Compositional Complexity in Dynamical Systems*

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Nontechnical abstract. The complexity of dynamical systems can be measured by observable quantities, or abstractly defined by conjunction of mathematical properties. Our complementary approach is based on the structure of systems. Using appropriate operators, systems are seen as composed systems, and dynamical properties as composed properties. This composition principle allows us to characterize the evolution of different types of complex systems. For instance, we study classical chaotic systems, spatio-temporal complex cellular automata, and the formal system generating paper foldings. The paper presents the framework and its applications informally, in order to emphasize the qualitative aspects of the approach.

Technical abstract. We propose a compositional characterization of complex behaviors, i.e. a way to generate complexity based on the structural composition of systems. A homomorphism between composition operators and composition of dynamical properties of systems is established, which allows us to analyze classical (Smale horseshoe map, Cantor relation), formal (paperfolding sequences) and spatially extended systems (cellular automata). This paper presents applications of the composition principle in an informal way, emphasizing the qualitative aspects of the approach.

Keywords: dynamical system, complexity, composition, invariance, attraction.

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1 How to study dynamical complexity?

Many directions have been explored to characterize complex behaviors in dynamical systems. In particular, chaos has become an important matter in the study of nonlinear systems.

Complexity. Chaos is usually associated with the time decrease of spatial correlations. Many different notions are used to measure complexity in this context: Lyapunov exponents, entropy, Markov chains, information flows, invariant densities [12, 15, 16].

Chaos can also be stated in mathematical terms. For instance, Devaney's widespread definition of chaos is based on three topological properties: unpredictability (sensitivity to initial conditions), space-undecomposability (topological transitivity), and regularity (density of periodic orbits) [4].

Recently, several authors have proposed new definitions and characterizations of (spatio-temporal) chaos in spatially extended systems like coupled map lattices, (probabilistic) cellular automata, ecologies [1, 10, 13, 17].

Surprisingly, only a few results exist on discrete-space chaos, although it seems appropriate to distinguish simple fixpoints and 2-cycles from 2^{100}-cycles that make systems seemingly complex [2, 23].

Despite these numerous complementary approaches, no objective definition of dynamical complexity covers all kinds of systems, so far.

Aim: compositional complexity. In order to bring new insights in understanding how dynamical complexity emerges, we introduce compositionality [7]. Basically, the compositional analysis of a dynamical system consists in determining some global property concerning dynamical or computational aspects of the system by combination of individual properties of its components, which are expected to be simpler. The following diagram illustrates the idea; S_i being the components of a composed system $S = \star_i S_i$, I denoting an individual property, G a global property, we want to find a way to combine the individual properties, viz. $\diamond$, to characterize the global property:

$$\begin{tikzcd} S_i \arrow[r, "I"] \arrow[d, "\star"'] & I(S_i) \arrow[d, "\diamond"] \\ \star_i S_i \arrow[r, "G"'] & \diamond_i I(S_i). \end{tikzcd}$$

In the following, we show how complexity can be explained in this light: composing elementary compatible systems attracting their state space to different regions leads to complex behaviors [7, 18, 19].

Outline of the paper. In §2, we recall the necessary background; in §3, we show how the composition principle is applied to characterize the complexity of systems in different contexts: classical dynamical systems, formal systems, and spatially extended systems; finally, in §4, we draw some conclusions.

2 Technical background

In terms of the above diagram, we now define dynamical systems (S_i), composition operators ($\star$), dynamical properties (I, G), and we recall two important theorems in the composition of properties ($\diamond$). We focus on discrete-time dynamical systems defined on sets of points: relations.

Definition 1 (Dynamical system)
A closed relation $f \subseteq X \times X$ on a compact metric space (X, d) defines a dynamical system (X, f):

$$\forall A \subseteq X, f(A) = \{y | \exists x \in A : (x, y) \in f\}.$$

Definition 2 (Composition operators)
Let f, g be dynamical systems on X.
Inversion:

$$f^{-1} = \{(y, x) | (x, y) \in f\}.$$

Sequential composition:

$$f; g = \{(x, z) | \exists y : (x, y) \in f \wedge (y, z) \in g\}.$$

Union: nondeterministic choice,

$$f \cup g = \{(x, y) | (x, y) \in f \vee (x, y) \in g\}.$$

Free product: without interaction,

$$f \times g = \{((x, y), (v, w)) | (x, v) \in f \wedge (y, w) \in g\}.$$

Connected product: with explicit interaction; $I \subseteq \mathbb{Z}$, $\forall i \in I$, X_i is a local space, $R(i) \subseteq I$ and $g_i \subseteq \times_{j \in R(i)} X_j \times X_i$,

$$\otimes_R = \{((x_i)_i, (y_i)_i) | \forall i, y_i = g_i((x_j)_{j \in R(i)})\}.$$

A composed system is thus obtained by recursive composition of dynamical systems based on the above operators (Def. 2).

