Irreversible Dynamos in Tori *

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Abstract

We study the dynamics of majority-based distributed systems in presence of permanent faults. In particular, we are interested in the patterns of initial faults which may lead the entire system to a faulty behaviour. Such patterns are called *dynamos* and their properties have been studied in many different contexts. In this paper we investigate dynamos for meshes with different types of toroidal closures. For each topology we establish tight bounds on the number of faulty elements needed for a system break-down, under different majority rules.

Keywords: Distributed Computing, Tori, Majority Rule, Fault Tolerance.

1 Introduction

Consider the following repetitive process on a *synchronous* network G: initially each vertex is in one of two states (colors), *black* or *white*; at each step, all vertices simultaneously (re)color themselves either black or white, each according to the color of the "majority" of its neighbors (majority rule). Different processes occur depending on how majority is defined (e.g., simple, strong, weighted) and on whether or not the neighborhood of a vertex includes that vertex. The problem is to study the initial configurations (assignment of colours) from which, after a finite number of steps, a monochromatic fixed point is reached, that is, all vertices become of the same colour (e.g., black). The initial set of black vertices is called *dynamo* (short for "dynamic monopoly") and their study has been introduced by Peleg [16] as an extension of the study of monopolies.

The dynamics of majority rules have been extensively studied in the context of cellular automata, and much effort has been concentrated on determining the asymptotic

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behaviors of different majority rules on different graph structures. In particular, it has been shown that, if the process is periodic, its period is at most two. Most of the existing research has focused, for example, on the study of the period-two behavior of symmetric weighted majorities on finite $\{0,1\}-$ and $\{0,\ldots,p\}$-colored graphs [8, 18], on the number of fixed points on finite $\{0,1\}$-colored rings [1, 2, 9], on finite and infinite $\{0,1\}$-colored lines [11, 12], on the behaviors of infinite, connected $\{0,1\}$-colored graphs [13]. Furthermore, dynamic majority has been applied to the immune system and to image processing [1, 7].

Although the majority rule has been extensively investigated, not much is known regarding dynamos. Some results are known in terms of *catastrophic fault patterns* which are dynamos based on "one-sided" majority for infinite chordal rings (e.g., [5, 14, 19]). Further results are known in the study of *monopolies*, that is dynamos for which the system converges to all black in a single step [3, 4, 15].

Other more subtle definitions have been posed. An *irreversible* dynamo is one where the initial black vertices do not change their colour regardless of their neighbourhood. This is opposed to *reversible* dynamos, for which a vertex may switch colour several times according to a changing neighborhood. Among the latter,*monotone* reversible dynamos are the ones for which black vertices remain always black because the neighborhood never forces them to turn white. Recently, some general lower and upper bounds on the size of monotone dynamos have been estabilished in [16], and a characterization of irreversible dynamos has been given for chordal rings in [6].

In this paper we consider irreversible dynamos in tori and we focus on their dimension, that is, the minimum number of initial black elements needed to reach the fixed point. The motivation for irreversible dynamos comes from fault-tolerance. Initial black vertices correspond to permanent faulty elements, and the white correspond to non-faulty. The faulty elements can induce a faulty behavior in their neighbors: if the majority of its neighbors is faulty (or has a faulty behavior), a non-faulty element will exhibit a faulty behavior and will therefore be indistinguishable from a faulty one. Irreversible dynamos are precisely those patterns of permanent initial faults whose occurrence leads the entire system to a faulty behavior (or catastrophe). In addition to its practical importance and theoretical interest, the study of irreversible dynamos gives insights on the class of monotone dynamos. In particular, all lower bounds established on the size of irreversible dynamos are immediately lower bounds for the monotone case.

The torus is one of the simplest and most natural way of connecting processors in a network. We consider different types of tori: the *toroidal mesh* (the classical architecture used in VLSI), the *torus cordalis* (also known as double-loop interconnection networks), and the *torus serpentinus* (e.g., used by ILIAC IV).

For each of these topologies we derive lower and upper bounds on the dimensions of dynamos. For a summary of results see table 1. The upper bounds are constructive, that is we derive the initial black vertices constituting the dynamo, and we also analyze the completion time (i.e., the time necessary to reach the fixed point).

