Symmetries and Sense of Direction in Labeled Graphs*

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Abstract

We consider distributed systems modeled by edge-labeled graphs. Properties of the labeling can be used in the design of efficient protocols; for example, sense of direction is known to have a strong impact on the communication complexity of many distributed problems.

In this paper, we analyze some relations between topology and symmetries in labeled graphs. In particular, we characterize the classes of completely symmetric and completely surrounding symmetric labeled graphs; we show that the former is a proper subset of the class of regular graphs, while the latter coincides with the class of Cayley graphs.

We then focus on the relationship between symmetries and sense of direction. We show an interesting link between minimal sense of direction in d-regular graphs (i.e., sense of direction that uses d labels) and Cayley graphs. Namely, we prove that a regular graph has a minimal symmetrical sense of direction iff it is a Cayley graph. We also discuss the relationship between minimal sense of direction and recently introduced group-based labelings.

1 Introduction

A *distributed system* is a collection of autonomous entities communicating by the exchange of messages. The communication topology of the system can be represented as an edge-labeled undirected graph where nodes correspond to the system entities, edges represent pairs of neighboring entities (i.e., entities which can communicate directly), and each node has a local label (called *port number*) associated to each of its incident edges. The entire system will be denoted by a labeled graph (G, λ). If all edges incident

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to a node are distinct, λ is said to be a *local orientation*. It is well known that, if λ satisfies also the set of consistency constraints called *sense of direction* [8], the communication complexity of several distributed problem is drastically reduced (e.g., see [7, 11, 13, 15, 16, 20]).

In this paper we consider symmetries of a graph (G, λ) with respect to the notions of *view* and *surrounding*. Our interest derives from the fact that the view describes what is computable in anonymous systems with local orientation [18, 24], and the surrounding represents what is computable in systems with sense of direction [9, 10]. We then analyze the relationship between distributed systems represented by labeled graphs, topology, symmetries, and sense of direction.

The first form of symmetry we consider is based on the notion of *view*; this notion has been introduced in [23, 24] and extensively studied sometimes with different names (e.g.,[1, 2, 9, 10, 12, 17, 19, 18, 21]). Informally, the view of a node in a graph G is a rooted tree "locally isomorphic" to G; i.e., there exists a map from the vertices of the tree to the vertices of G which preserves edges and edges-labels for a 2-neighborhood around each node. A graph is *completely view symmetric* when all the nodes have the same view, i.e., when all the views are indistinguishable.

We characterize the class of completely view symmetric graphs and prove that it is a proper subset of the class of regular graphs.

We then consider a stronger form of symmetry based on the notion of *surrounding* of a node which has been introduced and studied in [9, 10]. Informally, the d-surrounding of a node u in a graph G is a graph isomorphic to the subgraph of G containing all the nodes at distance less than d from u. A graph is *S_k-symmetric* when there is a partition of the nodes in k classes, such that all the nodes in each class have the same surroundings; it is *completely surrounding symmetric* when all the nodes have the same surroundings.

We characterize both the class of S_k-symmetric and that of completely surrounding symmetric graphs. In particular, we show that the class of completely surrounding symmetric graphs coincides with the class of Cayley graphs.

As a consequence of these characterizations, we point out an interesting link between minimal sense of direction in regular graphs (i.e., sense of direction that uses d labels where d is the degree of the graph) and completely surrounding symmetric graphs.

In fact, we prove that a regular graph has a minimal symmetrical sense of direction iff it is completely surrounding symmetric, i.e. iff it is a Cayley graph.

Finally, we discuss the relationship between sense of direction and the class of group-based labeling introduced in [22].

In the next section we discuss the framework and some basic properties. In Section 3, we introduce the notion of view and we characterize the class of completely symmetric graphs. In Section 4, we introduce the notion of surrounding, describe an intermediate situation of symmetry, and characterize the class of completely surrounding symmetric graphs. In Section 5, we show the relationship between sense of direction, completely surrounding symmetric graphs, and group-based labelings.

2 Framework and Basic Properties

Let $G = (V, E)$ be a graph where nodes correspond to entities and edges correspond to direct bidirectional communication links between entities. Let $E(x)$ denote the set of edges incident to node x. A *path* π in G is a sequence of edges in which the endpoint of one edge is the starting point of the next edge. Let $P[x]$ denote the set of all the paths with $x \in V$ as a starting point, let $P[x, y]$ denote the set of paths starting from node $x \in V$ and ending in node $y \in V$, and let $P^d[x, y]$ denote the set of paths of length at most d, that belong to $P[x, y]$.

Given a graph $G = (V, E)$ and a set Σ of labels, a *local edge-labeling* (or labeling) function for $x \in V$ is any function $\lambda_x : E(x) \to \Sigma$ which associates a label $l \in \Sigma$ to each edge $e \in E(x)$. The *labeling* λ of G is the set of local labeling functions, that is $\lambda = \{\lambda_x : x \in V\}$. By (G, λ) we shall denote a *labeled graph*, that is a graph G on which it is defined a labeling λ.

Given a labeling λ and a node $x \in V$, let $\Lambda_x : P[x] \to \Sigma^+$ be the *path-labeling function* defined as follows: for every path $\pi \in P[x_1]$ starting from x_1, $\Lambda_{x_1}(\pi) = [\lambda_{x_1}(\langle x_1, x_2\rangle), \ldots, \lambda_{x_m}(\langle x_m, x_{m+1}\rangle)]$ where $\pi = [\langle x_1, x_2\rangle, \ldots, \langle x_m, x_{m+1}\rangle]$. Let $\Lambda_{(G,\lambda)}[x] = \{\Lambda_x(\pi) : \pi \in P[x]\}$, $\Lambda_{(G,\lambda)}[x, y] = \{\Lambda_x(\pi) : \pi \in P[x, y]\}$, $\Lambda^d_{(G,\lambda)}[x, y] = \{\Lambda_x(\pi) : \pi \in P^d[x, y]\}$, and $\Lambda^d_{(G,\lambda)}[x] = \bigcup_{y \in V} \Lambda^d[x, y]$. When (G, λ) is clear from the contest, we will omit the subscript (G, λ) from the notation.

A labeled graph (G, λ) has *local orientation* iff $\forall x \in V, \forall e_1, e_2 \in E(x)$, $\lambda_x(e_1) = \lambda_x(e_2) \Leftrightarrow e_1 = e_2$. That is, a labeling is a local orientation when each node can distinguish among its incident edges.

Let $\mathcal{G}$ denote the set of all graphs, and let $\mathcal{O}$ denote the set of all labeled graphs with local orientation. By definition of local orientation, two different edges outgoing from x have two different labels. We can extend this property to paths of arbitrary length.

Property 1 *[10]*
If λ is a local orientation, then for each $\pi_1, \pi_2 \in P[x]$:
$\pi_1 = \pi_2 \Leftrightarrow \Lambda_x(\pi_1) = \Lambda_x(\pi_2)$

Given $(G, \lambda) \in \mathcal{O}$, let $\underset{(G,\lambda)}{\rightarrow}$ be the partial function $\underset{(G,\lambda)}{\rightarrow}: V \times \Sigma^* \to V$ such that $\underset{(G,\lambda)}{\rightarrow}(v, \alpha) = w \Leftrightarrow \exists \pi \in P[v, w] \wedge \Lambda_v(\pi) = \alpha$.

Note that by Property 1, $\underset{(G,\lambda)}{\rightarrow}$ is well defined. In the following, the symbol $\underset{(G,\lambda)}{\rightarrow}$ will be also used in infix notation (i.e. $\underset{(G,\lambda)}{\rightarrow}(v, \alpha) = v \underset{(G,\lambda)}{\rightarrow} \alpha$). Moreover, when (G, λ) is clear from the context, we will denote $\underset{(G,\lambda)}{\rightarrow}$ with $\rightarrow$.
By definition, the following property trivially holds.

Property 2 *[10] For each $u \in V$, $\alpha \in \Lambda[u] \Leftrightarrow u \rightarrow \alpha \in V$.*

A labeling λ has a *symmetric* function ψ if and only if for each $\langle x, y\rangle \in E$, $\lambda_y(\langle x, y\rangle) = \psi(\lambda_x(\langle x, y\rangle))$. In this case we say that the labeling is symmetric.

