Compositional Complexity in Cellular Automata: a Case Study

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Abstract

We relate two compositional approaches of dynamical systems showing the same emergence of dynamical complexity from the interaction of two simple and similar systems attracting their underlying space to different regions: two shifting cellular automata produce complexity, as well as classical dynamical systems like Cantor's relation or Smale's horseshoe map.

1 Introduction

Cellular automata (CA, for short) are discrete-time, discrete-space, massively parallel dynamical systems. The rich variety of their behaviors as well as their universal computation ability allow their use as models in many disciplines, ranging from parallel computing to ecology or physics. Indeed, they can exhibit very simple (destruction of information) to very complex (propagation of information following complex rules) dynamics, including spatiotemporal chaos.

Dynamics. In [4], the classification of CA dynamics of [32] was refined, and it was then formalized and structured in [9, 13]. This led to a hierarchy of behaviors, namely: periodicity, shifting and aperiodicity. Among these classes, aperiodic ones provide the most complex features, whereas periodic and shifting ones can be considered as phenomenologically simple. Indeed, even though shifting CA can be proved to be chaotic according to Devaney's definition of chaos [7, 1], they can be classified as simple using transfinite attraction and shifted Hamming distance.

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Compositional complexity. In order to bring new insights in understanding how complexity arises in such high-dimensional systems, we introduce compositionality [13]. Basically, the compositional analysis of a dynamical system consists in determining some global property concerning dynamical or computational aspects of the system by combination of individual properties of its components, which are expected to be simpler. The following diagram illustrates the idea; S_i being the components of a composed system $S = \star_i S_i$, I denoting an individual property, G a global property, we want to find a way to combine the individual properties, viz. $\diamond$, to characterize the global property:

$$\begin{array}{ccc} S_i & \xrightarrow{\;I\;} & I(S_i) \\ \star \big\downarrow & & \big\downarrow \diamond \\ \star_i S_i & \xrightarrow[\;G\;]{} & \diamond_i I(S_i). \end{array}$$

Aim. The goal of this paper is to present compositional arguments to evidence a conjecture on the emergence of complexity from the composition of elementary symmetric compatible systems [4, 10, 8]. The result is mainly based on union-invariant theorems elaborated in [26, 27, 12, 13], that we complement with physical measures of complexity.

Related work. In the literature on CA and related models, lots of papers study complexity. Among these ones, let us just mention important approaches developed in [14, 32, 6, 20], as well as the many references cited in [15, 11].

However, the notion of "chaos" is still not well defined in the context of such discrete-time discrete-space multi-dimensional dynamical systems. Several authors propose ways of defining complex behaviors in CA. This is one of the goals of classification. Other ones emphasize transition phenomena in the space of CA rules, allowing new classifications, too. In these latter ones, statistical measures are often used, together with information theory measures (entropy, activity, sensitivity to rule change, etc.), leading to definitions of complex behaviors, based on certain parameter values [14, 24, 22, 6, 25]. In [28], the author presents a classification of chaotic behaviors, based on notions of randomness, complexity measures, computability of initial conditions, and (non)determinism of rules. Finally, in [19], the author studies aperiodicity of some CA analytically.

Closer to our compositional approach, we find algebraic characterization of the behavior of some CA in [30, 5, 31]: composition operators are proposed, together with global results obtained by composition of local properties.

Outline. In §2, we recall useful notions from the compositional analysis of dynamical systems; in §3, we briefly present the CA framework; in §4, we give experimental conjectures on the disjunctive composition of CA; in §5, we compare a particular disjunction to Cantor relation chaotic dynamics; in §6, we define the successive steps of our analysis and we add complexity measures to our theoretical framework; in §7, we proceed to the analysis of the conjectures; finally, in §8, we conclude the paper.

2 Union composition of dynamical systems

Definition 1 (Dynamical system)
A dynamical system is a pair (Y, f) where Y is a compact metric space and $f \subseteq Y \times Y$ is a closed relation on Y.

The *deterministic execution* is defined as a function on the power set of Y:

$$\begin{array}{ll} f & : \ \mathbb{P}(X) \mapsto \mathbb{P}(X) \\ \text{s.t.} & f(A) = \cup_{a \in A} f(a). \end{array}$$

In this case, the *discrete-time dynamics* is given by the following recursively defined iteration scheme: $\forall A \subseteq X, n \in \mathbb{N}$,

$$\begin{aligned} f^0(A) &= A \\ f^{n+1}(A) &= f(f^n(A)) \\ f^{-n}(A) &= (f^{-1})^n(A). \end{aligned}$$

The *non-deterministic execution* chooses one possible path among all available ones, that is, one possible image at each step in an arbitrary way. Then, the dynamics is defined as the set of all possible evolutions the system can produce.

A useful composition operator is the union of two systems f and g, defined by their set-union.

