(\pi_1)) = c(\Lambda_y(\pi_2))$ but $x \neq y$. Let $\Lambda_x(\pi_1) = \alpha$ and $\Lambda_y(\pi_2) = \beta$.
By definition of d_b, we have that $c(\beta \cdot \Psi(\alpha)) = d_b(c(\beta), \Psi(\alpha))$; but $c(\beta) = c(\alpha)$, thus,

$$c(\beta \cdot \Psi(\alpha)) = d_b(c(\alpha), \Psi(\alpha)) = c(\alpha \cdot \Psi(\alpha)) \quad (1)$$

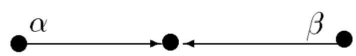

Figure 2: Proof of Theorem 5.

By consistency of c, we have that $c(\Psi(\beta) \cdot \beta \cdot \Psi(\alpha)) = c(\Psi(\alpha) \cdot \alpha \cdot \Psi(\alpha))$ (see Figure 2). By definition of d, $d(\Psi(\beta), c(\beta \cdot \Psi(\alpha))) = d(\Psi(\alpha), c(\alpha \cdot \Psi(\alpha)))$; by (1) $c(\beta \cdot \Psi(\alpha)) = c(\alpha \cdot \Psi(\alpha))$; thus, by Lemma 2, we have that $c(\Psi(\beta)) = c(\Psi(\alpha))$, which contradicts the consistency of c in z. □

Theorem 6 $\mathcal{ES}$, $\mathcal{H}$, $d_b \Rightarrow \mathcal{BC}$
Let (c,d) be a Sense of Direction in (G,λ). If there exists a backward decoding function for c and the system is homonymous, then c has backward consistency.

Proof Let (c,d) be a Sense of Direction in (G,λ) and let d_b be a backward decoding function for c. By contradiction, let $\exists x,y,z \in V$, $\pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$: $c(\Lambda_x(\pi_1)) = c(\Lambda_y(\pi_2))$ but $x \neq y$. Let $\Lambda_x(\pi_1) = \alpha$ and $\Lambda_y(\pi_2) = \beta$.
By definition of homonymity, we have that $c(\alpha \cdot \Psi(\alpha)) = c(\beta \cdot \Psi(\beta))$.
But, $d_b(c(\alpha), \Psi(\alpha)) = d_b(c(\beta), \Psi(\alpha))$ and thus, $c(\alpha \cdot \Psi(\alpha)) = c(\beta \cdot \Psi(\alpha))$. By the consistency of c, $c(\beta \cdot \Psi(\alpha)) \neq c(\beta \cdot \Psi(\beta))$; thus, $c(\alpha \cdot \Psi(\alpha)) \neq c(\beta \cdot \Psi(\beta))$ which contradicts the homonymity assumption. □

Theorem 7 $\mathcal{ES}$, $\mathcal{NS} \Rightarrow \mathcal{BC}$*: Backward Consistency is necessary for Name Symmetry.*

Proof Let (c,d) be a $\mathcal{SD}$ with edge and name symmetry and let ψ be the edge symmetry function. By contradiction, suppose the system does not have backward consistency; that is, suppose $\exists x,y,z$, $\pi_1 \in P[x,z]$, $\pi_2 \in P[y,z]$ with $x \neq y$, such that $c(\Lambda_x(\pi_1)) = c(\Lambda_y(\pi_2))$. Since there is name symmetry, by Theorem 3, we have that $c(\Psi(\Lambda_x(\pi_1))) = c(\Psi(\Lambda_y(\pi_2)))$. But $\Lambda_z(\overline{\pi}_1) = \Psi(\Lambda_x(\pi_1))$, $\Lambda_z(\pi_2) = \Psi(\Lambda_x(\pi_1))$ and $\overline{\pi}_1 \in P[z,x]$, $\overline{\pi}_2 \in P[z,y]$ which contradicts the consistency of c in z. □

4.3 Name Symmetry

Lemma 3 $\mu(\mu(x)) = x$

Proof Let (c, d) be a $\mathcal{SD}$ with name symmetry and let μ be the name symmetry function. Let $\alpha \in \mathcal{L}^+$. By definition of name symmetry function, we have that: $\mu(\mu(c(\alpha))) = \mu(c(\Psi(\alpha))) = c(\Psi(\Psi(\alpha))) = c(\alpha)$. □

Theorem 8 $\mathcal{ES}, \mathcal{NS} \Rightarrow d_b$
The existence of a consistent backward decoding function is necessary for Name Symmetry.