For $\pi \in P[x, y]$ let $\overline{\pi}$ denote the *reverse* path of π . If λ is symmetric, then let $\Psi : \Sigma^* \to \Sigma^*$ s.t. $\Psi(a_1 \cdot a_2 \cdot \ldots \cdot a_n) = \psi(a_n) \cdot \ldots \cdot \psi(a_1)$ where $a_i \in \Sigma$. Then the following easy follows:

Property 3 *Let λ be symmetric. For each $\pi \in P[x, y]$, $\Lambda_y(\overline{\pi}) = \Psi(\lambda_x(\pi))$.*

By Properties 1, 2, and 3, we have that:

Property 4 *If λ is symmetric, then $u \rightharpoonup (\alpha \cdot \Psi(\alpha)) = u$.*

3 Views and V-Symmetries

A crucial concept when computing on anonymous networks is the one of *view*, introduced in [24]. The view $T_{(G,\lambda)}(v)$ of a node v in a labeled graph (G, λ) is an infinite, labeled, rooted tree, defined recursively as follows. $T_{(G,\lambda)}(v)$ has the root x_0 corresponding to v. For each vertex v_i adjacent to v in G, $T_{(G,\lambda)}$ has a node x_i and an edge from x_0 to x_i with labels $\lambda_v(\langle v, v_i\rangle)$ and $\lambda_{v_i}(\langle v, v_i\rangle)$ at its x_0's and x_i's ends respectively. Node x_i is now the root of $T_{(G,\lambda)}(v_i)$ from v_i.

In other words, the view of (G, λ), $T_{(G,\lambda)}(v)$, of a node v is a rooted tree "locally isomorphic" to G; i.e., such that there exists a map from the vertices of the tree to the vertices of G which preserves edges and edges-labels for a 2-neighborhood around each node.

In the following, we shall refer to a node of a view by using the sequence of labels in the shortest path (in the view) from the root to that node. Since a view is a tree, such a naming in not ambiguous, and shall be called *canonical*. Thus, in a canonical naming, node x in view T is $x = \alpha$, where α is the (unique) sequence of edge labels in the shortest path in T from the root to x. Throughout the paper we shall exclusively use canonical naming of the nodes.

Let Σ be a (finite) set. Let Σ^* be the set of strings of element of Σ including the *empty* string ϵ, let $\Sigma^d \subseteq \Sigma^*$ be the set of strings of length at most d. Given $\alpha, \beta \in \Sigma^*$, let $\alpha \cdot \beta \in \Sigma^*$ be the *concatenation* of the α and β; given a set of strings A let $\alpha \cdot A = \{\alpha \cdot w : w \in A\}$. Given a pair of strings $\langle \alpha, \beta\rangle$ and a string γ, let $\gamma \cdot \langle \alpha, \beta\rangle = \langle \gamma \cdot \alpha, \gamma \cdot \beta\rangle$; given a set B of pair of strings, let $\gamma \cdot B = \{\gamma \cdot \langle \alpha, \beta\rangle : \langle \alpha, \beta\rangle \in B\}$.

When no ambiguity arises, we shall denote a view $T_{(G,\lambda)}(v)$ simply by $T(v)$. For any integer $d \geq 0$, let $T^d(v)$ denote the d-view of node v, i.e., $T(v)$ truncated to distance d. Given a labeled graph X, let $\mathcal{V}(X)$, $\mathcal{E}(X)$ and $\mathcal{L}(X)$ denote the vertices, the edges and the labeling of X, respectively.

By definition of view and of canonical naming, the following properties immediately follow.

Property 5 *[10] Given* $(G=(V,E),\lambda)$ *and* $u \in V$*:*

1. $T^0(u) = ((\{\epsilon\}, \emptyset), \emptyset)$;

2. $\mathcal{V}(T^{i+1}(u)) = \{\epsilon\} \cup \bigcup_{v:\langle u,v\rangle \in E} \Big(\lambda_u(\langle u,v\rangle) \cdot \mathcal{V}(T^i(v))\Big)$;

3. $\mathcal{E}(T^{i+1}(u)) = \{\langle \epsilon, \lambda_u(\langle u,v\rangle)\rangle : \langle u,v\rangle \in E\} \cup \bigcup_{v:\langle u,v\rangle \in E} \Big(\lambda_u(\langle u,v\rangle) \cdot \mathcal{E}(T^i(v))\Big)$;

4. $\mathcal{L}(T^{i+1}(u))_\alpha(\langle \alpha,\beta\rangle) = \begin{cases} l & \text{if } \alpha=\epsilon \wedge \beta = l \\ \lambda_{u\to l}(\langle u \to l, u\rangle) & \text{if } \alpha = l \wedge \beta=\epsilon \\ \mathcal{L}(T^i(u\to l))_{\alpha'}(\langle \alpha',\beta'\rangle) & \text{if } \alpha = l\cdot\alpha' \wedge \beta = l \cdot \beta' \end{cases}$

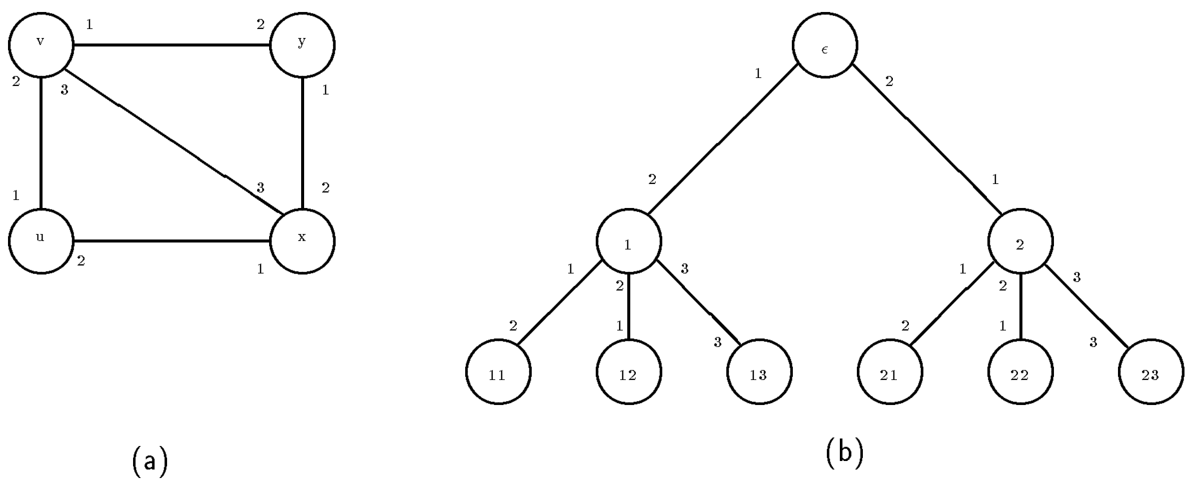

Figure 1: A labeled graph and its 2-view from node u.

Example 1 *A labeled graph* G *and its* 2*-view from node* u *are shown in Figure 1; each node in this view is uniquely identified by the sequence of labels corresponding to the shortest path from* u *in the tree.*

In [24], it has been proved that the cardinality of the set $\{v : T^{n-1}(v) = T^{n-1}(u)\}$ is the same for all u. Let $\sigma_{(G,\lambda)}$ denote such cardinality (called *symmetricity* of (G,λ)).

Definition 1 V-symmetry
A labeled graph $(G=(V,E),\lambda)$ *is* completely view symmetric *(or is* V*-symmetric) when all views are equal, that is* $\sigma_{(G,\lambda)} = |V|$.

The following lemmas express some relationships between nodes, edges and paths in the views.

Lemma 1 *[10]*
Let $T^d(u) = (T=(V_T,E_T),\overline{\lambda})$*:*

1. $\overline{\lambda}_\alpha(\langle\alpha,\alpha\cdot l\rangle)=l$. *If* λ *has symmetric function* ψ, *then* $\overline{\lambda}_{\alpha\cdot l}(\langle\alpha,\alpha\cdot l\rangle)=\psi(l)$.

2. $\exists\pi\in P_G^d[u,u\rightarrow\alpha]\wedge\Lambda_u(\pi)=\alpha\Leftrightarrow\alpha\in V_T$.

Lemma 2 *[24] Let* $|V|=n$. *For each* d, $\sigma_{(G,\lambda)}$ *divides* $\#_d(G)$, *where* $\#_d(G)$ *is the number of vertices in* G *with degree* d.

In the following theorem we characterize the class of V-symmetric graphs.