Definition 2 (Union)
Let f and g be two systems defined on Y; then

$$f \cup g = \{(x, y) \mid (x, y) \in f \vee (x, y) \in g\}.$$

Invariance and attraction are related to dynamical complexity of systems. Invariants are sets of states that have infinite internal histories, i.e. stable sets. Their structure represents trajectories between inner states; it organizes and, thus, strongly influences the resulting dynamics. Attraction establishes relations between initial and final or asymptotic states of infinite histories.

Definition 3 (Invariant)
The invariant J of a system (Y, f) is the greatest set verifying

$$S \subseteq f(S) \cap f^{-1}(S).$$

Definition 4 (Attraction)
Let (Y, f) be a system, and $P, Q \subseteq Y$; then P is attracted to Q by f, i.e. $P \overset{f}{\rightsquigarrow} Q$, iff

$$\cap_i \overline{\cup_{i \leq j} f^j(P)} = Q,$$

where $\overline{A}$ represents the closure of A in Y.

Finally, let us state a compositional result on the union of systems, the proof of which can be found in [13, Chap.6, Cor. 6.32].

Theorem 5 (Union invariant)
Let (Y, f) and (Y, g) be two injective compatible systems with distinct fixpoint invariants, such that $\gamma(f) + \gamma(g) < 1$ and $f^{-1}(Y) = g^{-1}(Y) = Y$ (or $\gamma(f^{-1}) + \gamma(g^{-1}) < 1$ and $f(Y) = g(Y) = Y$), then $f \cup g$ has a Cantor-set invariant.

The contractivity factor γ is defined by $\gamma(f) = \sup\limits_{x \neq y} \frac{d_H(f(x), f(y))}{d(x,y)}$, and d (d_H) is a (Hausdorff) metric on Y. Let us rephrase the result informally: complexity (Cantor-set structure) emerges from union composition of compatible systems attracting the space to different regions.

3 Cellular automata: preliminary notions

In this section, we briefly recall the definition of cellular automata, and the classification of CA behaviors used in the following.

3.1 Definitions

We consider one-dimensional CA, the cells of which being arranged on a linear bi-infinite lattice. Each cell takes its value from a local state space $X = \{0, 1, \ldots, k-1\}$. All cells are updated synchronously. At each step, every cell looks at the value of its neighbors (r to the left, r to the right) plus itself and computes its next value as a function of this neighborhood. This function is a *local transition function* $g : X^{2r+1} \mapsto X$. A *configuration* is a bi-infinite sequence of $X^{\mathbb{Z}}$ specifying a state for each cell, i.e. $x = (\cdots x_{-1}, x_0, x_1 \cdots)$. The *neighborhood* of a cell $i \in \mathbb{Z}$ is $(i-r, \ldots, i-1, i, i+1, \ldots, i+r)$ or, simply, $(i - r : i + r)$. The *global transition function* defines the next state of each cell as the local function applied to the states of its neighborhood:

$$\begin{array}{ll} f & : \; X^{\mathbb{Z}} \mapsto X^{\mathbb{Z}} \\ \text{s.t.} & \forall i \in \mathbb{Z}, f_i(x) = g(x_{i-r:i+r}). \end{array}$$

In the following, we restrict our attention to *elementary cellular automata*, i.e. with $r = 1$ and $k = 2$. We also use $\mathcal{C}$ instead of $2^{\mathbb{Z}}$ to denote the configuration space.

There are clearly $k^{k^{2r+1}}$ different local functions or *rules*. Using the above parameters, there are 256 different elementary cellular automata. Each function can be expressed as a *transition table* or *rule table* giving the output for each value of the neighborhood. Classically [32], the binary vector defining the function is translated into a decimal number: rule $\sum_{j=0}^{7} y_j \cdot 2^j$.

Example 6
The following table corresponds to rule $47 = 1 + 2 + 4 + 8 + 32$.

$x_{i-1}^t x_i^t x_{i+1}^t$	000	001	010	011	100	101	110	111
x_i^{t+1}	1	1	1	1	0	1	0	0

⊠

Finally, using the connected product introduced in [8, 13], we can define CA in the following way.

Definition 7 (Cellular automaton)
A cellular automaton f is structured as follows:

$$f = \otimes_R g$$

where $J = \mathbb{Z}$ is the lattice of cells; $R = \{(i, i-1), (i, i), (i, i+1) \mid i \in J\}$ describes the neighborhood of each cell; $\forall i \in J, X_i = \{0, 1\}$ is the local state space; $\forall i \in J, g_i = g : X^3 \mapsto X$ is the local transition function.

In general, the analysis of connected products is difficult. Here, we concentrate on the specific connected product describing CA, and we add complexity measures to theoretical results in order to analyze it. Let $\star$ and $\star'$ be two composition operators. Composition of several CA $\otimes_R g_i$ can be defined outside the connected products (this is the usual manner), or inside the product:

$$\star_i(\otimes_R g_i) \text{ or } \otimes_R (\star'_i g_i).$$

If G is a global property, and I an individual property, the objective is to find $\diamond$ such that

$$G(\otimes_R(\star'_i g_i)) = \diamond_i I(\otimes_R g_i).$$

The property composition operator $\diamond$ has to "jump" over the connected product. This is equivalent to finding intermediate G' and $\diamond'$ such that

$$G(\otimes_R(\star'_i g_i)) = G'(\diamond'_i(\otimes_R g_i)) = \diamond_i I(\otimes_R g_i).$$

In the particular case studied here, local rules are decomposed so that the previous equalities hold regarding complexity measures: the global property G and the individual property I both correspond to a complexity level in the classification introduced.