Proof Let (c, d) be a $\mathcal{SD}$ with edge and name symmetry and let ψ and μ be respectively the edge and name symmetry functions. Let $x, y \in V$, $\pi \in P[x, y]$, and $\omega = \Lambda_x(\pi)$. By definition of name symmetry function, we have that $c(\Psi(\omega)) = \mu(c(\omega))$; it follows that $\mu(c(\Psi(\omega))) = \mu(\mu(c(\omega)))$. By Lemma 3, we have that

$$\mu(c(\Psi(\omega))) = c(\omega) \tag{2}$$

Consider now the following backward decoding function: $\forall \pi \in P[x, y]$, $\omega = \Lambda_x(\pi)$ and for $\langle y, z\rangle = a \in E(y)$, $d_b(c(\omega), a) = \mu(d(\psi(a), \mu(c(\omega)))$.
By definition of name symmetry function $\mu(c(\omega)) = c(\Psi(\omega))$, thus, $\mu(d(\psi(a), \mu(c(\omega)) = \mu(d(\psi(a), c(\Psi(\omega))) = \mu(c(\Psi(w \cdot a))$. By (2), we have that $\mu(c(\Psi(w \cdot a)) = c(\omega \cdot a)$. It follows that $d_b(c(\omega), a) = c(\omega \cdot a)$, thus, d_b is consistent. □

Theorem 9 $\mathcal{ES} \not\Rightarrow \mathcal{NS}$: *Edge symmetry is not sufficient for name symmetry.*

Proof Consider the graph (G, λ) of Figure 1 where the labeling is a coloring; thus, it has edge symmetry. It is easy to see that there exist a $\mathcal{SD}$ (c, d) (G, λ). We will show that there is no name symmetry.
By consistency of c, we have that $c(2) = c(4)$, thus $d(1, c(2)) = d(1, c(4)$, which implies that $c(1 \cdot 2) = c(1 \cdot 4)$. On the other hand, by consistency of c, we have that $c(2 \cdot 1) \neq c(3)$, and $c(4 \cdot 1) = c(3)$, which implies that $c(2 \cdot 1) \neq (4 \cdot 1)$. By Theorem 3 it follows that there is no name symmetry in the system. □

Theorem 10 $\mathcal{NS} \not\Rightarrow \mathcal{ES}$: *Name symmetry is not sufficient for edge symmetry*

Proof In Figure 3, there is an example of a labeled graph (G, λ) in which λ does not have edge symmetry. It is easy to see that there exists a $\mathcal{SD}$ (c, d), where c is a chordal coding function (see [3]) without name symmetry. □

Theorem 11 $\mathcal{ES}, d_b \Rightarrow \exists\mathcal{NS}$
Let (c, d) be a Sense of Direction in (G, λ) with edge symmetry. If there exists a consistent backward decoding function d_b, then there exists a Sense of Direction $(\overline{c}, \overline{d_f})$ with Name Symmetry and a backward decoding function $\overline{d_b}$ for $\overline{c}$.

Proof Let $\overline{c} : \mathcal{L}^+ \to 2^{\mathcal{L}^+}$ be defined as follows:

$$\overline{c}(w) = c^{-1} \circ c(w) \cap \Psi \circ c^{-1} \circ c \circ \Psi(w).$$

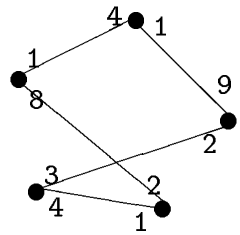

Figure 3: A labeling which has $\mathcal{NS}$ but does not have $\mathcal{ES}$.