Theorem 1 *A labeled graph* (G,λ) *is* V*-symmetric if and only if* G *is* d*-regular for some* d *and* λ *is a symmetric local labeling using* d *labels.*

Proof Let $G=(V,E)$ and $n=|V|$.
$\Rightarrow$
By definition of completely symmetric, $s(G,\lambda)=n$; by Lemma 2, $\sigma_{(G,\lambda)}$ divides $\#_i(G)$ for each i. Moreover, $\sum_{i=1}^{n}\#_i(G)=n$; thus, there exists $\overline{i}$ such that $\#_{\overline{i}}(G)=n$ and $\#_i(G)=0$ for each $i\neq\overline{i}$. That is, G is $\overline{d}$ regular.
By contradiction, suppose that λ is not symmetric. Thus, there exist $\langle x,y\rangle,\langle u,z\rangle\in E$, such that $\lambda_x(\langle x,y\rangle)=\lambda_u(\langle u,z\rangle)$ but $\lambda_y(\langle x,y\rangle)\neq\lambda_z(\langle u,z\rangle)$. Let $\lambda_x(\langle x,y\rangle)=l$. $\langle\epsilon,l\rangle$ is the only edge from the root ϵ of $T^{n-1}(x)$ with label l, and $\langle\epsilon,l\rangle$ is the only edge from the root ϵ of $T^{n-1}(u)$ with label l. Thus, $\lambda_y(\langle x,y\rangle)\neq\lambda_z(\langle u,z\rangle)$ implies $\overline{\lambda}_l(\langle\epsilon,l\rangle)=\lambda_y(\langle x,y\rangle)\neq\lambda_z(\langle u,z\rangle)=\overline{\lambda}_l(\langle\epsilon,l\rangle)$. That is $T(x)\neq T(u)$ contradicting that all views are equal.
Suppose that more than d labels are used by λ. Since G is d-regular, this implies that there exist (at least) two different nodes that use a different set of labels. Let u and v be such nodes and let $l\in\Lambda(u)-\Lambda(v)$. It follows that l is a vertex of $T^1(u)$ but not of $T^1(v)$, contradicting that all views are equal.
$\Leftarrow$
Since the graph is d regular, and λ uses only d labels, then $\Lambda(u)=\Lambda(x)=\Sigma$.
We will show by induction on i that for each x: 1) $\mathcal{V}(T^i(x))=\Sigma^i$; 2) $\mathcal{E}(T^i(x))=\langle\{\epsilon\},\Sigma\rangle\cdot\Sigma^{i-1}$; 3) $\mathcal{L}(T^i(x))=\overline{\lambda}$ (a set of functions independent of x).
($i=0$:) By definition, $\mathcal{V}(T^0(x))=\{\epsilon\}$, $\mathcal{E}(T^0(x))=\emptyset$.
($i\Rightarrow i+1$:) Let $\mathcal{V}(T^i(v))=\Sigma^i$, $\mathcal{E}(T^i(v))=\langle\{\epsilon\},\Sigma\rangle\cdot\Sigma^{i-1})$, and $\mathcal{L}(T^i(v))=\overline{\lambda}^i$. Thus, there is an edge e from u with label l iff there is an edge e' from x with label l. This implies that for each $x\in V$:
$$\mathcal{V}(T^{i+1}(x))=\{\epsilon\}\cup\bigcup_{\forall l\in\Lambda(u)}\left(l\cdot\mathcal{V}(T^i(v))\right)=\{\epsilon\}\cup\bigcup_{\forall l\in\Lambda(u)}\left(l\cdot L^i\right)=\{\epsilon\}\cup\bigcup_{\forall l\in\Sigma}l\cdot\Sigma^i=\Sigma^{i+1}.$$
$$\mathcal{E}(T^{i+1}(x))=\{\langle\epsilon,l\rangle:l\in\Lambda(u)\}\cup\bigcup_{\forall l\in\Lambda(u)}\left(l\cdot\mathcal{E}(T^i(x))\right)=\bigcup_{\forall l\in\Lambda(u)}\left(\{\langle\epsilon,l\}\cup l\cdot\langle\{\epsilon\},\Sigma\rangle\cdot\Sigma^{i-1})\right)=$$
$$\bigcup_{\forall l\in\Sigma}\left(\{\langle\epsilon,l\}\cup l\cdot\langle\{\epsilon\},\Sigma\rangle\cdot\Sigma^{i-1})\right)=\langle\{\epsilon\},\Sigma\rangle\cdot\Sigma^i.$$
By Lemma 1.1, if λ is symmetric, then $\mathcal{L}(T^{i+1}(x))$ is univocally determined by the edges in $\mathcal{E}(T^{i+1}(x))$. Since, as just shown, the set of edges is independent of x, also $\mathcal{L}(T^{i+1}(x))$ is independent of x. □

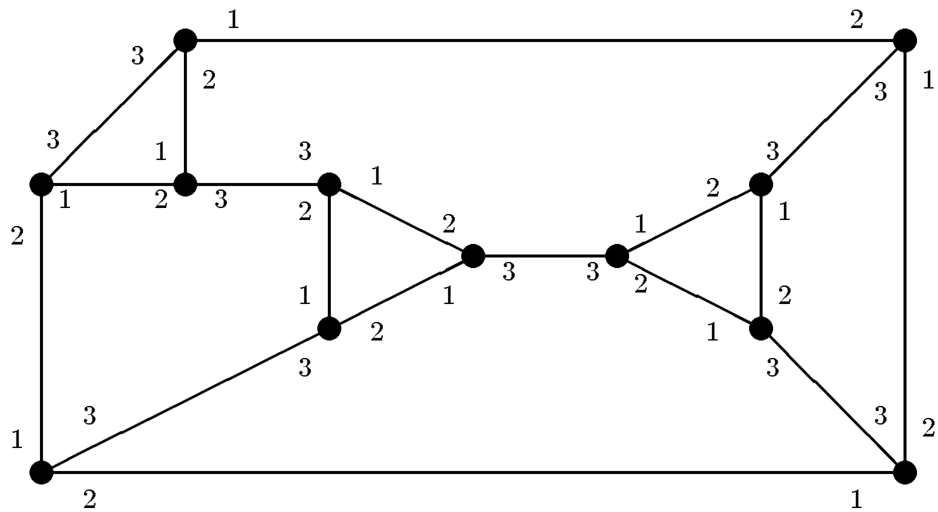

Figure 2: A V-symmetric graph.

Example 2 *Consider the labeled graph (I, δ) shown in Figure 2 (also called minimum identity graph [3]). This graph is V-symmetric. In fact, the labeling δ is symmetric with the following function ψ: $\psi(1) = 2$, $\psi(2) = 1$ and $\psi(3) = 3$. Moreover, δ uses only 3 labels. By Theorem 1, the graph is completely symmetric.*

4 Surroundings and S-Symmetries

4.1 Surroundings

We now recall the concept of *surrounding* of a node in a labeled graph defined in [10]; this notion is stronger than the notion of view considered in the previous Section.

Informally, the d-surrounding of (G, λ) $N^d_{(G,\lambda)}(u)$, from a node u is a labeled graph isomorphic to the subgraph of G containing all the nodes at distance less than d from u. Each node v in the surrounding is denoted by the sequences of labels corresponding to the set of paths in G of length at most d starting from u and ending in v (see Figure 1). Formally:

Definition 2 Surrounding
*Given a labeled graph $(G = (V, E), \lambda)$, an integer $d \geq 0$, and node $u \in V$, the d-*surrounding *of u is the labeled graph $N^d_{(G,\lambda)}(u)$ where*

1. $\mathcal{V}(N^d(u)) = \{[\![\alpha]\!]^d_u : \alpha \in \Lambda^d_{(G,\lambda)}[u]\}$;

2. *given α and β, $\langle [\![\alpha]\!]^d_u, [\![\beta]\!]^d_u \rangle \in \mathcal{E}(N(u))$ if and only if $e = \langle u \rightarrow \alpha, u \rightarrow \beta \rangle \in E$ and $d_G(u, e) \leq d$. The edge e it will be called the* corresponding *edge of $\langle [\![\alpha]\!]^d_u, [\![\beta]\!]^d_u \rangle$.*

3. $\mathcal{L}(N(u))_{[\![\alpha]\!]}(\langle[\![\alpha]\!],[\![\beta]\!]\rangle) = \lambda_{u\rightarrow\alpha}(\langle u \rightarrow \alpha, u \rightarrow \beta\rangle$ *(i.e. the label of the corresponding edge).*

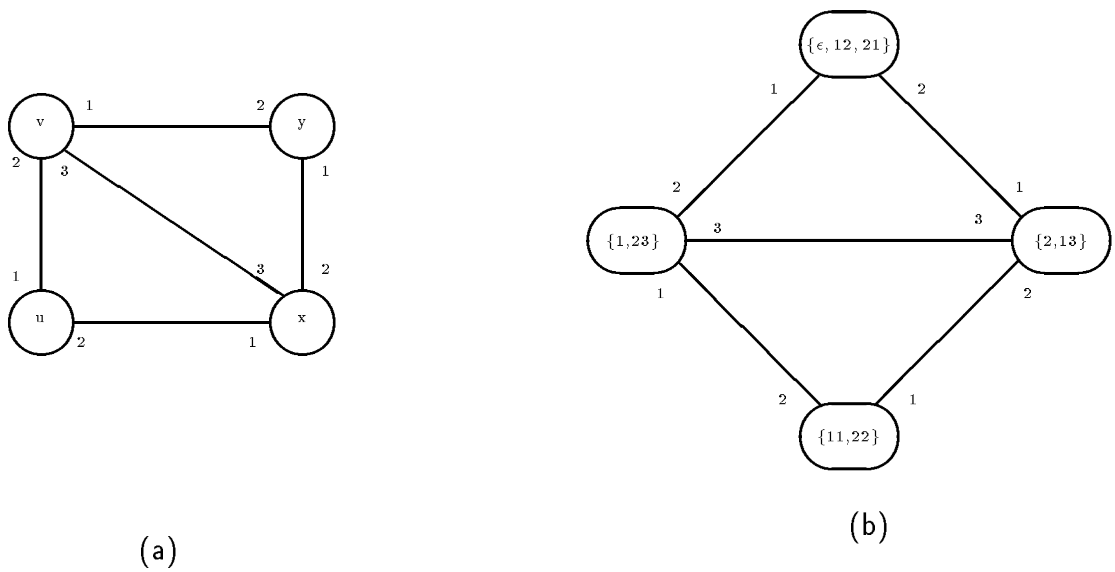

Figure 3: A labeled graph and its 2-surrounding from node u.