Limiting the discussion to meshes with toroidal connections avoids to examine border

	Simple majority		Strong majority	
	Lower Bound	*Upper Bound*	*Lower Bound*	*Upper Bound*
Toroidal mesh	$\lceil \frac{m+n}{2} \rceil - 1$	$\lceil \frac{m+n}{2} \rceil - 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{H}{3} \rceil (K+1)$
Torus cordalis	$\lceil \frac{n}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor + 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{m}{3} \rceil (n+1)$
Torus serpentinus	$\lfloor \frac{N}{2} \rfloor$, for $\lceil \frac{M}{3} \rceil \geq \lfloor \frac{N}{2} \rfloor$ $\lfloor \frac{M}{3} \rfloor$, for $\lceil \frac{M}{3} \rceil < \lfloor \frac{N}{2} \rfloor$	$\lfloor \frac{N}{2} \rfloor + 1$	$\lceil \frac{mn+1}{3} \rceil$	$\lceil \frac{H}{3} \rceil (K+1)$

Table 1: Bounds on the size of irreversible dynamos for tori of $m \times n$ vertices, with $M = max\{m,n\}$, $N = min\{m,n\}$, and $H,K = m,n$ or $H,K = n,m$ (choose the alternative that yields stricter bounds).

effects for some vertices, as it would occur in simple meshes. Our results and techniques can be easily adapted to simple meshes.

2 Basic Definitions

Let us consider an $m \times n$ mesh M and denote with $\mu_{i,j}$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, a vertex of M. The differences among the considered topologies consist only in the way that border vertices (i.e., $\mu_{i,0}, \mu_{i,n-1}$ with $0 \leq i \leq m-1$ and $\mu_{0,j}, \mu_{m-1,j}$ with $0 \leq j \leq n-1$) are linked to other processors. The vertices $\mu_{i,n-1}$ on the last column are usually connected either to the opposite ones on the same rows (i.e. to $\mu_{i,0}$), thus forming ring connections in each row, or to the opposite ones on the successive rows (i.e. to $\mu_{i+1,0}$), in a snake-like way. The same linking strategy is applied for the last row.

In the toroidal mesh rings are formed in rows and columns; in the torus cordalis there are rings in the columns and snake-like connections in the rows; finally, in torus serpentinus there are snake-like connections in rows and columns. Formally, we have:

Definition 1 Toroidal Mesh
A toroidal mesh of $m \times n$ vertices is a mesh where each vertex $\tau_{i,j}$, with $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ is connected to the four vertices $\tau_{(i-1) \bmod m, j}$, $\tau_{(i+1) \bmod m, j}$, $\tau_{i,(j-1) \bmod n}$, $\tau_{i,(j+1) \bmod n}$ (mesh connections).

Definition 2 Torus Cordalis
A torus cordalis of $m \times n$ vertices is a mesh where each vertex $\tau_{i,j}$, with $0 \leq i \leq m-1, 0 \leq j \leq n-1$ has mesh connections except for the last vertex $\tau_{i,n-1}$ of each row i,which is connected to the first vertex $\tau_{(i+1) \bmod m, 0}$ of row $i+1$.

Notice that this torus can be seen as a chordal ring with one chord.

Definition 3 Torus Serpentinus
A torus serpentinus of $m \times n$ *vertices is a mesh where each vertex* $\tau_{i,j}$*, with* $1 \leq i \leq m-1, 0 \leq j \leq n-1$ *has mesh connections, except for the last vertex* $\tau_{i,n-1}$ *of each row* i *which is connected to the first vertex* $\tau_{(i+1) \bmod m, 0}$ *of row* $i+1$*, and for the last vertex* $\tau_{m-1,j}$ *of each column* j *which is connected to the first vertex* $\tau_{0,(j-1) \bmod n}$ *of column* $j-1$*.*

Majority will be defined as follows:

Definition 4 Irreversible-majority rule. *A vertex* v *becomes black if the majority of its neighbours are black. In case of tie* v *becomes black* (simple majority)*, or keeps its color* (strong majority)*.*

In the tori, simple (or strong) irreversible-majority asks for at least two (or three) black neighbours. We can now formally define dynamos:

Definition 5 *A* simple *(respectively:* strong*)* irreversible dynamo *is an initial set of black vertices from which an all black configuration is reached in a finite number of steps under the simple (respectively: strong) irreversible-majority rule.*

Simple and strong majorities will be treated separately because they exhibit different properties and are treated by different techniques.