Claim 1 *$\alpha \in \overline{c}(w)$ iff $c(\alpha) = c(w)$ and $c \circ \Psi(\alpha) = c \circ \Psi(w)$.*

Proof $c^{-1} \circ c(w) = \{\alpha : c(\alpha) = c(w)\}$ and $\Psi \circ c^{-1} \circ c \circ \Psi(w) = \Psi(\{\alpha' : c(\alpha') = c \circ \Psi(w)\}) = \{\Psi(\alpha') : c(\alpha') = c \circ \Psi(w)\} = \{\Psi \circ \Psi(\alpha) : \Psi(\alpha) = \alpha' \wedge c(\alpha') = c \circ \Psi(w)\} = \{\alpha : c \circ \Psi(\alpha) = c \circ \Psi(w)\}$. Thus $\overline{c}(w) = \{\alpha : c(\alpha) = c(w)\} \cap \{\alpha' : c \circ \Psi(\alpha') = c \circ \Psi(w)\} = \{\alpha : c(\alpha) = c(w) \wedge c \circ \Psi(\alpha) = c \circ \Psi(w)\}$. □

Claim 2 *For each selection function s:*

$$c \circ s \circ \overline{c} = c \quad \textit{and} \quad c \circ \Psi \circ s \circ \overline{c} = c \circ \Psi.$$

Proof $s \circ \overline{c}(w) \in \overline{c}(w)$ by definition of selection function. By Property 1, $c \circ s \circ \overline{c}(w) = c(w)$ and $c \circ \Psi \circ s \circ \overline{c}(w) = c \circ \Psi(w)$. □

We now show that $\overline{c}$ is consistent. $\forall x, y, z, \pi_1 \in P[x,y], \pi_2 \in P[x,z]$ $\overline{c}(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_2))$ implies $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_2))$ by Claim 1. By consistency of c, it follows that $y = z$. Thus we have proved that $\overline{c}(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_2))$ implies $y = z$. If $y = z$, then $\overline{\pi_1}, \overline{\pi_2} \in P[y,x]$. Thus, by consistency of c, $c(\Lambda_y(\overline{\pi_1})) = c(\Lambda_y(\overline{\pi_2}))$. By property 4, $\Lambda_y(\overline{\pi_1})) = \Psi(\Lambda_x(\pi_1))$ and $\Lambda_y(\overline{\pi_2})) = \Psi(\Lambda_x(\pi_2))$. Then $\overline{c}(\Lambda_x(\pi)) = c^{-1} \circ c(\Lambda_x(\pi_1)) \cap \Psi \circ c^{-1} \circ c \circ \Psi(\Lambda_x(\pi_1)) = c^{-1} \circ c(\Lambda_x(\pi_2) \cap \Psi \circ c^{-1} \circ c \circ \Psi(\Lambda_x(\pi_2)) = \overline{c}(\Lambda_x(\pi_2))$, concluding the proof of the consistency of $\overline{c}$.

Let $\overline{d_f} : \mathcal{L} \times 2^{\mathcal{L}^+} \to 2^{\mathcal{L}^+}$ be defined as follows:

$$\overline{d_f}(a, \overline{c}(w)) = c^{-1}(d_f(a, c \circ s(\overline{c}(w))) \cap \Psi \circ c^{-1}(d_b(c \circ \Psi \circ s(\overline{c}(w), \psi(a))))$$

where s is a selection function.

Now we prove that $\overline{d_f}$ is a consistent decoding function for $\overline{c}$; that is, $\forall x, y, z, \langle x, y\rangle\pi \in P[x,z]$ $\overline{d_f}(\lambda_x(\langle x,y\rangle), \overline{c}(\Lambda_y(\pi))) = \overline{c}(\Lambda_x(\langle x,y\rangle\pi))$. By definition of $\overline{d_f}$, $\overline{d_f}(\lambda_x(\langle x,y\rangle), \overline{c}(\Lambda_y(\pi))) = c^{-1}(d_f(\lambda_x(\langle x,y\rangle), c \circ s(\overline{c}(\Lambda_y(\pi))) \cap \Psi \circ c^{-1}(d_b(c \circ \Psi \circ s(\overline{c}(\Lambda_y(\pi)), \psi(\lambda_x(\langle x,y\rangle))))))$. By Claim 2 $c^{-1}(d_f(\lambda_x(\langle x,y\rangle), c \circ s(\overline{c}(\Lambda_y(\pi))) \cap \Psi \circ c^{-1}(d_b(c \circ \Psi \circ s(\overline{c}(\Lambda_y(\pi)), \psi(\lambda_x(\langle x,y\rangle)))))) =$ $c^{-1}(d_f(\lambda_x(\langle x,y\rangle)c(\Lambda_y(\pi))) \cap \Psi \circ c^{-1}(d_b(c \circ \Psi(\Lambda_y(\pi)), \psi(\lambda_x(\langle x,y\rangle)))))$. By consistency of d_f and d_b, $c^{-1}(d_f(\lambda_x(\langle x,y\rangle), c(\Lambda_y(\pi))) \cap \Psi \circ c^{-1}(d_b(c \circ \Psi(\Lambda_y(\pi)), \psi(\lambda_x(\langle x,y\rangle))))) = c^{-1}(c(\lambda_x(\langle x,y\rangle) \cdot \Lambda_y(\pi))) \cap \Psi \circ c^{-1}(\Psi(\Lambda_y(\pi)) \cdot \psi(\lambda_x(\langle x,y\rangle))))$. Finally, $c^{-1}(c(\lambda_x(\langle x,y\rangle) \cdot \Lambda_y(\pi))) \cap \Psi \circ c^{-1}(\Psi(\Lambda_y(\pi)) \cdot \psi(\lambda_x(\langle x,y\rangle)))) = c^{-1}(c(\Lambda_x(\langle x,y\rangle\pi))) \cap \Psi \circ c^{-1}(\Psi(\Lambda_x(\langle x,y\rangle\pi))) = \overline{c}(\Lambda_x(\langle x,y\rangle\pi))$. Thus $\overline{d_f}$ is consistent.