Example 3 *A labeled graph G and its 2-surrounding from node u are shown in Figure 3. Notice the difference between the surrounding and the view (shown in Figure 1).*

Definition 3 *S-Symmetry*
A labeled graph is S_k-symmetric *when there are k classes of nodes such that all the nodes in each class have the same surroundings.*
It is completely surrounding symmetric *(or S-symmetric) when all surroundings are equal (i.e. $k = 1$).*

4.2 S-symmetries

In this section we fully characterize the class of S-symmetric graphs and show that it coincides with the class of Cayley graphs with Cayley labeling.

Given a set of generators Ω for a finite group Γ, a Cayley graph is a graph $N_\Gamma = (V, E)$, where the vertices correspond to the elements of the group ($V = \Gamma$) and the edges correspond to the action of the generators; that is $\langle x, y\rangle \in E$ iff $\exists g \in \Omega : x \circ g = y$, where $\circ$ is the operation of the group. The set of generators is closed under inverses; so we can consider the graph undirected.

Let $\Sigma = \Omega$; the natural labeling λ for a Cayley graph N_Γ is the following: $\forall\langle x, y\rangle \in E(x)$, $\lambda_x(\langle x, y\rangle) = g$, where g is the generator such that $y = x \circ g$. In the following we shall call this labeling *Cayley labeling.*

Theorem 2 *A labeled graph (G, λ) is S-symmetric iff G is a Cayley graph and λ is a Cayley labeling.*

Proof

$\Rightarrow$) Let $\Gamma = \{[\![\alpha]\!] : \alpha \in \Sigma^*\}$ and let $\cdot : \Gamma \times \Gamma \to \Gamma$ be such that $\cdot([\![\alpha]\!], [\![\beta]\!]) = [\![\alpha \cdot \beta]\!]$. The function $\cdot$ is well defined because its value does not depend on the choice of the strings α and β. By contradiction, suppose that $\alpha, \alpha' \in a \in \Gamma$, $\beta, \beta' \in b \in \Gamma$ and $\alpha \cdot \beta \in c \in \Gamma$ (where $a, b, c \in \Gamma$) while $\alpha' \cdot \beta' \notin c$. This implies that starting from u, the two paths β and β' reach the same node: thus, $\beta \in [\![\beta']\!]_u$. On the other hand, the two paths β and β' starting from $u \rightharpoonup \alpha$ reach two different nodes: thus $\beta \notin [\![\beta']\!]_{u \rightharpoonup \alpha}$. These last two facts imply $N(u) \neq N(u \rightharpoonup \alpha)$. Contradiction.
We now show that $(\Gamma, \cdot)$ is a group. First, we prove that $\cdot$ is associative: in fact, we have that $([\![\alpha]\!] \cdot [\![\beta]\!]) \cdot [\![\gamma]\!] = [\![\alpha \cdot \beta]\!] \cdot [\![\gamma]\!] = [\![(\alpha \cdot \beta) \cdot \gamma]\!] = [\![\alpha \cdot (\beta \cdot \gamma)]\!] = [\![\alpha]\!] \cdot [\![\beta \cdot \gamma]\!] = [\![\alpha]\!] \cdot ([\![\beta]\!] \cdot [\![\gamma]\!])$. We now show that $[\![\epsilon]\!]$ is the identity of Γ: in fact, for each $a \in \Gamma$, let α s.t. $[\![\alpha]\!] = a$, $a \cdot [\![\epsilon]\!] = [\![\alpha \cdot \epsilon]\!] = [\![\alpha]\!] = a = [\![\epsilon \cdot \alpha]\!] = [\![\epsilon]\!] \cdot a$. With analogous consideration as in the proof of Theorem 1, we have that completely surrounding symmetry implies that λ is symmetric. Let $[\![\alpha]\!]^{-1} = [\![\Psi(\alpha)]\!]$, then $[\![\alpha]\!] \cdot [\![\alpha]\!]^{-1} = [\![\alpha \cdot \Psi(\alpha)]\!] = [\![\epsilon]\!]$ since $u \rightharpoonup \alpha \cdot \Psi(\alpha) = u = u \rightharpoonup \epsilon$.
Let $\Omega = \{[\![a]\!] : a \in \Sigma\}$; Ω is a set of generator for Γ. Since $a \in \Gamma$ then, by definition, there exists $\alpha \in \Sigma^*$ s.t. $[\![\alpha]\!] = [\![\alpha_1]\!] \cdot [\![\alpha_2]\!] \cdot \ldots \cdot [\![\alpha_{|\alpha|}]\!] = a$ where $[\![\alpha_i]\!] \in \Omega$. Moreover, if $l \in \Sigma$, then $\Psi(l) = \psi(l) \in \Sigma$. That implies that, if $a = [\![l]\!] \in \Omega$, then $a^{-1} = [\![\psi(l)]\!] \in \Omega$. Finally, $[\![\epsilon]\!] \notin \Omega$ because there are no self-loops. Thus, $N_{(G,\lambda)}(u)$ is the Cayley graph of the group $(\Gamma, \cdot)$ with generator Ω. To conclude the proof, by Lemma 3, $N_{(G,\lambda)}(u)$ is lg-isomorphic to (G, λ).

$\Leftarrow$) It directly from the definition of Cayley graphs that, if (G, λ) a Cayley graph, then (G, λ) is completely surrounding symmetric.

□

We now examine the relationship between S-symmetry and labelings which are *Regular Coloring and Orientation* [14].

Given a regular graph G, a coloring and orientation of G with d colors $s_1, \ldots s_d$ is a symmetric labeling of the edges such that for each node x in G, the edges in $E(x)$ are mapped one-to-one to $s_1, \ldots s_d$ ([14]). In other words, a coloring and orientation is a symmetric labeling with local orientation that uses d labels.
A coloring and orientation is *regular* if, for any two paths π_1 and π_2 with the same sequence of labels, π_1 is a cycle iff π_2 is a cycle.

The following theorem is proved in [14].

Theorem 3 *[14]*
A d-regular graph G has a regular coloring and orientation λ iff G is a Cayley graph on d generators and λ is the Cayley labeling.

Thus, the notions of S-symmetry and of regular coloring and orientation coincide:

Theorem 4 *A labeled graph (G, λ) is S-symmetric iff λ is a regular coloring and orientation.*

Proof It follows from Theorems 2 and 3. □

As a corollary, we obtain that:

Corollary 1 *A labeled graph (G, λ) is S-symmetric iff $\forall \alpha \in \Sigma^*$: $x \underset{(G,\lambda)}{\rightarrow} \alpha = x$ if and only if $y \underset{(G,\lambda)}{\rightarrow} \alpha = y$.*

4.3 S_k-Symmetries

In this section, we give a necessary and sufficient condition for two nodes to have the same surrounding. Using this result, we then characterize the class of S_k symmetric graphs; that is the graphs in which there are k different surroundings.

Definition 4 lg-isomorphism
Given two labeled graphs $(G = (V, E), \lambda)$ and $(G' = (V', E'), \lambda')$, a bijective function $\chi : V \rightarrow V'$ is a labeled graph isomorphism *for (G, G') iff:*
1) $\langle u, v\rangle \in E \Leftrightarrow \langle \chi(u), \chi(v)\rangle \in E'$;
2) $\lambda(\langle u, v\rangle) = \lambda'(\chi(u), \chi(v)\rangle)$.

Lemma 3 *Let (G, λ) be a connected labeled graph. There exists a graph isomorphism χ for $(N^{e(u)}, (G, \lambda))$ such that $\chi([\![\epsilon]\!]) = u$.*

The following theorem gives a necessary and sufficient condition for two nodes to have the same surrounding.

Theorem 5 *For all $u, v \in V$ the following two conditions are equivalent:*
1) $\forall \alpha, \beta \in \Sigma^*$: $u' = u \rightarrow \alpha$ is defined iff $v' = v \rightarrow \alpha$, and $u' \underset{(G,\lambda)}{\rightarrow} \beta = u' \Leftrightarrow v' \underset{(G,\lambda)}{\rightarrow} \beta = v'$;
2) $N_{(G,\lambda)}(u) = N_{(G,\lambda)}(v)$.