3 Irreversible Dynamos with Simple Majority

Network behaviour changes drastically if we pass from simple to strong majority. We start our study from the former case.

3.1 Toroidal Mesh

Consider a toroidal mesh with a set T of $m \times n$ vertices, $m, n > 2$. Each vertex has four neighbors, then two black neighbors are enough to color black a white vertex. Let $S \subseteq T$ be a generic subset of vertices, andR_S be the *smallest rectangle* containing S. The size of R_S is $m_S \times n_S$. If S is all black, a *spanning set* for S (if any) is a *connected* black set $\sigma(S) \supseteq S$ derivable from S with consecutive applications of the simple majority rule. We have:

Proposition 1 *Let* S *be a black set,* $m_S < m-1$*,* $n_S < n-1$*. Then, any (non necessarily connected) black set* B *derivable from* S *is such that* $B \subseteq R_S$*.*
Proof No vertex $p \notin R_S$ may become black as a result of black propagation from S. In fact, to become black, p must be adjacent to at least another black vertex $p' \notin R_S$ (and possibly to a black vertex of the boundary of R_S). Since there are no black vertices external to R_S the process cannot start. □

Proposition 2 *Let S be a black set. The existence of a spanning set $\sigma(S)$ implies $|S| \geq \lceil \frac{(m_S+n_S)}{2} \rceil$.*

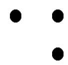
Proof Let S be a set of minimal cardinality such that $\sigma(S)$ exists. By induction on the size of R_S.

Basis We include four cases:

$m_S = n_S = 1$

$m_S = 2, \; n_S = 1$ 

$m_S = 1, \; n_S = 2$

$m_S = n_S = 2$ 

The hypothesis on $|S|$ is verified. Note that in the first three cases $S = \sigma(S)$.

Induction step. $m_S > 2$ and/or $n_S > 2$. First observe that a minimal set S is not connected. In fact, a connected set with $m_S > 2$ and/or $n_S > 2$ should contain a chain of three adjacent black vertices xyz and could be eliminated from S (and then reconstructed from xz in $\sigma(S)$) without preventing the construction of $\sigma(S)$. Consider now an algorithm A that builds $\sigma(S)$ from S. (We do not know A, but speculate on its properties). Starting from (the non connected) S, consider the first operation (a white vertex z turning black) that generates a set $\alpha(S)$, with $\alpha(S) \subseteq \sigma(S)$. We have $\alpha(S) = \sigma(P) \cup \{z\} \cup \sigma(Q)$, where P and Q are disjoint black sets, $P \subset S$, $Q \subset S$; $\sigma(P)$, $\sigma(Q)$ are spanning sets for P and Q, (with $\sigma(P) \subset \sigma(S)$, $\sigma(Q) \subset \sigma(S)$); and z is linked to two black vertices $p \in \sigma(P)$ and $q \in \sigma(Q)$. We have two possibilities:

(i) p, z, q are on the same (horizontal or vertical) straight-line segment;

(ii) the chain p, z, q is bent.

Recall that $\sigma(P) \subseteq R_P$, $\sigma(Q) \subseteq R_Q$. In case (i) we have: $m_P + m_Q \geq m_S + 1, n_P + n_Q \geq n_S - 1$. In case (ii) we have: $m_P + m_Q \geq m_S, n_P + n_Q \geq n_S$. Hence, in both cases:

$$m_P + n_P + m_Q + n_Q \geq m_S + n_S. \tag{1}$$

By the inductive hypothesis we also have:

$$|P| \geq \lceil \frac{(m_P + n_P)}{2} \rceil, |Q| \geq \lceil \frac{(m_Q + n_Q)}{2} \rceil. \tag{2}$$