Definition 3 (Dynamics)
Let (X, f) be a system. Its dynamics is recursively defined by:

$$\begin{aligned} f^0 &= \mathcal{I}_X \\ f^{\pm(n+1)} &= f^{\pm n}; f^{\pm} \end{aligned}$$

where $\mathcal{I}_X$ is the identity relation on X, i.e. $\forall A \subseteq X, \mathcal{I}_X(A) = A$.

Remark 4
Equivalently, the dynamics can be defined as the set of all trajectories the system can follow:

$$\theta(A, f) = \{s \in X^{\mathbb{Z}} \mid (s_0 \in A) \wedge (\forall n \in \mathbb{Z}, (s_n, s_{n+1}) \in f)\}.$$

This definition reminds of the notion of trace in parallelism semantics [5].

Invariance and attraction are related to dynamical complexity of systems. Invariants are sets of states that have infinite internal histories, i.e. stable sets. Their structure represents trajectories between inner states; it organizes and, thus, strongly influences the resulting dynamics. Attraction establishes relations between initial and final or asymptotic states of infinite histories.

Definition 5 (Invariant)
The invariant J of a system (Y, f) is the intersection of the greatest sets verifying

$$S \subseteq f(S) \quad \text{and} \quad S \subseteq f^{-1}(S).$$

By simple application of Tarski's lattice-fixpoint theorem [22], we get the following proposition.

Proposition 6
The invariant J of f is computed by limits of successive iterations:

$$J = (\cap_n f^n(X)) \cap (\cap_n f^{-n}(X)).$$

Remark 7
Our notion of invariance is symmetric in time. The relational framework allows us to treat systems and their inverses in a uniform way.

Definition 8 (Attraction)
Let (Y, f) be a system, and $P, Q \subseteq Y$; then P is attracted to Q by f, i.e. $P \overset{f}{\rightsquigarrow} Q$, iff

$$\cap_i \overline{\cup_{i \leq j} f^j(P)} = Q,$$

where $\overline{A}$ represents the closure of A in Y.

A rich invariant structure together with attraction can entail rich dynamical properties. The notion of packed invariance summarizes this.

Definition 9 (Packed invariance)
The invariant J of f is packed *iff f or f^{-1} attracts the space to J, and J has a Cantor-set structure.*

What is complexity? The distinction between "simple" and "complex" is not sharp. As emphasized by the last definition, three factors play a role: the type and size of invariants, and the attractors of the system. Packed invariance covers a wide range of structurally complex behaviors, from apparently simple ones (single attracting fixpoints) to more complex ones (chaotic attractors), due to the attraction to a Cantor set. Using [7, Prop(s). 5.33, 5.34], it can be proved that packed invariance entails topological transitivity and sensitivity to initial conditions, which define Knudsen chaos [11].

Proposition 10 (Chaos)
If the invariant J of a system f is packed, f is (Knudsen) chaotic on J.

In general, we order different dynamical behaviors from simple to complex [6, 9]:

$$\text{fixpoint} \ < \ \text{cycle} \ < \ (\text{shift} <) \ \text{aperiodic behavior}$$

where shifting behaviors concern spatially extended systems only.

When systems are composed together, we want to derive the complexity of composed systems from the complexity of their components. In this sense, some compositions *do not increase complexity* (e.g. ; and $\times$), others *do not decrease complexity* (e.g. $\cup$). The case of $\otimes$ cannot be treated homogeneously. Let us now state two important compositional results [7, Chap. 6, Cor. 6.32, Prop(s). 6.34, 6.35].