Proof
$1 \Rightarrow 2)$
In order to prove the thesis we prove the following claim:

Claim: *Condition 1) implies that u and v are lg-transitive.*

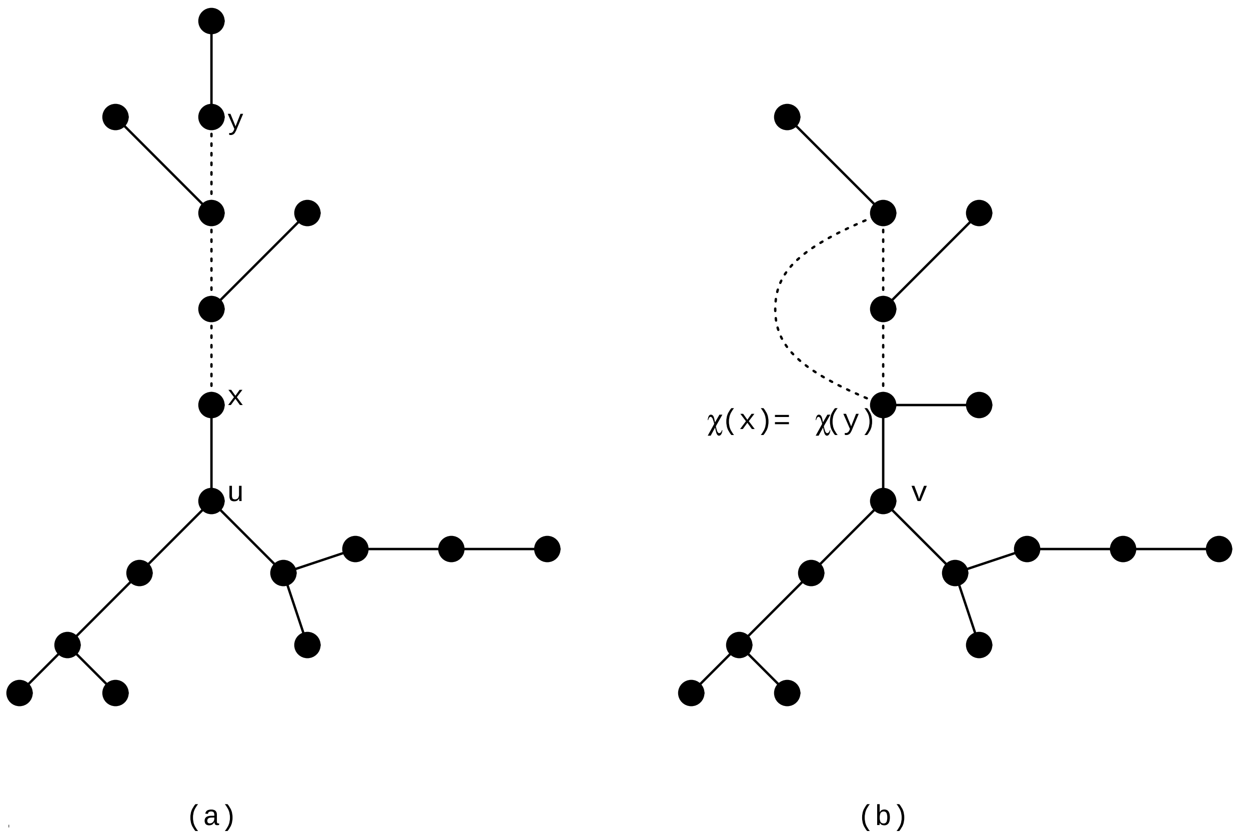

Figure 4:

Proof of Claim. We will show that there exists a labeled graph automorphism χ that maps u into v. Let S be a spanning tree of (G, λ). Then for every $x, y \in V$ there is only one shortest path $p[y, x]$ in $P_S[y, x]$. Let $s(y, x) = \lambda_y(p[y, x])$ be the labeling of that shortest path. The fact that S is a spanning tree implies that s is defined for each node of V.
By definition of $\rightarrow$ and s, we have that:

$$x = y \rightarrow s(y, x) \quad (*)$$

By induction on the length of $s(u', x)$ it is easy to see that: If $u' = u \rightarrow \alpha$ and $v' = v \rightarrow \alpha$, then

$$s(u', x) = s(v', y) \Leftrightarrow s(x, u') = (y, v') \quad (**)$$

We first prove that χ is an edge preserving morphism from S onto (G, λ).
Let $\chi : V \longrightarrow V$ be such that $\chi(x) = v \rightarrow s(u, x)$. The function is well defined because $s(u, x)$ is defined for each x, and by hipotesis $v \rightarrow s(u, x)$ is also defined. Thus, $\chi(x) = v \rightarrow s(u, x)$ is defined for each $x \in V$. Note that $\chi(u) = \chi(u \underset{(G,\lambda)}{\rightarrow} \epsilon) = v \underset{(G,\lambda)}{\rightarrow} \epsilon = v$.
The fact that S is a spanning tree implies that $\langle x, y \rangle \in \mathcal{E}(S) \Rightarrow s(u, y) = s(u, x) \cdot l$ for some $l \in \Sigma$. By Property 1, $s(u, y) = s(u, x) \cdot l \Rightarrow \langle v \rightarrow s(u, x), v \rightarrow s(u, y) \rangle \in \mathcal{E}(G)$ (otherwise, there will be two different paths labeled $s(u, x)$ starting from v). By definition of χ, $\langle x, y \rangle \in \mathcal{E}(S) \Rightarrow \langle \chi(x), \chi(y) \rangle \in \mathcal{E}(G)$. Thus, we have proved that χ is an edge preserving morphism from S onto (G, λ).

We now prove that χ is one-to-one. By contradiction suppose that there exist x and y such that $x \neq y$ and $\chi(x) = \chi(y)$ (as shown in Figure 4). By (*) and by definition of χ, $x = u \rightarrow s(u,x)$ and $\chi(x) = v \rightarrow s(u,x)$. By hypothesis, for each β and $\alpha = s(u,x)$, $x \rightarrow \beta = x \Leftrightarrow \chi(x) \rightarrow \beta = \chi(x)$. Moreover, by (**), $s(y,x) = s(\chi(u) \rightarrow s(u,y), \chi(x)) = s(\chi(y), \chi(x))$. Then, $x \underset{(G,\lambda)}{\rightarrow} s(y,x) = y \neq x$ that implies $\chi(x) \underset{(G,\lambda')}{\rightarrow} s(y,x) \neq \chi(x)$, which is a contradiction because $\chi(x) \underset{(G,\lambda')}{\rightarrow} s(y,x) = \chi(y) = \chi(x)$. Thus χ is one-to-one and it is also bijective because of the finites of V.
By definition of morphism we have that, if S is a connected subgraph of (G,λ), then $\chi(S)$ is a connected subgraph of (G,λ). Thus, the bijectivity of χ implies that $\chi(S)$ is a connected subgraph of (G,λ) that covers all the vertices. Thus, $\chi(S)$ is a spanning tree of (G,λ), and $\langle x,y\rangle \in \mathcal{E}(S) \Leftrightarrow \langle \chi(x), \chi(y)\rangle \in \mathcal{E}(\chi(S))$.
We will now prove that $\langle x,y\rangle \in \mathcal{E}(G) - \mathcal{E}(S) \Leftrightarrow \langle \chi(x), \chi(y)\rangle \in \mathcal{E}(G) - \mathcal{E}(\chi(S))$.
The fact that S is a spanning tree implies that $e = \langle y,x\rangle \in \mathcal{E}(G) - \mathcal{E}(S)$ iff $s(x,u)\cdot s(u,y)\cdot \lambda_y(e)$ is the label of a cycle starting and ending in x. That is, $x \rightarrow s(x,u)\cdot s(u,y)\cdot\lambda_y(e)) = x$. It follows that $\chi(x) \rightarrow s(x,u)\cdot s(u,y)\cdot \lambda_y(e)$ is defined; that is, $s(x,u)\cdot s(u,y)\cdot\lambda_y(e)$ is a path starting from $\chi(x)$. By (**), $s(u,x) = s(v,\chi(x))$ implies that $s(x,u) = s(\chi(x), v)$. By Property 1, there is only one path in $P[\chi(x), v]$ labeled with $s(x,u)$. By hypothesis and the definition of χ, we have that $s(u,y)$ is the labeling of a path starting from v and ending in $\chi(y)$. Thus, $s(x,u)\cdot s(u,y)$ is a labeling of a path starting from $\chi(x)$ and ending in $\chi(y)$. By hypothesis, $\chi(x) \underset{(G,\lambda)}{\rightarrow} (s(x,u)\cdot s(u,y)\cdot \lambda_y(e)) = \chi(x)$ that implies $\langle \chi(x), \chi(y)\rangle \in \mathcal{E}(G)$ and $\lambda_{\chi(x)}(\langle \chi(x), \chi(y)\rangle) = l = \lambda_x(\langle x,y\rangle)$. Thus, we have proved that $\langle x,y\rangle \in \mathcal{E}(G) \Rightarrow \langle \chi(x), \chi(y)\rangle \in \mathcal{E}(G)$. Following a similar argument, it is easy to prove the other implication, concluding the proof of the Claim.