Combining relations 1, 2, and noting that $|S| \geq |P| + |Q|$, we immediately obtain:

$$|S| \geq \lceil \frac{(m_P + n_P)}{2} \rceil + \lceil \frac{(m_Q + n_Q)}{2} \rceil \geq \lceil \frac{(m_S + n_S)}{2} \rceil. \tag{3}$$

□

From propositions 1 and 2 we immediately derive our first lower bound result:

Theorem 1 *Let S be a simple irreversible dynamo for a toroidal mesh $m \times n$. We have:*
(i) $m_S \geq m-1, n_S \geq n-1$;
(ii) $|S| \geq \lceil \frac{(m+n)}{2} \rceil - 1$.

To build a matching upper bound we need some further results.

Proposition 3 *Let S be a black set, such that a spanning set $\sigma(S)$ exists. Then a black set R_S can be built from S. (note: R_S is then a spanning set of S).*
Proof If $\sigma(S) = R_S$ we are done (this includes the case $m_S = 1$ and/or $n_S = 1$). Otherwise, there must exists a white vertex $p \in R_S - \sigma(S)$ in a concave corner of $\sigma(S)$, that is p has two adjacent black vertices (this is inevitable because $\sigma(S)$ spans R_S). Color p black and iterate on the white vertices. □

Definition 6 *An* alternating chain C *is a sequence of adjacent vertices starting and ending with black. The vertices of* C *are alternating black and white, however, if* C *has even length there is exactly one pair of consecutive black vertices somewhere.*

Theorem 2 *Let S be a black set consisting of the black vertices of an alternating chain C, with $m_S = m-1$ and $n_S = n-1$. Then, the whole torus can be colored black starting from S, that is S is a simple irreversible dynamo.*
Proof By construction, using the following algorithm.

1. Color black the whole chain C, thus obtaining a spanning set $\sigma(S)$.
2. Color black R_S as indicated in proposition 3. The whole torus is now black, except for one row r and one column c.
3. Color black the vertices of r and c that are adjacent to two vertices of R_S. In fact all the vertices of r and c are in this condition, except for the one at their crossing.
4. Color black the last vertex. □

An example of alternating chain of proper length, and the phases of the algorithm, are illustrated in figure 1. From theorem 2 we have:

Corollary 1 *Any $m \times n$ toroidal mesh admits a simple irreversible dynamo S with $|S| = \lceil \frac{(m+n)}{2} \rceil - 1$.*
Proof Let S consist of the black vertices of an alternating chain C placed on a vertical and a horizontal side of the torus, excluding the extremes. That is $|C| = m+n-3$. From theorem 2, S is a dynamo. Refer now to figure 2. If m even and n even, or m odd and n odd, we have $|C|$ odd, hence S contains $\lceil \frac{|C|}{2} \rceil = \lceil (m + \frac{n}{2}) \rceil - 1$ vertices. If m even and n odd, or vice-versa, we have $|C|$ even, hence $|S|$ contains $\frac{|C|}{2} + 1 = \lceil \frac{(m+n)}{2} \rceil - 1$. □

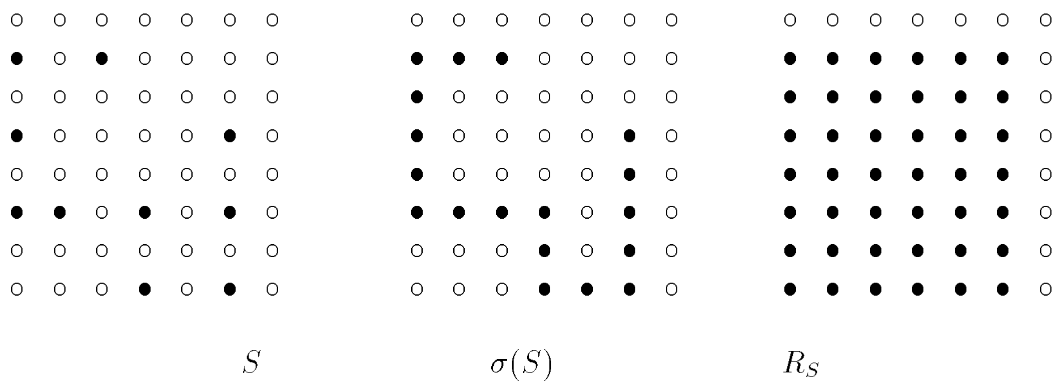


Figure 1: The initial set S, the spanning set $\sigma(S)$ and the smallest rectangle R_S containing S.