Theorem 11 (Product invariant)
Let (X, f) and (Y, g) be dynamical systems whose invariants are respectively J^f and J^g. The invariant of $(X \times Y, f \times g)$ is $J = J^f \times J^g$. Moreover, if J^f and J^g are packed, then J is packed, too.

Theorem 12 (Union invariant)
Let (Y, f) and (Y, g) be two injective compatible systems with distinct (packed) fixpoint invariants, such that $\gamma(f)+\gamma(g) < 1$ and $f^{-1}(Y) = g^{-1}(Y) = Y$ (or $\gamma(f^{-1})+\gamma(g^{-1}) < 1$ and $f(Y) = g(Y) = Y$), then $f \cup g$ has a packed invariant.

The contractivity factor γ is defined by $\gamma(f) = \sup\limits_{x \neq y} \frac{d_H(f(x), f(y))}{d(x,y)}$, and d (d_H) is a (Hausdorff) metric on Y. Let us rephrase the last result informally: complexity (Cantor-set structure) emerges from union composition of simple (i.e. fixpoint) compatible (i.e. individually and globally contracting or expanding) systems attracting the space to different regions.

3 Case studies in compositional complexity

In this section, we present applications of our approach to classical, formal, and spatially extended systems. We give the reader an intuitive "flavour" of the approach, while further technical details can be found in [6, 7, 18, 19].

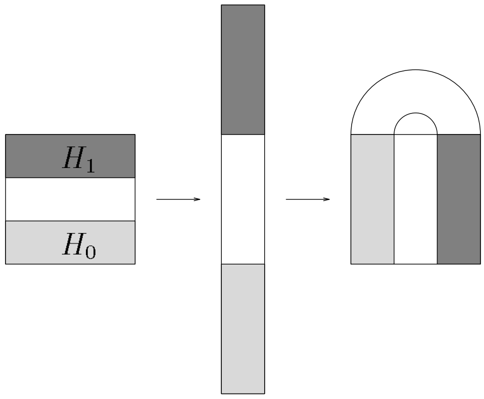

Figure 1: Effect of f on $[0,1]^2$: y-stretch, x-squeeze, fold

Smale horseshoe map. This two-dimensional system is important because it constitutes a paradigmatic example of chaotic behavior. Indeed, sensitivity to initial conditions results from successive stretching and folding of the state space. The dynamical process yields mixing via filamentation, as a baker who kneads a blob dough [20, 21]. Many other chaotic systems like Lorentz equations [14] or Hénon map [8], are also subject to mixing via such spatial deformations, which seem to be important factors of complexity. We analyze this phenomenon by rewriting the system as a union of two products, and explain it by using the invariant theorems.

Let $0 < \lambda < \frac{1}{2}$ and $0 < \frac{1}{\mu} < \frac{1}{2}$. We choose $\lambda = \frac{1}{3}$ and $\frac{1}{\mu} = \frac{1}{3}$. The function f is defined on $[0,1]^2$ (see Fig. 1):

$$f(x,y) = \begin{cases} (\lambda x, \mu y) & \text{on} \quad H_0 = A_x \times B_y \\ (-\lambda x + 1, -\mu y + \mu) & \text{on} \quad H_1 = A_x \times C_y \end{cases}$$

with the following notational conventions: $A_x = [0,1]$, $B_y = [0, \frac{1}{\mu}]$, $C_y = [1 - \frac{1}{\mu}, 1]$. The shape of the codomain $f([0,1]^2)$ explains why f is the so-called "horseshoe map".

We remark that the state spaces are disjoint: x and y act independently. A straightforward decomposition of f is the following: $f = f_0 \cup f_1$, with $f_0 = R \times S$ on H_0, $f_1 = V \times W$ on H_1, and (see Fig. 2): $R(x) = \lambda x$ on A_x, $S(y) = \mu y$ on B_y, $V(x) = -\lambda x + 1$ on A_x, and $W(y) = -\mu y + \mu$ on C_y.

Since $\lambda < \frac{1}{2}$, components R and V are contracting in the future (with contractivity factors λ). In the same way, since $\mu > 2$, S and W are expanding in the future (with contractivity factors μ). The individual invariants are fixpoints: $J^R = J^S = 0$, $J^V = \frac{1}{1+\lambda}$, and $J^W = \frac{\mu}{1+\mu}$.