3.2 Phenomenological classification

Here is the classification of long-term CA behaviors presented in [4, 9, 13], which determines five phenomenological classes grouped into three families: periodic (types $\mathcal{N}$, $\mathcal{F}$, $\mathcal{P}$), shifting (type $\mathcal{S}$), and aperiodic (type $\mathcal{A}$) behaviors; typical evolutions are illustrated in Fig. 1.

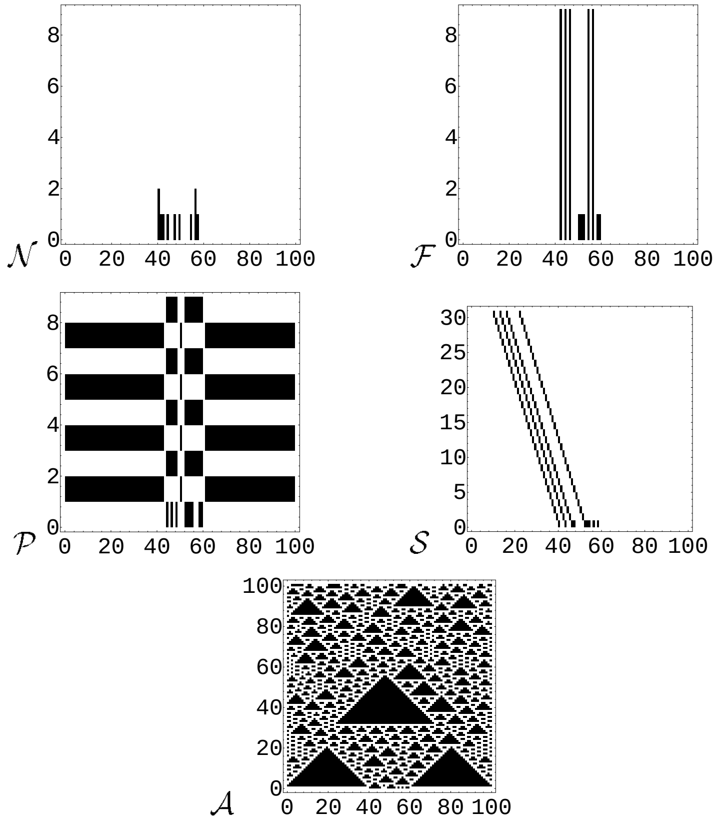

Figure 1: CA classification: typical evolutions of CA classes $\mathcal{N}$, $\mathcal{F}$, $\mathcal{P}$, $\mathcal{S}$, $\mathcal{A}$. In this figure, the horizontal axis represents a portion of the CA state space, the vertical axis represents the temporal evolution (bottom-up); black dots are 1's and white dots are 0's.

Type $\mathcal{N}$: CA evolving to null configurations. This class contains CA that quickly evolve to homogeneous configurations, i.e. without information (all ones or zeroes), after finite transients.

Type $\mathcal{F}$: CA evolving to fixed points. This second class contains CA that evolve to fixed points after finite transients. Of course, class $\mathcal{N}$ is a particular case of class $\mathcal{F}$.

Type $\mathcal{P}$: CA with periodic behaviors. This class contains CA that evolve to periodic configurations, after finite transients. It contains the two previous ones.

Type $\mathcal{S}$: CA with generalized subshift behaviors. This class contains CA that, starting from an initial configuration in particular subspaces of the configuration space, evolve to configurations where a generalized alternating subshift behavior occurs. Here is a definition of this behavior, generalizing [3].

Definition 8 (Generalized alternating subshift)
A CA is a generalized alternating subshift rule if the corresponding global function f is such that there is a closed invariant subset Σ_1 of $\mathcal{C}$ such that

$$\forall x \in \Sigma_1, f^n(x) = \rho^m(x)$$

where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, and $\rho : \mathcal{C} \mapsto \mathcal{C}$ is the classical shift, i.e. $\forall x \in \mathcal{C}, \forall i, \rho(x)_i = x_{i+1}$.

On the complement of Σ_1, where the system evolves like a shift, its behavior can be regular or totally irregular.

Type $\mathcal{A}$: CA with complex or aperiodic behaviors. A configuration is aperiodic if it is not eventually periodic (neither periodic nor one of its forward iterations). Qualitatively, what we observe is a number of different patterns growing, vanishing and moving toward the future. In general, a broad range of behaviors can show up: from random noise, total disorder, and spatio-temporal chaos to some kind of regularity or intermittency in which diverse forms can propagate. Aperiodicity entails that almost the whole domain is visited through successive iterations.