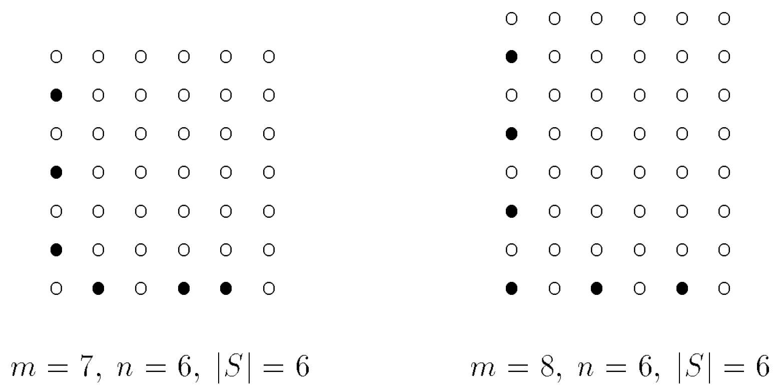


Figure 2: Examples of alternating chains.

Figure 3: Examples of dynamos requiring $\lceil \frac{(m+n)}{2} \rceil$ steps.

Note that the lower bound of theorem 1 matches the upper bound of corollary 1. From the proof of this corollary we see that a dynamo of minimal cardinality can be built on an alternating chain. Furthermore we have:

Corollary 2 *There exists a simple irreversible dynamo S of minimal cardinality starting from which the whole toroidal mesh can be colored black in $\lceil \frac{(m+n)}{2} \rceil$ steps.*

Proof For $m = n$, $|S| = \lceil \frac{(m+n)}{2} \rceil - 1 = m - 1$, let S consist of the vertices of one of the main diagonals, except for one (figure 3, left side). Starting from this configuration, at the first step the two diagonals, adjacent to the main one, become black. The propagation continues on two new diagonals at each step. Therefore in $m-2$ steps R_S is black. Two more steps are needed to color black the still white vertices on the last row and column. Consider now $m > n$ (the case $m < n$ is symmetric. Starting from the vertex in row $\lfloor \frac{(m-n)}{2} \rfloor + 1$ and column 1, place $n-1$ black vertices along the diagonal D heading down and right (figure 3, right side). Starting now from the top vertex of this diagonal place other vertices in column 1 on an alternating chain heading to the top of the mesh, up to row 1. Similarly place other vertices in column $n-1$ on an alternating chain heading down. A black wave propagates from the two sides of D. The number of steps to color black the whole graph equals the number $\lceil (m+n)/2 \rceil - 2$ of diagonals encountered on each side of D, until R_S is made black. Two more steps are needed to complete the border. □

An interesting observation is that the coloring mechanism shown in corollary 2 works also for asynchronous systems. In this case, however, the number of steps looses its significance.

3.2 Torus Cordalis

If studied on a torus cordalis, simple irreversible-majority is quite easy. We now show that any simple irreversible dynamo must have size $\geq \lceil \frac{n}{2} \rceil$, and that there exist dynamos

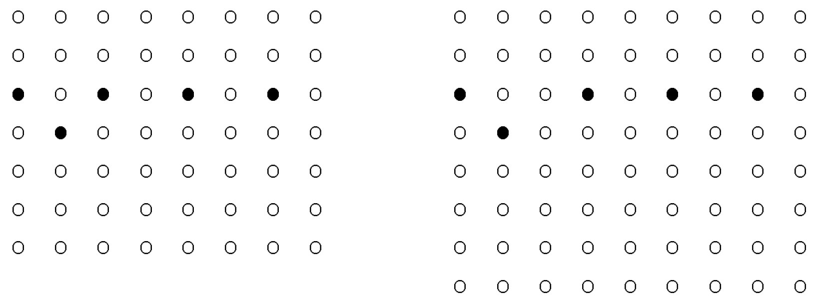


Figure 4: Simple irreversible dynamos of $\lfloor \frac{n}{2} \rfloor + 1$ vertices for tori cordalis, with n even and n odd.

with almost optimal size.