The two products $R \times S$ and $V \times W$ are *hyperbolic systems*, i.e. contracting in opposite temporal directions (past or future) on different axes. However, the two subsystems f_0 and f_1 have compatible dynamics (contracting on x and expanding on y). By Thm. 11, their invariants are distinct fixpoints: $J^{f_0} = (0,0)$, $J^{f_1} = (\frac{1}{1+\lambda}, \frac{\mu}{1+\mu})$, and, since the components are trivially packed, both product invariants are packed.

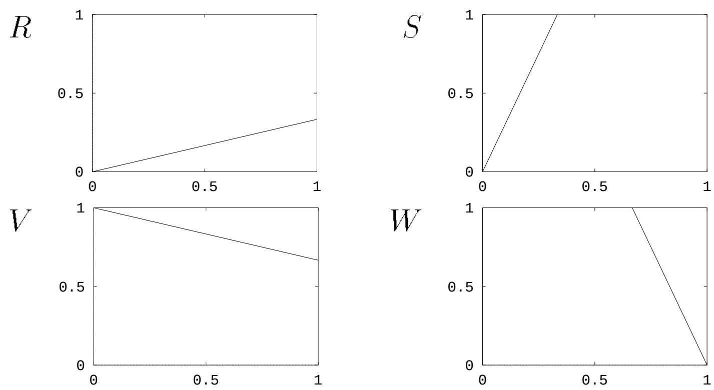

Figure 2: Graphs of R, S, V, and W

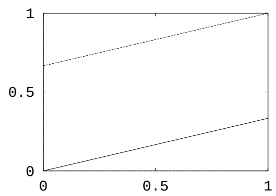

Figure 3: Graph of f

Finally, to characterize the global invariant J^f of f and its structure, we use Thm. 12 whose assumptions are verified. The conclusion is: the invariant of the Smale horseshoe map f is packed, which explains its chaotic dynamics. We have thus a structurally simple system principally based on a combination of linear pieces, union and free product, which shows a complex dynamics.

Cantor relation. This system (see Fig. 3) is interesting because it generates the well-known Cantor middle-thirds set (see Fig. 4) as invariant, which is a typical example of fractal. Despite its nondeterminism, the system can be treated easily in our relational framework: it suffices to reverse the execution, and to consider past invariance instead of future invariance. Of course, the global invariant takes both directions into account, which permits to avoid any consideration of time direction. Union is again the appropriate composition operator to study the Cantor relation.

On $[0,1]$, we define $f_1(x) = \frac{1}{3}x$, $f_2(x) = \frac{1}{3}x + \frac{2}{3}$, and $f = f_1 \cup f_2$.

Let us now describe the dynamics of these systems (see Fig. 5). The behavior of f_1 on $[0,1]$ is very simple: every point is attracted to the fixpoint 0, in an infinite number of iterations. Function f_2 has also a unique attracting fixpoint on $[0,1]$, which is 1. Both functions are contracting with a factor $\frac{1}{3}$.

Thus, according to Thm. 12, *the behavior of their union is chaotic*, as the union invariant is a Cantor set. Each subsystem attracts the space to a different fixpoint; the global system attracts the space to a highly structured region where it is chaotic.

Figure 4: Iterative construction of Cantor's middle-thirds set: recursive elimination of middle thirds intervals

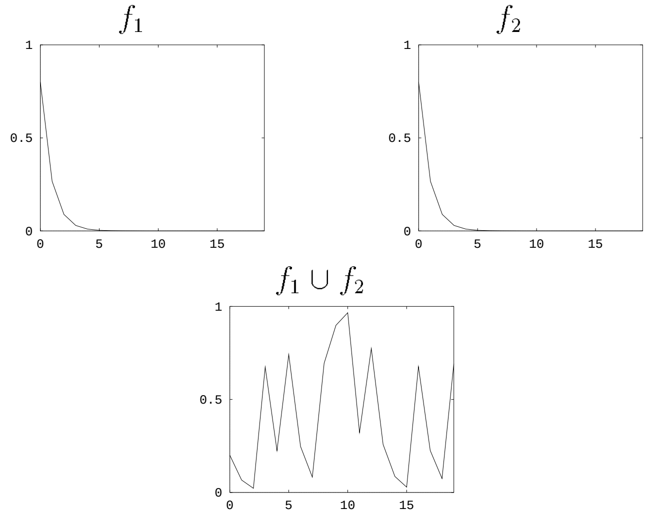

Figure 5: Evolution of f_1, f_2: fixpoint attraction, and $f = f_1 \cup f_2$: chaotic dynamics

Formal systems: paperfoldings. Paperfolding sequences are the landscapes obtained by infinite sequences of up and down foldings of an imaginary sheet of paper (see Fig. 6).