Based on this phenomenological classification and its formalization using transfinite attraction and shifted Hamming distance [9, 13], the following complexity hierarchy can be defined.

Definition 9 (Complexity)
On the set $\{ \mathcal{N}, \mathcal{F}, \mathcal{P}, \mathcal{S}, \mathcal{A} \}$, the following ordering is established: $\mathcal{N} < \mathcal{F} < \mathcal{P} < \mathcal{S} < \mathcal{A}$. The complexity $\Upsilon(g)$ of a CA based on the local rule g is the place of its long-term behavior in the defined ordering.

4 Conjectures in CA composition

Each CA rule can be represented by a vector of eight binary components, corresponding to all possible neighborhood configurations (see §3.1). Using the componentwise logical disjunction, it is possible to generate all elementary rules from the basis $\{1, 2, 4, 8, 16, 32, 64, 128\}$. In [9, 8], the dynamics of these particular CA rules was further studied separately and under composition. From the systematic compositions of basis rules, conjectures were proposed that very much resembles the results presented in §2 on union invariants. The aim of this section is to motivate a deeper study of disjunction by comparison to union.

Basis rules. The basis contains elements from each of the first four classes defined in §3.2: using [4]'s notations, it gives

$$(1 \in p, 2 = s^+, 4 \in f, 8 = n^+, 16 = s^-, 32 \in n, 64 = n^-, 128 \in n)$$

and, using our standard notation,

$$(8, 32, 64, 128 \in \mathcal{N}, 4 \in \mathcal{F}, 1 \in \mathcal{P}, 2, 16 \in \mathcal{S}).$$

Remark 10
Rules denoted by n^+, n^- and s^+, s^- are symmetric: there local transition functions are equal under left-to-right transformation, for instance, $g_{n^+}(a, b, c) = g_{n^-}(c, b, a)$; this property is important when observing the composition of rules.

Disjunction. This composition is easy to compute: Table 1 shows the disjunction of rules 2 and 16, the result being rule 18.

Table 1: Local rule tables of CA 2, 16 and 18

Rule	000	001	010	011	100	101	110	111
2	0	1	0	0	0	0	0	0
16	0	0	0	0	1	0	0	0
18	0	1	0	0	1	0	0	0

Local and global disjunctions are equivalent: let g_1 and g_2 be two local rules, and x be a configuration of $\mathcal{C}$, then

$$(\otimes_R g_1)(x) \vee (\otimes_R g_2)(x) = (\otimes_R (g_1 \vee g_2))(x).$$

Notice that this will allow us to merge G and G' as well as $\star$ and $\diamond'$ in the abstract expressions of §3.1.

Complexity and disjunction. Let us now present all binary disjunctions of basis rules in terms of the classification given in §3.2. We intend to derive information about the dynamics of CA considering the dynamics of the single basis rules in the composition. The abstract expression presenting CA composition in §3.1 becomes

$$\Upsilon(\otimes_R(\vee_i g_i)) = \Upsilon(\vee_i(\otimes_R g_i)) = \diamond_i \Upsilon(\otimes_R g_i),$$

where $\diamond$ is obtained by exhaustive simulation of all possible binary disjunctions (see Table 2, where symmetric cases are omitted for clarity), and Υ refers to Def. 9.

Table 2: Local $\vee$-composition

		8 n^+ $\mathcal{N}$	32 n $\mathcal{N}$	64 n^- $\mathcal{N}$	128 n' $\mathcal{N}$	4 f $\mathcal{F}$	1 p $\mathcal{P}$	2 s^+ $\mathcal{S}$	16 s^- $\mathcal{S}$
8	$\mathcal{N}$	-	$\mathcal{N}$	$\boxed{\mathcal{F}}$	$\mathcal{N}$	$\mathcal{F}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$
32	$\mathcal{N}$	-	-	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{F}$	$\mathcal{P}$	$\mathcal{S}$	$\mathcal{S}$
64	$\mathcal{N}$	-	-	-	$\mathcal{N}$	$\mathcal{F}$	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$
128	$\mathcal{N}$	-	-	-	-	$\mathcal{F}$	$\mathcal{A}$	$\mathcal{S}$	$\mathcal{S}$
4	$\mathcal{F}$	-	-	-	-	-	$\mathcal{P}$	$\mathcal{S}$	$\mathcal{S}$
1	$\mathcal{P}$	-	-	-	-	-	-	$\mathcal{S}$	$\mathcal{S}$
2	$\mathcal{S}$	-	-	-	-	-	-	-	$\boxed{\mathcal{A}}$
16	$\mathcal{S}$	-	-	-	-	-	-	-	-

Conjectures. Looking at Table 2, we see that when two symmetric rules are composed together, complexity grows:

$$\begin{aligned} \underbrace{\otimes_R 8}_{n^+\in\mathcal{N}} \vee \underbrace{\otimes_R 64}_{n^-\in\mathcal{N}} &= \underbrace{\otimes_R(8 \vee 64)}_{72\in\mathcal{F}} \\ \underbrace{\otimes_R 2}_{s^+\in\mathcal{S}} \vee \underbrace{\otimes_R 16}_{s^-\in\mathcal{S}} &= \underbrace{\otimes_R(2 \vee 16)}_{18\in\mathcal{A}}. \end{aligned}$$

Apart from this, type $\mathcal{F}$ is obtained from type $\mathcal{F}$, and periodicity entails type-$\mathcal{P}$, type-$\mathcal{S}$, and type-$\mathcal{A}$ behaviors. Two important complexity laws were conjectured in [4, 8].