In a similar way, it can be shown that the function $\overline{d_b}(\overline{c}(w),a) = c^{-1}(d_b(\overline{c}(w),a)) \cap \Psi \circ c^{-1}(d_f(\psi(a), c\circ\Psi\circ s(\overline{c}(w))))$ is a backward consistent decoding function for $\overline{c}$.

We now prove that $\mu = \Psi : 2^{\mathcal{L}} \to 2^{\mathcal{L}}$ is the symmetry function; that is, $\forall x,y,\pi \in P[x,y]$ $\Psi(\overline{c}(\Lambda_x(\pi)) = \overline{c}(\Lambda_y(\overline{\pi}))$. By Property 4 $\overline{c}(\Lambda_y(\overline{\pi})) = \overline{c}(\Psi(\Lambda_x(\pi)))$. By definition of $\overline{c}$,$\overline{c}(\Psi(\Lambda_x(\pi))) = c^{-1}\circ c(\Psi(\Lambda_x(\pi)))\cap\Psi\circ c^{-1}\circ c\circ\Psi(\Psi\Lambda_x(\pi))$. By Property 5$c^{-1}\circ c(\Psi(\Lambda_x(\pi)))\cap \Psi\circ c^{-1}\circ c\circ\Psi(\Psi\Lambda_x(\pi)) = \Psi\circ\Psi\circ c^{-1}\circ c(\Psi(\Lambda_x(\pi)))\cap\Psi\circ c^{-1}\circ c\circ\Lambda_x(\pi)$. By definition of $\overline{c}$, $\Psi(\Psi\circ c^{-1}\circ c(\Psi(\Lambda_x(\pi)))\cap c^{-1}\circ c\circ\Lambda_x(\pi)) = \Psi\circ\overline{c}(\Lambda_x(\pi))$ which conclude the proof.

□

4.4 Homonymity

Lemma 4 *[5] Let (G,λ) be a system with edge symmetry. If there exist $x,y,z \in V$, $y \neq z$ and $\pi_1 \in P[x,x]$, $\pi_2 \in P[y,z]$ such that $\Lambda_x(\pi_1) = \Lambda_y(\pi_2)$ then there does not exist any consistent coding function for (G,λ).*

Theorem 12 $\mathcal{ES} \Rightarrow \exists\mathcal{H}$
Let λ be Edge Symmetric and let (c,d) be a Sense of Direction in (G,λ). There exist a Sense of Direction with homonymy.

Proof Let $c : \mathcal{L}^+ \to \mathcal{N}$ and let $\bigcirc$ be a new element (i.e. $\bigcirc \notin \mathcal{N}$) and $\overline{c} : \mathcal{L}^+ \to \mathcal{N}\cup\{\bigcirc\}$ s.t.

$$\overline{c}(w) = \begin{cases} \bigcirc & \text{if } \exists x,\pi \in P[x,x] \wedge \Lambda_x(\pi) = w \\ c(w) & \text{otherwise} \end{cases}$$

Note: by Lemma 4, if $\pi \in P[x,y] \wedge x \neq y$, then $\overline{c}(\Lambda_x(\pi)) = c(\Lambda_x(\pi))$.