These folded landscapes are sequences or words generated by specific grammars. Formally, let $\mathcal{L} = \{V, \Lambda\}$ be the set of profiles; applying foldings to "clean" unfolded papers gives $U(\varepsilon) = V$ and $D(\varepsilon) = \Lambda$; for any landscape (sequence of profiles) $\forall w = w_1 w_2 \cdots w_n \in \mathcal{L}^*$ where n is assumed to be odd to be reachable from the "clean" unfolded paper ε, and $\forall i, w_i \in \mathcal{L}$, foldings are given by

$$\begin{aligned} U(w) &= V w_1 \Lambda w_2 V \cdots V w_n \Lambda \\ D(w) &= \Lambda w_1 V w_2 \Lambda \cdots \Lambda w_n V. \end{aligned}$$

The extension of this definition to infinite landscapes and applications of is left to the reader.

Interestingly, *the combination of up and down foldings* can be seen *as a union composition of dynamical systems* that verifies the assumptions of Thm. 12 in the space of infinite symbol sequences $\mathcal{L}^\omega$. The invariant of this system is the set of all possible landscapes.

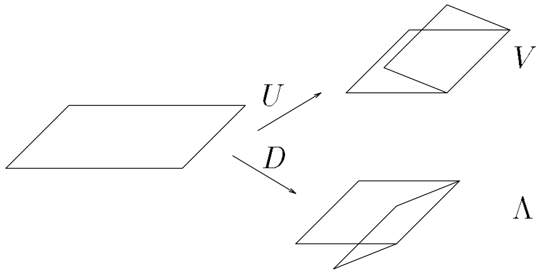

Figure 6: Two possible foldings, up (U) and down (D), and the resulting profiles V and Λ

Starting from the empty landscape or any other word, the repetitive application of U leads to a fixed landscape L_U. The infinite application of D leads to another fixed landscape L_D.

The combination of U *and* D as $U \cup D$ or its inverse *leads to a Cantor-set invariant* on which *the global system is chaotic.*

This result is not surprising as the folding process behind the abstract mathematical terms used to describe these infinite symbolic sequences reproduces the "stretch-and-squeeze" effect of Smale's horseshoe map and other hyperbolic dynamical systems that show chaotic behaviors due to iterative foldings of their underlying state spaces. The formal language generation of paperfolding sequences allows us to extend the result to a whole family of formal systems which are also proved to be complex in this sense.

Spatially extended systems: cellular automata. The connected product seems a natural operator for describing massively distributed systems with interactions between components. Among those, elementary one-dimensional cellular automata are simple and powerful models: they are computationally universal, and cover a wide range of applications, from biology to physics and theoretical computer science [24].

Elementary CA are defined by local Boolean functions of three variables, $g : \{0,1\}^3 \mapsto \{0,1\}$, often named by their unique decimal representation $\sum_{a,b,c \in \{0,1\}} g(a,b,c) \cdot 2^{4a+2b+c}$. Their global dynamics f is defined by the synchronous update of all cells of a bi-infinite one-dimensional lattice: $\forall x \in 2^{\mathbb{Z}}, i \in \mathbb{Z}, f(x)_i = g(x_{i-1}, x_i, x_{i+1})$. In other words, using the connected product: $I = \mathbb{Z}$, $\forall i, X_i = \{0,1\}, R(i) = \{i-1, i, i+1\}, g_i = g$, and $f = \otimes_R g$.