Theorem 3 *Let S be a simple irreversible dynamo for a torus cordalis $m \times n$. We have: $|S| \geq \lceil \frac{n}{2} \rceil$.*
Proof Immediate from the observation that, in a torus cordalis with simple majority, the existence of two consecutive white columns prevents them to be colored black (recall that the first and the last columns are consecutive). $\square$

Theorem 4 *Any $m \times n$ torus cordalis admits a simple irreversible dynamo S with $|S| = \lfloor \frac{n}{2} \rfloor + 1$. Starting from S the whole torus can be colored black in $\lfloor \frac{m-3}{2} \rfloor n + 3$ steps.*
Proof Immediate from a detailed inspection of figure 4. $\square$

3.3 Torus Serpentinus

Since the torus serpentinus is symmetric with respect to rows and columns, we assume without loss of generality that $m \geq n$, and derive lower and upper bounds to the size of any simple irreversible dynamo. For $m \leq n$ simply exchange m with n in the expressions of the bounds.

Consider a *white cross*, that is a set of white vertices arranged as in Figure 5, with height m and width n. The parallel white lines from the square of nine vertices at the center of the cross to the borders of the mesh are called *rays*. Note that each vertex of the cross is adjacent to three other vertices of the same cross, thus implying:

Proposition 4 *In a torus serpentinus the presence of a white cross prevents a simple irreversible dynamo to exist.*

We then have:

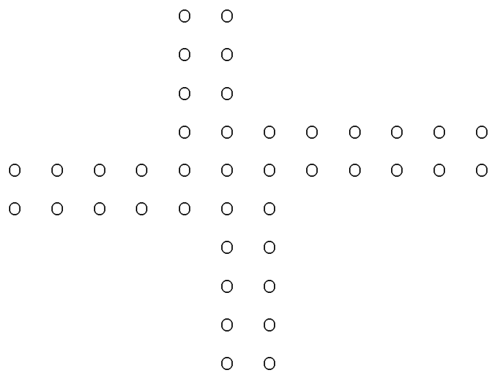


Figure 5: The white cross configuration.

Proposition 5 *If a torus serpentinus contains three consecutive white rows, any simple irreversible dynamo must contain* $\geq \lfloor \frac{n}{2} \rfloor$ *vertices.*

Proof Let r_i, r_{i+1}, r_{i+2} be the three consecutive white rows. Consider the square at the intersection of the first three columns c_0, c_1, c_2 with such rows. For a dynamo to exist, this square cannot be the center of a white cross by proposition 4. Since the square lies on the white rows r_i, r_{i+1}, r_{i+2}, at least one of the potential vertical rays in columns c_0, c_1, or c_1, c_2, must contain a black vertex. The same reasoning applies to all the squares obtained by shifting the given square horizontally, one column at a time. Therefore to prevent that a cross of white vertices appears in the torus, at least one black vertex every two columns must be present. □

Based on proposition 5 we obtain the following lower bounds:

Theorem 5 *Let* S *be a simple irreversible dynamo for a torus serpentinus* $m \times n$. *We have:*

1. $|S| \geq \lfloor \frac{n}{2} \rfloor$, *for* $\lceil \frac{m}{3} \rceil \geq \lfloor \frac{n}{2} \rfloor$;
2. $|S| \geq \lceil \frac{m}{3} \rceil$, *for* $\lceil \frac{m}{3} \rceil < \lfloor \frac{n}{2} \rfloor$.