We will now prove that $[\![\alpha]\!]_u^d = [\![\alpha]\!]_v^d$. By definition, $[\![\alpha]\!]_u^d = \Lambda^d_{(G,\lambda)}[u, u \underset{(G,\lambda)}{\rightarrow} \alpha] = \Lambda^d_{(G,\lambda)}[u,x]$ for some x and $[\![\alpha]\!]_u^d = \Lambda^d_{(G,\lambda)}[v, v \underset{(G,\lambda)}{\rightarrow} \alpha] = \Lambda^d_{(G,\lambda)}[v, \chi(x)]$. The fact that χ is a lg-isomorphism implies that $u \rightarrow l = y \Leftrightarrow \chi(u) \rightarrow l = \chi(y)$. Thus by induction, it can be proved that the strings corresponding to the paths starting from u and ending in x are the same as the strings corresponding to the paths starting from $v = \chi(u)$ and ending in $\chi(x)$. Thus, $\Lambda^d_{(G,\lambda)}[u,x] = \Lambda^d_{(G,\lambda)}[v,\chi(x)]$. Thus, $[\![\alpha]\!]_u^d = [\![\alpha]\!]_v^d$ for each α implies that $N_{(G,\lambda)}(u) = N_{(G,\lambda)}(v)$.
$2 \Rightarrow 1$
By contradiction, suppose that $N_{(G,\lambda)}(u) = N_{(G,\lambda)}(v)$ and exists α,β s.t. either: 1) $u \rightarrow \alpha$ is defined but $v \rightarrow \alpha$ is not, or 2) $u \rightarrow \alpha = u'$, $v \rightarrow \alpha = v'$, $u' \rightarrow \beta = u'$, but $v' \rightarrow \beta \neq v'$.
Case 1): $[\![\alpha]\!] \in \mathcal{V}(N_{(G,\lambda)}(u))$, but $[\![\alpha]\!] \notin \mathcal{V}(N_{(G,\lambda)}(v))$.
Case 2): $u' \rightarrow \beta = u'$ implies $\alpha\cdot\beta \in [\![\alpha]\!]_u$, while $v' \rightarrow \beta \neq v'$ implies $\alpha\cdot\beta \notin [\![\alpha]\!]_v$.
In both cases a contradiction follows. □

We can now characterize the class of S_k-symmetric graphs.

Theorem 6 *A labeled graph (G,λ) is S_k-surrounding symmetric iff there exists a partition $P = (P_1,\ldots,P_k)$ of the nodes such that $\forall i$, $\forall x,y \in P_i$, $\forall \delta \in \Sigma^*$:*

a) $x \rightarrow \delta \in P_j$ *iff* $y \rightarrow \delta \in P_j$*; and*
b) $x = x \rightarrow \delta$ *iff* $y = y \rightarrow \delta$*.*

Proof
($\Rightarrow$): Let $N_1, \ldots, N_k$ be an arbitrary order of the k different surroundings, and let $P = (P_1, \ldots, P_k)$ be a partition of V such that $P_i = \{u : N(u) = N_i\}$. By contradiction, suppose that either a) or b) does not hold; that is, $\exists i,\ x, y \in P_i,\ \delta \in \Sigma^*$ such that:
a') $x \rightarrow \delta \in P_j$ but $y \rightarrow \delta \notin P_j$, or b') $x \rightarrow \delta = x$ but $y \rightarrow \delta \neq y$.
Case a'): By construction, $N(x \rightarrow \delta) \neq N(y \rightarrow \delta)$; this implies the existence of β such that $[\![\beta]\!]_{x \rightarrow \delta} \neq [\![\beta]\!]_{y \rightarrow \delta}$. Thus, $[\![\delta \cdot \beta]\!]_x \neq [\![\delta \cdot \beta]\!]_y$ that implies $N(x) \neq N(y)$: a contradiction.
Case b'): In this case, $\delta \in [\![\epsilon]\!]_x$ but $\delta \notin [\![\epsilon]\!]_y$; this implies $[\![\epsilon]\!]_x \neq [\![\epsilon]\!]_y$, and, thus, $N(x) \neq N(y)$: a contradiction.
($\Leftarrow$): Let $P = (P_1, \ldots, P_k)$ be a partition of the node for which conditions a) and b) hold. We will show that for each $u, v \in P_i$, $N(u) = N(v)$. In order to show this, we prove that: $\forall \alpha, \beta \in \Sigma$:
1) $u' = u \rightarrow \alpha$ is defined iff $v' = v \rightarrow \beta$ is defined, and
2) $u' \rightarrow \beta = u'$ iff $v' \rightarrow \beta = v'$.
By condition a) with $\delta = \alpha$, $x = v$ and $y = v$, we have that $u \rightarrow \alpha \in P_j$ iff $v \rightarrow \alpha \in P_j$, which implies $u \rightarrow \alpha$ is defined iff $v \rightarrow \alpha$ is defined (this proves 1), and that $u', v' \in P_l$ for some l. Thus by condition b) with $\delta = \beta$, $x = u'$ and $y = v'$, we have that $u' = u' \rightarrow \beta$ iff $v' = v' \rightarrow \beta$, that proves 2). It now follows from Theorem 5 that $N(v) = N(u)$. $\Box$

5 Symmetries and Sense of Direction

In this section, we introduce the definition of sense of direction, we show the existing link between symmetries and minimal sense of direction in regular graphs, and we discuss the relationship with the CG-labelings of a graph, a particular class of labelings based on commutative groups.

5.1 Sense of Direction

Given a labeled graph (G, λ), the system is said to have *Sense of Direction* when λ satisfies a set of global consistency constraints which allow to understand, from the labels associated to the edges, whether different paths from any given node x end in the same node or in different ones. More precisely, sense of direction involves the existence of a consistent coding and a consistent decoding function.

Given (G, λ), a *consistent coding function* $\mathbf{c}$ for λ is any function with domain Σ^+, such that $\forall x, y, z \in V$, $\forall \pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$

$$\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_x(\pi_2)) \quad \Leftrightarrow \quad y = z.$$

In other words, in a consistent coding function paths originating from the same node are mapped to the same value (called *local name*) if and only if they end in the same node.

Given a consistent coding function $\mathbf{c}$, a consistent decoding function $\mathbf{d}$ for $\mathbf{c}$ is any function such that $\forall\langle x,y\rangle \in E(x)$, $\pi \in P[y,z]$

$$\mathbf{d}(\lambda_x(\langle x,y\rangle), \mathbf{c}(\Lambda_y(\pi))) = \mathbf{c}(\lambda_x(\langle x,y\rangle)\cdot\Lambda_y(\pi))$$

where $\cdot$ is the concatenation operator.

Definition 5 $\mathcal{WSD}$ - Weak Sense of Direction
Given a labeled graph (G,λ)*, a coding function* $\mathbf{c}$ *for* λ*,* (G,λ) *has* Weak Sense of Direction $\mathbf{c}$ *iff* $\mathbf{c}$ *is consistent. Alternatively, we shall say that* $\mathbf{c}$ *is a* $\mathcal{WSD}$ *in* (G,λ).

Definition 6 $\mathcal{SD}$ - Sense of Direction
Given a labeled graph (G,λ)*, a coding function* $\mathbf{c}$ *for* λ *and a decoding function* $\mathbf{d}$ *for* $\mathbf{c}$*,* (G,λ) *has* Sense of Direction $(\mathbf{c},\mathbf{d})$ *iff* $\mathbf{c}$ *and* $\mathbf{d}$ *are consistent. Alternatively, we shall say that* $(\mathbf{c},\mathbf{d})$ *is a* $\mathcal{SD}$ *in* (G,λ).

Notice that there exist labeled graphs with $\mathcal{WSD}$ but without $\mathcal{SD}$. An example of such graphs is shown in Figure 5.

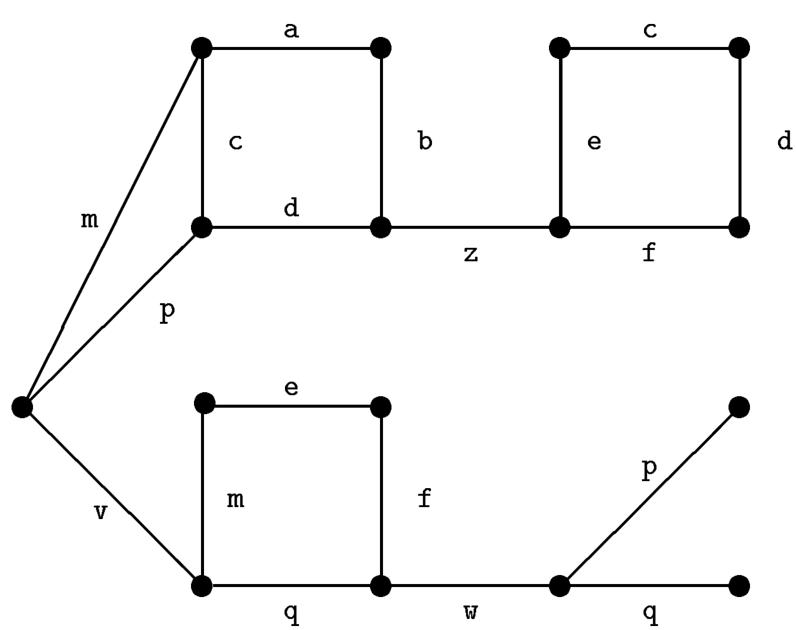

Figure 5: A labeled graph with $\mathcal{WSD}$, but without $\mathcal{SD}$.