Conjecture 11 (Law of complexity conservation)
In the local disjunctive composition of two basis rules, the complexity never decreases:

$$\forall x, y \in \{\mathcal{N}, \mathcal{F}, \mathcal{P}, \mathcal{S}\}, \Upsilon(x \vee y) \geq \max(\Upsilon(x), \Upsilon(y)).$$

Conjecture 12 (Law of complexity increase)
The complexity of the local disjunctive composition of two basis rules increases (e.g., $\Upsilon(x \vee y) > \max(\Upsilon(x), \Upsilon(y))$) iff one of the following conditions holds:

- $x = n^+$, $y = n^-$; *(symmetric null rules)*
- $x = s^+$, $y = s^-$; *(symmetric shifting rules)*
- $x \in \mathcal{N}\backslash\{32\}$, $y \in \mathcal{P}$. *(periodicity)*

We see that two particular cases appear to increase complexity: concurrent symmetry and periodicity. In the rest of this section, we concentrate on the first case, which seems to behave as prescribed by union-invariant theorems.

5 Complexity by composition of shifts

Among the experimental facts described in the previous section, let us pay attention to Conj. 12, and to its second statement in particular: disjunction of shifts leads to complexity.

Here, we proceed to a naive comparison of this conjecture to the compositional analysis of the Cantor relation or any other complex system obtained by union of elementary systems attracting the space to different fixpoint invariants.

5.1 Rules 2 and 16

The rule tables of 2 and 16 are represented in Table 1, together with their disjunction, viz. rule 18. Rule 2 gives 1 iff the local configuration is $(0, 0, 1)$, 0 in the other cases; rule 16 gives a 1 in the symmetric configuration: 1 iff the local configuration reads $(1, 0, 0)$, 0 otherwise. We then compute the disjunction of rules 2 and 16, which gives rule 18: 1 iff the local configuration is $(1, 0, 0)$ or $(0, 0, 1)$, and 0 otherwise.

The behavior of rules 2 and 16 is very simple (type $\mathcal{S}$): configurations are shifted along the lattice, to the left or to the right. The behavior of rule 18 is less simple, we classify it as a complex, type-$\mathcal{A}$ CA. Examples are represented in Fig. 2.

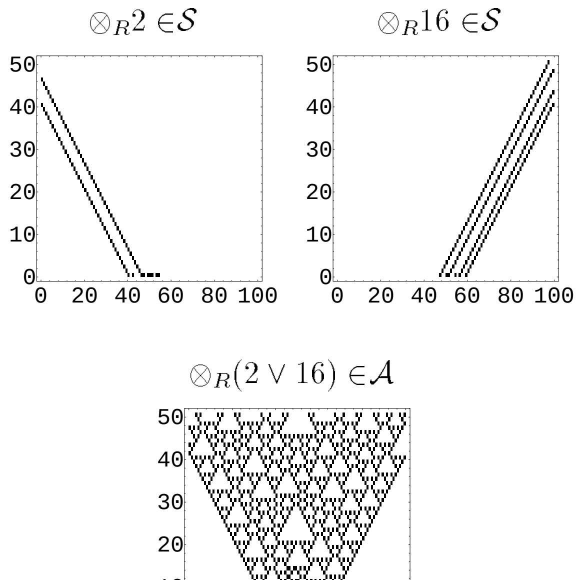

Figure 2: Evolutions of CA rules 2 (left) and 16 (right) from a random initial condition show simple shifting behaviors, their disjunction 18 (bottom) shows a complex type-$\mathcal{A}$ dynamics.