Proof 1. By contradiction, let $|S| < \lfloor \frac{n}{2} \rfloor$. Since we have $\lfloor \frac{n}{2} \rfloor \leq \lceil \frac{m}{3} \rceil$, there must exist at least three consecutive white rows, thus contradicting proposition 5.
2. By contradiction, let $|S| < \lceil \frac{m}{3} \rceil$. This implies that there are at least three consecutive white rows. The hypothesis $\lceil \frac{m}{3} \rceil < \lfloor \frac{n}{2} \rfloor$ then contradicts proposition 5. □

The upper bound for the torus serpentinus is identical to the one already found for the torus cordalis. In fact we have:

Theorem 6 *Any* $m \times n$ *torus serpentinus admits a simple irreversible dynamo* S *with* $|S| = \lfloor \frac{n}{2} \rfloor + 1$. *Starting from* S *the whole torus can be colored black in* $\lfloor \frac{m-3}{2} \rfloor n + 3$ *steps.*

Proof The configuration reported in figure 4 is a simple irreversible dynamo also for a torus serpentinus. The proof follows as the one of theorem 4. □

4 Irreversible Dynamos with Strong Majority

A strong majority argument allows to derive a significant lower bound valid for the three considered families of tori (simply denoted by tori). Since these tori have a neighbourhood of four, three adjacent vertices are needed to color black a white vertex under strong majority. We have:

Theorem 7 *Let S be a strong irreversible dynamo for a torus $m \times n$. Then $|S| \geq \lceil \frac{mn+1}{3} \rceil$.*

Proof Let T and E be the sets of vertices and edges of the torus; $S \subseteq T$ be the set of black vertices; R be the restriction of the torus to the subset $T - S$ of white vertices. If R contains a cycle, each of its vertices is adjacent to two other vertices of it, hence the cycle can never turn black. That is, if S is a dynamo R is a forest, then its set E_R of edges is such that $|E_R| \leq |T| - |S| - 1$. Note that each edge of the set $E - E_R$ has at least one extreme in S. Since each vertex of T has four neighbours, we have $|E| - |E_R| \leq 4|S|$. We then have:

$|E| = 2mn \leq |T| - |S| - 1 + 4|S| = mn + 3|S| - 1$

and the bound follows. □

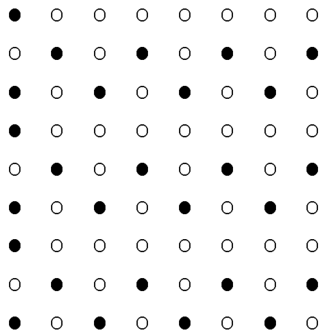
We now derive an upper bound also valid for all tori.

Theorem 8 *Any $m \times n$ torus admits a strong irreversible dynamo S with $|S| = \lceil \frac{m}{3} \rceil (n+1)$. Starting from S the whole torus can be colored black in $\lfloor \frac{n}{2} \rfloor + 1$ steps.*

Proof Let each group of three consecutive rows have the following configuration of colors: in the first row all vertices are white, except for the first one; in the second and in the third row place alternating colors, respectively starting with a white vertex, and a black vertex (see Figure 6). If m is not a multiple of three, make the same configuration without the first row, or the first and the second row. We have $\lceil \frac{m}{3} \rceil (n+1)$ black vertices. It can be easily seen that, after one step, the second row, the third row, and the last column become black. At each consecutive step two new columns, adjacent to the columns already colored black, become also black. Therefore the whole torus is colored black in $\lfloor n/2 \rfloor + 1$ steps. □

For particular values of m and n the bound of theorem 8 can be made stricter for the toroidal mesh and the torus serpentinus. In fact these networks are symmetrical with respect to rows and columns, hence the pattern of black vertices reported in figure 6 can be turned of 90 degrees, still constituting a dynamo. We immediately have:

Corollary 3 *Any $m \times n$ toroidal mesh or torus serpentinus admits a strong irreversible dynamo S with $|S| = max\{\lceil \frac{m}{3} \rceil (n+1), \lceil \frac{n}{3} \rceil (m+1)\}$.*



$$m = 9,\ n = 8,\ |S| = \lceil \frac{m}{3} \rceil (n+1) = 27$$

Figure 6: A strong irreversible dynamo for toroidal mesh, torus cordalis or torus serpentinus.

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