We will first show that $\overline{c}$ is consistent. For every $x,y,z \in V, \pi_1 \in P[x,y], \pi_2 \in P[x,z]$ only four cases are possible:

1. $x = y = z$; thus $\pi_1,\pi_2 \in P[x,x]$ which implies $\bigcirc = \overline{c}(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_2)) = \bigcirc$; that is, $\overline{c}(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_2)) \Leftrightarrow y = z$.

2. $x = y \neq z$; thus $\pi_1 \in P[x,x]$ and $\pi_2 \in P[x,z]$ which imply $\bigcirc = \overline{c}(\Lambda_x(\pi_1)) \neq \overline{c}(\Lambda_x(\pi_2)) = c(\lambda_x(\pi_2)) \in \Lambda$. That is, $\overline{c}(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_2)) \Leftrightarrow y = z$.

3. $x = z \neq y$ As in case 2 above switching y and z.

4. $x \neq y$ and $x \neq z$; thus, $c(\Lambda_x(\pi_1)) = \overline{c}(\Lambda_x(\pi_1))$ and $c(\Lambda_x(\pi_2)) = \overline{c}(\Lambda_x(\pi_2))$. By consistency of c, $c(\Lambda_x(\pi_1)) = c(\Lambda_x(\pi_2)) \Leftrightarrow y = z$.

Let $\overline{d} : \mathcal{L}\times(\mathcal{N}\cup\{\bigcirc\}) \to \mathcal{N}\cup\{\bigcirc\}$ be defined as follows:

$$\overline{d}(a,n) = \begin{cases} c(a) & \text{if } n = \bigcirc \\ \bigcirc & \text{if } \exists x,y,\langle x,y\rangle\pi \in P[x,x] \wedge \Lambda_x(\langle x,y\rangle) = a\cdot w \wedge c(w) = n \\ d(a,n) & \text{otherwise.} \end{cases}$$

Claim 3 *For each $a \in \mathcal{L}$ and each $n \in \mathcal{N}$: if $\exists x',y',z',x' \neq z', \langle x',y'\rangle\pi' \in P[x',z']$: $\Lambda_{x'}(\langle x',y'\rangle\pi') = a\cdot w' \wedge c(w') = n$, then $\not\exists x,y,\langle x,y\rangle\pi \in P[x,x] : \Lambda_x(\langle x,y\rangle\pi) = a\cdot w \wedge c(w) = n$*

Proof Let $x, y \in V$ and $\langle x,y\rangle\pi \in P[x,x]$ be s.t. $\Lambda_x(\langle x,y\rangle\pi) = a \cdot w$; and let $x', \underline{y', z'} \in V(x' \neq z')$ and $\langle x', y\rangle\pi' \in P[x', z']$ be s.t. $\Lambda_y(\langle x', y'\rangle\pi') = a \cdot w' \wedge c(w') = n$. Since $\overline{\langle x, y\rangle} \in P[y,x]$ and $\pi \in P[y,x]$ by consistency of c, $c(\lambda_y(\langle y,x\rangle)) = c(\psi(a)) = c(\Lambda_y(\pi)) = c(w)$. since $\overline{\langle x', y'\rangle} \in P[y', x']$ and $\pi' \in P[y', z']$, by consistency of c and since $x' \neq y'$, it follows that $c(\lambda_{y'}(\langle y', x'\rangle)) = c(\psi(a)) \neq c(\Lambda_{y'}(\pi')) = c(w') = n$. Thus $n \neq c(w)$. □

We now prove that $\overline{d}$ is a consistent decoding function for $\overline{c}$. For every $x, y, z \in V, \pi \in P[y,z]$; there are only three possible cases:

1. $y = z$; thus, $\pi \in P[y,y]$ and, by definition , $c(\Lambda_y(\pi)) = \bigcirc$. Now, $\overline{d}(\lambda_x(\langle x,y\rangle), \overline{c}(\Lambda_y(\pi)) = \overline{d}(\lambda_x(\langle x,y\rangle), \bigcirc) = c(\lambda_x(\langle x,y\rangle))$. Since $\langle x,y\rangle\pi \in P[x,y]$, by consistency of c, $c(\lambda_x(\langle x,y\rangle)) = c(\Lambda_x(\langle x,y\rangle\pi)) = \overline{c}(\Lambda_x(\langle x,y\rangle\pi))$.