5.2 Cantor relation

Let us now briefly summarize the compositional analysis of the Cantor relation f, in order to emphasize how close its dynamics is to the disjunctive composition (see Fig. 3). The relation is defined by union composition:

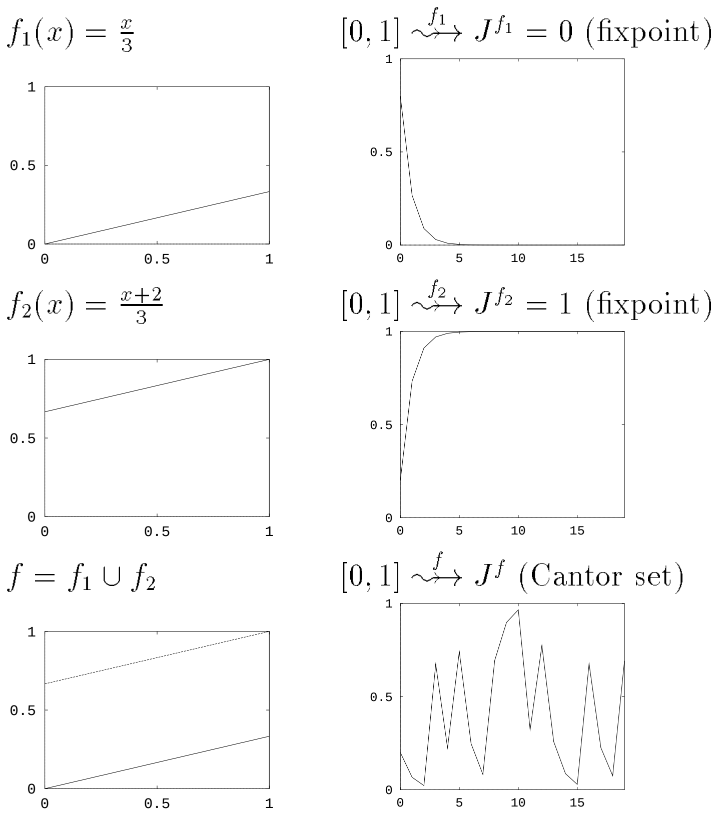

Figure 3: Compositional analysis of the Cantor relation: f_1 and f_2 show simple dynamics (attraction to their respective fixpoints); f shows a complex (direct and reverse) dynamics, i.e. chaotic on the Cantor set invariant.

$$f_1 = \frac{x}{3} \tag{1}$$

$$f_2 = \frac{x+2}{3} \tag{2}$$

$$f = f_1 \cup f_2. \tag{3}$$

Proposition 13
Cantor relation $f = f_1 \cup f_2$ has a complex dynamics determined by a Cantor-set invariant.

PROOF. The behavior of f_1 on $[0,1]$ is very simple: every point is asymptotically attracted to the fixed point 0. Function f_2 has also a unique attracting fixed point on

$[0,1]$, which is 1. These fixpoints are different. Moreover, f_1 and f_2 are both contracting, with a coefficient $\gamma(f_1) = \gamma(f_2) = \frac{1}{3}$, and they verify $f_1^{-1}([0,1]) = f_2^{-1}([0,1]) = [0,1]$. All assumptions of Theorem 5 are verified, which concludes the proof.
□

The behavior of the union is much richer than its individual components: it is has a Cantor-set invariant attracting the whole space.

5.3 Comparison

Rule 2 and rule 16 behave in a very simple way: every configuration is shifted to the left or to the right. Everything seems to be attracted to the same point, situated at infinity. Regarding f_1 and f_2, every point of $[0,1]$ is continuously attracted to 0 or 1.

Rule 18 is obtained by disjunction of rules 2 and 16. Globally, one can consider that at some places of a configuration, the behavior of 2 is executed, whereas at other places the behavior of 16 is executed. The same thing appears when executing the union of two systems. An important difference is that the choice in the first case is directed by the opportunity to activate a transition, while in the second case the choice is directed by an oracle (pure randomness in the future, but recall the system is deterministic backward and yet complex).

Finally, when rules 2 and 16 on one hand, or f_1 and f_2 on the other hand, are composed together such that both components can be applied on the configuration at the same time, complexity arises. In the first case, this is only an experimental conjecture (Conj. 12) which is sort of evidenced by simulation, whereas in the second case, there is a theorem to prove the observation (Theorem 5).

5.4 A more precise conjecture

Considering the first experimental results and rephrasing the essence of union-invariant theorems in the context of CA, Conj. 12 can be refined as follows.

Conjecture 14 (Complex dynamics by $\vee$-composition – 1)
Rules 2 *and* 16 *have compatible dynamics (contracting in the future), their disjunction is globally contracting in the future, and their invariants are different fixpoints. Then* $\otimes_R(2 \vee 16)$ *has a rich invariant and a complex dynamics.*

Remark 15
This complex dynamics is not necessarily related to chaos, as mentioned in §1.

6 Qualitative analysis and complexity measures

Now we have more precisely stated that the disjunctive composition of shifting CA leads to complexity (Conj. 14), the questions are:

- Does rule 18 really behave like the Cantor relation?

- How to prove that the resulting dynamics is complex?

The problem is that we do not have any theorem to treat this case of local composition. In other words:

- Does rule 18 behave like a global union?

- In mathematical terms: $\Upsilon(\otimes_R 18) = \Upsilon((\otimes_R 2) \cup (\otimes_R 16))$?

Probably not! In fact, rule 18 certainly entails more complexity than a global union because it acts locally. We thus investigate the intermediate case between a global union and this local disjunction: a local union.