Example 4 *It is easy to verify that in the labeled graph (G, λ) of Figure 5 there exists a consistent coding function* $\mathbf{c}$. *However, it has been proved in [5] that in (G, λ) there does not exist any consistent decoding function for* $\mathbf{c}$; *thus, (G, λ) does not have $\mathcal{SD}$.*

Let (G, λ) have $\mathcal{SD}$. If λ is a symmetric labeling, i.e., there exists a precise relation between the labels at the two sides of an edge, we say that (G, λ) has a *symmetric* $\mathcal{SD}$.

A sense of direction is called *homonymous* when $\forall x, y \in V$, $\pi_1 \in P[x, x]$,$\pi_2 \in P[y, y]$: $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(\Lambda_y(\pi_2))$; that is, when the coding function associates the same value to all the cycles in the graph.

5.2 Minimal Sense of Direction

A *Minimal (weak) sense of direction* is a (weak) sense of direction that uses d labels, where d is the maximum degree of the graph.

A problem that has been recently studied [6] is the characterization of the class of regular graphs with minimal sense of direction. In [6] a necessary condition for a graph to have a minimal sense of direction has been given. However, a complete characterization of this class of graphs has not been found yet.

In the following we partially answer this question; in fact, we completely characterize the class of graphs with minimal *symmetric* $\mathcal{SD}$.

Consider first the following lemma.

Lemma 4 *[10] Let (G, λ) be a system with sense of direction. (G, λ) is S-symmetric iff it is V-symmetric.*

We can now characterize the class of graphs with minimal symmetric (weak) sense of direction. Let $\mathcal{W}$ be the class of labeled regular graphs with minimal symmetric $\mathcal{WSD}$, let $\mathcal{S}$ be the class of labeled regular graphs with minimal symmetric $\mathcal{SD}$, let $\mathcal{C}$ be the class of Cayley graphs with Cayley labeling. We have that a regular labeled graph (G, λ) has minimal symmetric (weak) sense of direction iff G is a Cayley graph and λ is a Cayley labeling. That is:

Theorem 7

$$\mathcal{W} = \mathcal{S} = \mathcal{C}.$$

Proof

($\mathcal{W} \subseteq \mathcal{C}$): Let λ be a minimal symmetric labeling for graph G and c be the consistent coding function for λ. If λ is minimum, then it uses d labels. Thus by Theorem 1, (G, λ) is V-symmetric. By Lemma 4, (G, λ) is S-symmetric. By Theorem 2, (G, λ) is a Cayley graph with Cayley labeling.

($\mathcal{C} \subseteq \mathcal{S}$): In [6], it has been proved that every Cayley graph has a minimal symmetric sense of direction.

($\mathcal{S} \subseteq \mathcal{W}$): It trivially follows from definition. □

Moreover, in regular graphs with minimal sense of direction, symmetry of the labels and homonimity coincides. In fact, we have that:

Theorem 8 *Let (G, λ) be a regular labeled graph with minimal sense of direction. λ is symmetric iff the sense of direction is homonymous.*

Proof Let (G, λ) be a d-regular labeled graph with minimal sense of direction.
($\Rightarrow$): Let ψ be the symmetry function. Let $\pi_1 \in P[x, x]$ and $\pi_2 \in P[y, y]$, with $x \neq y$. The graph is d-regular and uses d labels $(l_1, \ldots, l_d)$, thus, by definition of consistent coding function, we have that: $\mathbf{c}(\Lambda_x(\pi_1)) = \mathbf{c}(l_i \cdot \psi(l_i)) = \mathbf{c}(\Lambda_y(\pi_2))$, where $1 \leq i \leq d$.
($\Leftarrow$): Let $l_1, \ldots, l_d$ be the d labels used. By contradiction, suppose λ is not symmetric. In this case, we would have two edges $\langle x, y\rangle$ and $\langle z, w\rangle$ such that $\lambda_x(\langle x, y\rangle) = a$, $\lambda_y(\langle y, x\rangle) = b$, $\lambda_z(\langle z, w\rangle) = a$, $\lambda_w(\langle w, z\rangle) = c$, with $c \neq a$.
By definition of homonimity, we have that $\mathbf{c}(\Lambda_x(\langle x, y\rangle \cdot \langle y, x\rangle)) = \mathbf{c}(\Lambda_z(\langle z, w\rangle \cdot \langle w, z\rangle))$, that is

$$\mathbf{c}(a, b) = \mathbf{c}(a, c) \tag{1}$$

Since the graph is d-regular and uses d labels, there must exists an edge $\langle y, y'\rangle$, with $y' \neq x$ such that $\lambda_y(\langle y, y'\rangle) = c$. But, by definition of consistent coding function, we have that: $\mathbf{c}(\Lambda_x(\langle x, y\rangle \cdot \langle y, x\rangle)) \neq \mathbf{c}(\Lambda_x(\langle x, y\rangle \cdot \langle y, y'\rangle))$, that is:

$$\mathbf{c}(a, b) \neq \mathbf{c}(a, c)$$

that contradicts (1). □

Thus, in Theorem 7, the assumption of symmetry can be replaced by the assumption of homonimity. The result of Theorem 7, with the assumption of homonimity, has been independently discovered by [4].

Note that the symmetric condition of λ in Corollary 7 is necessary to prove that a labeled graph is a Cayley graph with Cayley labeling. Consider, for example, the labeled graph (G, δ) shown in Figure 6. It is easy to verify that (G, δ) has a sense of direction. G is a Cayley graph but δ is not a Cayley labeling; in fact, for node x $a \cdot a = I$; however, for node y, $a \cdot a \neq I$.

5.3 Group Sense of Direction

A particular labelings (here called *Commutative Group labeling*, or CG-labeling) has been defined in [22]; in this section, we describe the relations between sense of direction, CG-labeling, and Cayley graphs.

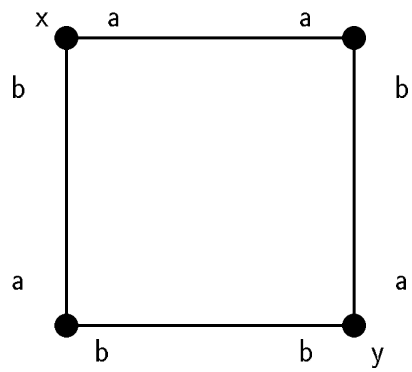

Figure 6: A non-Cayley labeling of C_4 with minimum number of labels.

5.3.1 CG-labeling

Let Γ be a commutative group with binary operation $+$ and identity 0. A labeled graph $(G = (V, E), \lambda)$ has a *CG-labeling* (based on Γ) iff 1) there exists a bijection $\mathcal{N} : V \rightarrow \Gamma$, and 2) $\lambda_u(\langle u, v\rangle) + \mathcal{N}(u) = \mathcal{N}(v)$ [22].

Thus, given any graph G, it is always possible to construct a CG-labeling by choosing appropriate commutative groups. Observe that a CG-labeling satisfies the *anti-symmetry* property $\lambda_u(\langle u, v\rangle) = -\lambda_v(\langle v, u\rangle)$, that is a special case of symmetric labeling. For each $a_0 \cdot a_1 \cdot \ldots \cdot a_n = \alpha \in \Gamma^*$, let $\sum : \Gamma^* \rightarrow \Gamma$ be such that $\sum(\alpha) = a_0 + a_1 + \ldots + a_n$.

It is easy to see that a CG-labeling has sense of direction; as we show in the next theorem.