2. $x = z$; thus, $\langle x,y\rangle\pi \in P[x,x]$ and by definition of $\overline{d}$ $\overline{d}(\lambda_x(\langle x,y\rangle), \overline{c}(\Lambda_y(\pi))) = \bigcirc$. By definition of $\overline{c}$, $c(\Lambda_x(\langle x,y\rangle\pi)) = \bigcirc$.

3. $x \neq z \neq y$; thus, $\overline{c}(\Lambda_y(\pi)) = c(\Lambda_y(\pi))$ and, by Claim 3, $\overline{d}(\lambda_x(\langle x,y\rangle), \overline{c}(\Lambda_y(\pi))) = d(\lambda_x(\langle x,y\rangle), c(\Lambda_y(\pi)))$. Since d is a consistent decoding function for c, then $(\lambda_x(\langle x,y\rangle), c(\Lambda_y(\pi))) = c(\Lambda_x(\langle x,y\rangle\pi)) = \overline{c}(\Lambda_x(\langle x,y\rangle\pi))$.

Thus $(\overline{c}, \overline{d})$ is a Sense of Direction. Since, for every cycle the sequence of labels are mapped by $\overline{c}$ to the same symbol $\bigcirc$, $\overline{c}$ is homonymous. □

Theorem 13 *In connected graphs, all colorings $\mathcal{SD}$ are homonymous.*

Proof Let (c,d) be a $\mathcal{SD}$ in (G,λ), where λ is a coloring. By contradiction. Suppose that the system is not homonymous; that is there exist u,$v \in V$, $\pi_1 \in P[u,u]$, $\pi_2 \in P[v,v]$, such that $c(\Lambda_u(\pi_1)) \neq c(\Lambda_v(\pi_2))$. Consider a path connecting u and v (such a path exists since G is connected). Let $\pi = [u = x_0, x_1, \ldots x_i, \ldots, x_m = v]$ be the path, and let $\Lambda_u(\pi) = (a_1, \ldots, a_m)$. By consistency of c, $c(\Lambda_u(\pi_1)) = c(\lambda_{x_0}(\langle x_0, x_1\rangle) \cdot \lambda_{x_1}(\langle x_1, x_0\rangle)) = c(a_0 \cdot a_0)$, and analogously $c(\Lambda_v(\pi_2)) = c(\lambda_{x_m}(\langle x_m, x_{m-1}\rangle) \cdot \lambda_{x_1}(\langle x_1, x_0\rangle)) = c(a_m \cdot a_m)$. We want to show that $c(a_0 \cdot a_0) = c(a_i \cdot a_i)$ (and thus, $c(a_0 \cdot a_0) = c(a_m \cdot a_m)$, which would contradict the homonymity).
By induction on i.
Clearly, $c(a_0 \cdot a_0) = c(a_1 \cdot a_1)$.
Let $c(a_0 \cdot a_0) = c(a_i \cdot a_i)$, we will show that $c(a_0 \cdot a_0) = c(a_{i+1} \cdot a_{i+1})$. Since λ is a coloring, we have that $c(\Lambda_{x_{i-1}}(\langle x_{i-1}, x_i\rangle) \cdot \lambda_{x_i}(\langle x_i, x_{i-1}\rangle)) = c(\Lambda_{x_i}(\langle x_i, x_{i-1}\rangle) \cdot \lambda_{x_{i-1}}(\langle x_{i-1}, x_i\rangle))$; by consistency of c, we have that $c(\Lambda_{x_i}(\langle x_i, x_{i-1}\rangle) \cdot \lambda_{x_{i-1}}(\langle x_{i-1}, x_i\rangle)) = c(\Lambda_{x_i}(\langle x_i, x_{i+1}\rangle) \cdot \lambda_{x_{i+1}}(\langle x_{i+1}, x_i\rangle))$, thus, it follows that $c(\Lambda_{x_{i-1}}(\langle x_{i-1}, x_i\rangle) \cdot \lambda_{x_i}(\langle x_i, x_{i-1}\rangle)) = c(\Lambda_{x_i}(\langle x_i, x_{i+1}\rangle) \cdot \lambda_{x_{i+1}}(\langle x_{i+1}, x_i\rangle))$, that is, $c(a_i \cdot a_i) = c(a_{i+1} \cdot a_{i+1})$ and, thus, $c(a_0 \cdot a_0) = c(a_{i+1} \cdot a_{i+1})$.
□

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