The reason why we decide to study this intermediate case amounts to the fact that these three versions (local disjunction, local union, global union) can be seen as particular cases of a more general composition type that very much resembles probabilistic CA.

Generally, we can define a *probabilistic CA* as follows: at each step t, each cell i applies a local rule g depending on t and i. This local rule is a probabilistic choice between (at least) two possibilities: g_1 with probability p and g_2 with probability $1-p$. The model can be reduced to four specific cases, according to when and where the probabilistic choice is made. This leads to Table 3, where we mention an (approximately) equivalent nonprobabilistic model.

Table 3: Different models of probabilistic CA

place of choice	time of choice	choice	$\approx$ model
local	each step	g_i^t	local union
local	once	g_i	interleaved [8]
global	each step	g^t	global union
global	once	g	classical CA

We have to characterize the complexity of different systems. Generally, we base our intuitive understanding of complexity on the notions of invariance and attraction. Here, we focus on the richness of the attractor, but we introduce other tools from the experimental part of CA studies:

- Boolean derivative and weight;

- generalized mean-field theory;

- entropy.

Boolean derivative and weight. In the following, we apply the notion of Boolean derivative to CA [29]. Given a Boolean function f, its *Boolean derivative* is defined by

$$\begin{aligned} \frac{\partial f}{\partial x_j} &= f(\cdots \widehat{x_j} \cdots) \oplus f(\cdots x_j \cdots) \\ \dot{f} &= (\frac{\partial f}{\partial x_{j-1}}, \frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial x_{j+1}}) \end{aligned}$$

where $\oplus$ is the exclusive disjunction, and $\widehat{\cdot}$ stands for a logical negation.

Since every elementary rule can be expressed as an exclusive disjunction of basic rules, the derivative is easy to compute: R_i being basic rules, c_i being binary coefficients,

$$f = \oplus_i c_i R_i \Rightarrow \dot{f} = \oplus_i c_i \dot{R}_i.$$

The *weight* of a function f is the average number of 1's in the partial derivatives of f. It is related to the sensitivity to initial conditions of the global function. The bigger is the weight, the more sensitive the function is.

Generalized mean-field theory. To evaluate the disorder induced by a rule, several methods are conceivable. In [18, 17], the authors present the *generalized mean-field theory*.

- The 0th order corresponds to Langton's λ parameter [21, 16]: it takes the proportion of 1's in the image of all triples of the rule.
- The 1st order stands for a mean-field approximation of the 0th order.
- The 2nd order is an extension of the first idea to larger neighborhoods: it involves spatial correlations.
- As further orders converge to an invariant measure, they mimic the behavior of cellular automata with more and more fidelity.

Entropy. A notion of entropy is also useful to characterize the complexity or disorder induced by some rules (this kind of notion has been extensively studied in [23, 2]). If $(p_i)_{i\in I}$ is a probability distribution corresponding to a set of possible events I, the *entropy* of this distribution is:

$$S = -\sum_{i\in I} p_i \log(p_i).$$

In this case, probabilities could be Markov approximations of CA rules as given by the generalized mean-field theory.

7 Compositional analysis of complex CA

In this section, we systematically analyze the local disjunction, the local union, and the global union using the complexity measures introduced in §6. This allows us to reinforce Conj. 14.

7.1 Local disjunction, local union, and global union

Because of the model itself, it is obvious that the complexity of the global union is smaller than the complexity of the local union:

$$\Upsilon((\otimes_R 2) \cup (\otimes_R 16)) < \Upsilon(\otimes_R (2 \cup 16)).$$

What we would like to show goes in the same direction:

$$\Upsilon(\text{global union}) < \Upsilon(\text{local union}) < \Upsilon(\text{local disjunction}).$$

To compare these different cases, we use the complexity measures introduced above. For each rule, we have:

- a table with, in each column,
  - a typical neighborhood configuration;
  - the local image the rule will produce after one iteration;
  - a label given to the neighborhood;
  - the possible labels appearing when the label itself is embedded in a bigger neighborhood of radius 2 (instead of radius 1), after one iteration;
  - the probability to find the label after one iteration, starting from a neighborhood of radius 2;
- the labels that are never reached in an evolution;
- the Boolean derivative of the rule, and its corresponding weight;
- the proportion of 1's in the image of the rule (Langton's λ parameter);
- the entropy of the rule, based on the probabilities given above.

In Tables 4 to 7, we detail the analysis of four different rules, namely rules 2, 16, their local disjunction and their local union.