Two compositions are conceivable in this kind of model: global or local. Let $\star$ and $\star'$ be two composition operators. Composition of several CA $\otimes_R g_k$ can be defined outside the connected products (this is the usual manner), or inside the product:

$$\star_k(\otimes_R g_k) \text{ or } \otimes_R (\star'_k g_k).$$

The former does not add anything to the previous type of systems. Our concern is the study of local composition composed with the connected product: f and g being two local transition functions, $\star$ being a local composition operator, we study the system

Nbd	000	001	010	011	100	101	110	111
$\otimes 2$	0	1	0	0	0	0	0	0
$\otimes 16$	0	0	0	0	1	0	0	0
$\otimes 18 = \otimes(2 \vee 16)$	0	1	0	0	1	0	0	0
$\otimes(2 \cup 16)$	0	$\{0,1\}$	0	0	$\{0,1\}$	0	0	0

Table 1: Local rule tables of CA 2, 16, $2 \vee 16$ and $2 \cup 16$, where triples represent neighborhoods of cells

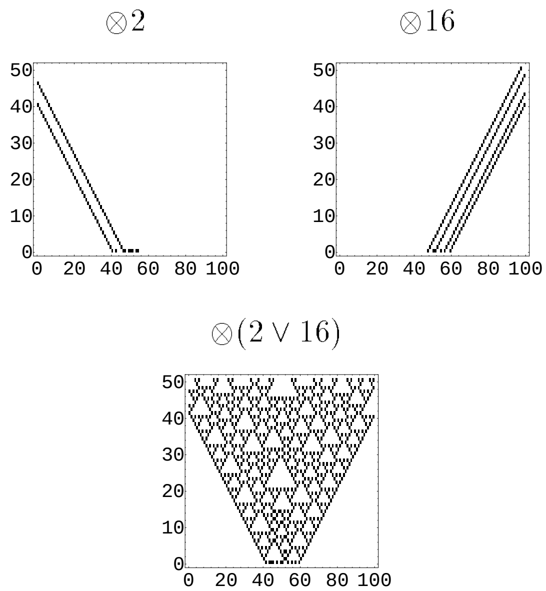

Figure 7: Evolution of CA $\otimes 2$, $\otimes 16$: shifting behavior, and $\otimes 18 = \otimes(2 \vee 16)$: complex dynamics

$\otimes(f \star g)$. If G is a global property, and I an individual property, the objective is to find $\diamond$ such that

$$G(\otimes_R(\star'_k g_k)) = \diamond_k I(\otimes_R g_k).$$

This kind of local composition adds a difficulty to the analysis of systems, because the property composition operator $\diamond$ has to "jump" over the connected product. This is equivalent to finding intermediate G' and $\diamond'$ such that

$$G(\otimes_R(\star'_k g_k)) = G'(\diamond'_k(\otimes_R g_k)) = \diamond_k I(\otimes_R g_k).$$

It has been conjectured that the disjunction of local rule tables of certain cellular automata gives rise to chaos [3]. In particular, the behavior of rule 18 (disjunction of 2 and 16) is complex, whereas its components are shifts, i.e. very simple systems (see Table 1 and Fig. 7).

Comparing the dynamics represented in Fig(s). 7 and 5 (rules 2 and 16 vs f_1 and f_2, rule $18 = 2 \vee 16$ vs $f = f_1 \cup f_2$), we would like to use Thm. 12 again to prove that rule 18 is complex. Though it is possible, it is not straightforward.

Firstly, if γ denotes the complexity of a system as proposed above (see p. 4), with the help of complexity measures (boolean derivatives, Markov approximations, entropy), one proves

$$\gamma(\otimes(f \vee g)) > \gamma(\otimes(f \cup g)) > \gamma(\otimes f \cup \otimes g).$$

Secondly, using Thm. 12, we show that the behavior of the global union $\otimes f \cup \otimes g$ is complex. Hence, the local disjunction $\otimes(f \vee g)$ is also complex.

Thus, *the disjunctive composition of two cellular automata having distinct simple shift-like dynamics has a complex behavior.*

4 Conclusion

In different kinds of classical one- or two-dimensional systems, our compositional approach allows us to recover old results regarding complexity in a clear and effective way. In high-dimensional systems or formal grammars generating symbolic languages, the compositional approach leads to new results; we detect the same emergence of complex behaviors: concurrent attraction to different invariants of the space gives rise to a complex behavior. Structurally, the composed systems are simple but their behavior is in each case very complicated.

In conclusion, let us propose a structural definition of chaos: *complexity arises when two simple systems having compatible but opposite dynamics interact in the same space.* Even very simple systems can generate complex behaviors if these assumptions are present.

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