Theorem 9 *A system (G, λ), where λ is a CG-labeling has SD $(\sum, \sum)$.*

Proof Let λ be a CG-labeling based on Γ in (G, λ). By Theorem 6 in [22], λ has the closed path property, that is for each $\pi \in P[u, v]$, $\sum(\lambda_u(\pi)) = 0 \Leftrightarrow u = v$. By contradiction suppose that $\sum$ has the closed path property and it is not a consistent coding function for λ. This means that there exist three nodes x, y, z and two path $\pi_1 \in P[x, y]$, $\pi_2 \in P[x, z]$ s.t. $\sum(\lambda_x(\pi_1)) = \sum(\lambda_x(\pi_2)) \not\Leftrightarrow y = z$. Let π be the reverse path $\overline{\pi_1}$ of π_1 concatenated with π_2, then $\sum(\pi) = \sum(\lambda_y(\overline{\pi_1} \cdot \pi_2)) = \sum(\lambda_y(\overline{\pi_1})) + \sum(\lambda_x(\pi_1))$. By the extension of the anti-symmetry property, $\sum(\lambda_y(\overline{\pi_1})) = -\sum(\lambda_x(\pi_1))$. By the closed path property on π, $\sum(\lambda_y(\pi)) = 0 \Leftrightarrow y = z$, that implies $\sum(\lambda_x(\pi_1)) = \sum(\lambda_x(\pi_2)) \Leftrightarrow y = z$ which is a contradiction. Analogously, it is easy to see that $\sum$ is also a consistent decoding function; thus $(\sum, \sum)$ is a sense of direction in (G, λ). □

In the following we shall refer to such a sense of direction as a *CG-sense of direction.*

5.3.2 CG-$\mathcal{SD}$ and $\mathcal{SD}$

In [22], Tel posed the question of whether any $\mathcal{SD}$ is a CG-$\mathcal{SD}$. In the following, we will settle this question with a negative answer.

To verify that there are $\mathcal{SD}$ which are not CG-sense of direction, it suffices to choose labelings which are not anti-symmetric. However, even when considering only anti-symmetric labeling, there are $\mathcal{SD}$ which are not CG-$\mathcal{SD}$, as shown by the following theorem.

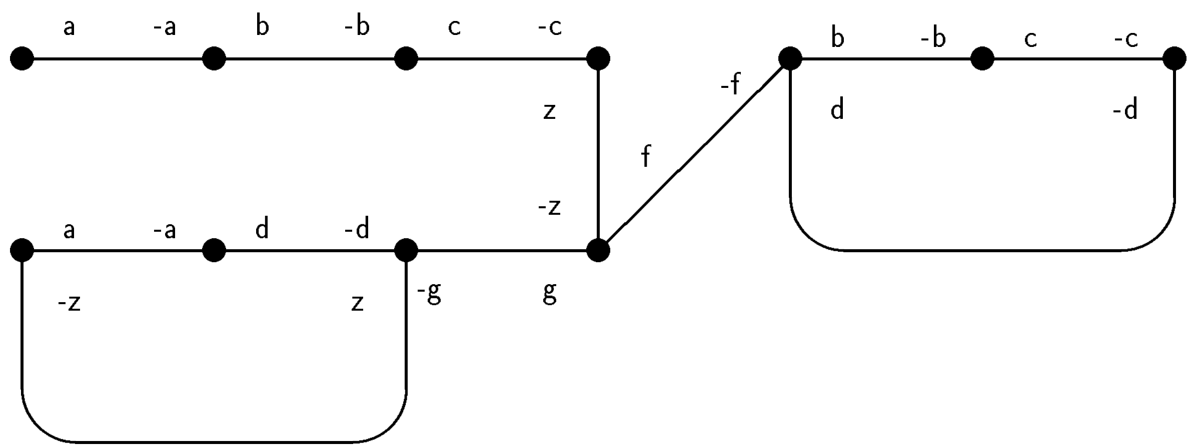

Figure 7: A sense of direction that is not CG-sense of direction.

Theorem 10 *The class of labeled graphs with CG-sense of direction is a proper subset of the class of anti-symmetrically labeled graphs with sense of direction.*

Proof By Theorem 9 any labeled graph with CG-sense of direction has sense of direction.
Consider the labeled graph (G, λ) of Figure 7. This labeled graph does not have a CG-sense of direction. In fact, suppose, by contradiction, that there is a CG-sense of direction. We must have that 1) $b + c = d$ (since $b \cdot c \cdot -d$ is a cycle); 2) $a + d = -z$ (since $a \cdot d \cdot z$ is a cycle); while, 3)$a + b + c + z \neq 0$ (since $a \cdot b \cdot c \cdot z$ is not a cycle). Substituting 1) in 3), we have that: $a + b + z \neq 0$, which contradicts 2).
However, in this graph there exists an homonymous sense of direction that satisfies the anti-symmetric property. In fact, consider the following coding function $\mathbf{c} : \Sigma^+ \to \Sigma^+$ such that $\forall \alpha = a_0 \cdot a_1 \cdot \ldots \cdot a_k \in \Sigma^+$, $\mathbf{c}(a_0 \cdot a_1 \cdot \ldots \cdot a_k) = a_0 \ominus a_1 \ominus \ldots \ominus a_k$, where $\ominus$ is a *non commutative* operator such that $\forall x \in \Sigma$: $x \ominus \epsilon = x$, $\epsilon \ominus x = x$, $x \ominus -x = \epsilon$, $-x \ominus x = \epsilon$; moreover: $a \ominus d = -z$; $b \ominus c = d$. It is easy to verify that $\mathbf{c}$ is a consistent coding function in (G, λ).
Furthermore, it is easy to see that $\mathbf{d} : \Sigma \times \Sigma^+ \to \Sigma^+$: $\forall a \in \Sigma$, $\alpha \in \Sigma^+$, $\mathbf{d}(a, \mathbf{c}(\alpha)) =$

$a \ominus \mathbf{c}(\alpha)$ is a consistent decoding function for $\mathbf{c}$. Thus, $(\mathbf{c}, \mathbf{d})$ is a sense of direction in (G, λ). □

5.3.3 Minimal $\mathcal{SD}$ and Uniform CG-$\mathcal{SD}$

We now consider the relationship between Minimal sense of direction in regular graphs and a particular case of *CG*-sense of direction called *uniform*. A CG-sense of direction (based on Γ) is *uniform* if every node has the same collection of local labels Ω [22].

First observe that any graph (G, λ) with a group $\mathcal{SD}$ is a subgraph of a commutative Cayley graph with a Cayley labeling. Consider now uniform CG-sense of direction.

Clearly, a uniform CG-sense of direction is defined only for regular graphs; by definition, it is a minimal and symmetric sense of direction and, thus, it is a Cayley labeling. In particular, a uniform CG-sense of direction based on Γ is a Cayley labeling of the commutative Cayley graph Γ with set of generators Ω.
In other words, the class of labeled graphs with uniform CG-sense of direction coincides with the class of *commutative* Cayley graphs with Cayley labeling.

On the other hand, there exist graphs with minimal $\mathcal{SD}$ for which there is no uniform *CG*-$\mathcal{SD}$, as will be shown in the next theorem.

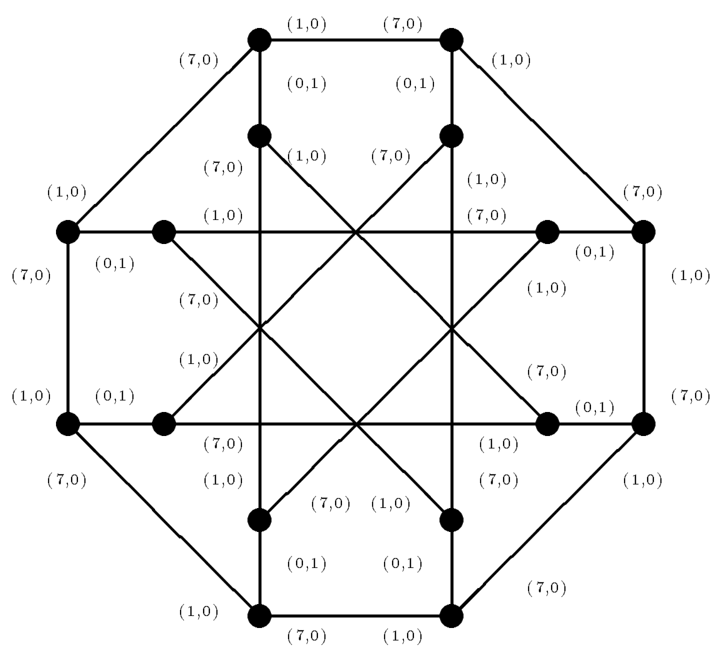

Figure 8: A non-commutative Cayley graph.

Theorem 11 *The class of labeled regular graphs with uniform CG-sense of direction is a* proper subset *of the class of labeled regular graphs with minimal sense of direction.*

Proof From Theorem 7, it follows that every labeled graph with minimal CG sense of direction has also a minimal sense of direction. We now show that the converse is not true. Consider the labeled graph (G, λ) shown in Figure 8. (G, λ) is a Cayley graph based on the group $\mathbf{Z}_8 \times \mathbf{Z}_2$ with non-commutative operation $+$ such that $(i, x)+(j, y) = (i + j \cdot 3^x \bmod 8, x + y \bmod 2)$ and set of generators $\{(1, 0), (7, 0), (0, 1)\}$. Note that the absence of a cycle of length four in the graph implies that it *cannot* be generated by a commutative group. By Theorem 7, (G, λ) has a minimal sense of direction. However, in this graph there is not a uniform CG-sense of direction, because the group cannot be generated by a commutative group. Thus, any CG-sense of direction on G must use more than 3 labels. □

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