Table 4: Rule 2, $\otimes_R 2$

	000	001	010	011	100	101	110	111
image	0	1	0	0	0	0	0	0
label	a	b	c	d	e	f	g	h
next	a, b	c	a, e	a, e	a	a	a	a
proba	$\frac{20}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	0	$\frac{4}{32}$	0	0	0

Table 5: Rule 16, $\otimes_R 16$

	000	001	010	011	100	101	110	111
image	0	0	0	0	1	0	0	0
label	a	b	c	d	e	f	g	h
next	a, e	a, e	a, b	a	c	a	a, b	a
proba	$\frac{20}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	0	$\frac{4}{32}$	0	0	0

7.2 Comparison and summary of the results

Basing our comparison on the tables presented above, we get the following results (see Table 8):

- proportion of residual 1's

$$0 < P_2 = P_{16} = P_{2\cup 16} < P_{2\oplus 16} < P_{orig} = \frac{1}{2};$$

- entropy

$$0 < S_2 = S_{16} < S_{2\cup 16} < S_{2\oplus 16} < S_{orig} = 2.07944.$$

Table 6: Rule 18, $\otimes_R(2 \vee 16)$

	000	001	010	011	100	101	110	111
image	0	1	0	0	1	0	0	0
label	a	b	c	d	e	f	g	h
next	a, b, e, f	c, g	a, b, e, f	a, e	c, d	a	a, b	a
proba	$\frac{14}{32}$	$\frac{4}{32}$	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	0

Table 7: Local union, $\otimes_R(2 \cup 16)$

	000	001	010	011	100	101	110	111
image	0	$\{0,1\}$	0	0	$\{0,1\}$	0	0	0
label	a	b	c	d	e	f	g	h
next	a, b e, f	a, c e, g	a, b e, f	a, e	a, b c, d	a	a, b	a
proba	$\frac{21.5}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{0.5}{32}$	$\frac{3}{32}$	$\frac{0.5}{32}$	$\frac{0.5}{32}$	0

Table 8: Summarized compositional analysis

Local rule	2	16	$2 \vee 16$	$2 \cup 16$
Unreached	d, f, g, h	d, f, g, h	h	h
Derivative	$\dot{R}_2$	$\dot{R}_{16}$	$\dot{R}_2 \oplus \dot{R}_{16}$	–
$\dot{R}(x_1, x_2, x_3)$	$(\widehat{x}_2 x_3, \widehat{x}_1 x_3, \widehat{x}_1 \widehat{x}_2)$	$(\widehat{x}_2 \widehat{x}_3, x_1 \widehat{x}_3, x_1 \widehat{x}_2)$	$(\widehat{x}_2, x_1 \widehat{x}_3 \oplus \widehat{x}_1 x_3, \widehat{x}_2)$	–
Weight	$(2, 2, 2)$	$(2, 2, 2)$	$(4, 4, 4)$	–
Average	2	2	4	–
Prop. 1's	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
Entropy	1.07354	1.07354	1.66132	1.12789

Thus, the local composition $2 \cup 16$ induces more order than $2 \vee 16$, which seems to be a stronger composition than the first one. Hence, we have the second part of the inequality presented above, i.e.

$$\Upsilon(\text{local union}) < \Upsilon(\text{local disjunction}).$$

Proposition 16 (Complex dynamics by $\vee$-composition – 2)

$$\Upsilon(\otimes_R(2 \vee 16)) > \Upsilon(\otimes_R(2 \cup 16)) > \Upsilon((\otimes_R 2) \cup (\otimes_R 16)) = \textit{ complex behavior.}$$

PROOF. Using Theorem 5, we establish the complex behavior of the global union; evidently, the local union is more complex than its global counterpart; with the help of our complexity measures, we state that the local disjunction introduces more disorder than the local union; this permits to conclude that this local disjunction has a globally complex behavior.
□

8 Conclusion

In this paper, we carried out a "compositional experimental analysis" combining experimental complexity measures with theoretical compositional results. This led to interesting conjectures on the emergence of type-$\mathcal{A}$ complex behaviors by composition of type-$\mathcal{S}$ simple behaviors: Conj(s). 14 and 16, refining Conj. 12 of [4].

What is the interest of the previous developments? Establishing the complexity of the local disjunction is not a breakthrough because it is already clear visually, under simulation. But, we analyzed it by composition with the help of common experimental tools in the field of cellular automata, viz. complexity measures.

The result is: merging two very simple compatible behaviors attracting the space to different parts of it entails the emergence of complexity, as the composition realizes an opportunistic or oracle choice between different components.

An increase in complexity can come from the explosion of dimensions of the state space, or from the mixing generated by the neighborhood relation R and the local transition functions g_i's. In fact, the situation is not totally black or white. Complexity can arise even without mixing, just by the union or another operator. Simplicity can also arise from the connected product. All depends on the way systems are composed, together with some important properties: the attraction properties to different regions of the global state space. In this case, complexity can be created from very simple systems composed together.

Two axes could be further investigated: first, we could make use of other complexity measures to refine our results (e.g. higher orders of generalized mean-field theory, variants of entropy); second, in addition to union (and disjunction), other composition operators could be applied to CA and analyzed in the light of complexity measures, too. Partial experimental results have already been published in [8] on sequential composition, that confirm the power of mixing theoretical compositional results with complexity measures.

Finally, a last question concerns connected products in general: these first compositional experimental results should be strengthened and other techniques for the compositional analysis of connected products should be elaborated.

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