Testing the Quality of Manufactured Balls*

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Abstract

We consider the problem of testing the roundness of a manufactured ball, using the finger probing model of Cole and Yap [3]. When the center of the object is known, a procedure requiring $O(n^2)$ probes and $O(n^2)$ computation time is described. (Here $n = \lfloor 1/q \rfloor$, where q is the quality of the object.) When the center of the object is not known, the procedure requires $O(n^2)$ probes and $O(n^4)$ computation time. We also give lower bounds which show that the number of probes used by these procedures is optimal.

1 Introduction

The field of metrology is concerned with measuring the quality of manufactured objects. A basic task in metrology is that of determining whether a given manufactured object is of acceptable quality. Usually this involves probing the surface of the object using a measuring device such as a coordinate measuring machine to get a set S of sample points, and then verifying, algorithmically, how well S approximates an ideal object.

A special case of this problem is determining whether an object is *round*, or *spherical*. For our purposes, an object I is *good* if there exists two concentric spheres I_{in} and I_{out} of radius $1 - \epsilon$ and $1 + \epsilon$, respectively, such that I_{in} is entirely contained in I and I is entirely contained in I_{out}, and *bad* otherwise. We call the problem of deciding whether an object is good or bad the *roundness classification problem.* See Figure 1 for examples of good and bad objects.

In the field of computational geometry, the algorithmic side of the roundness classification problem has received considerable attention and efficient algorithms for testing the roundness of a set of 2D [1, 4, 5, 6, 8, 9, 11, 12, 13, 14] and 3D [5] sample points are known. However, very little research has been done on probing strategies for the roundness classification problem. Notable exceptions are the work by Mehlhorn, Shermer, and Yap [10], in which planar objects (disks) are considered, Bose and Morin [2], in which disks and cylinders are considered, and Fu and Yap [7] in which a probing strategy for finding the near-center of a d-dimensional ball using $d(d + 1)$ probes is presented.

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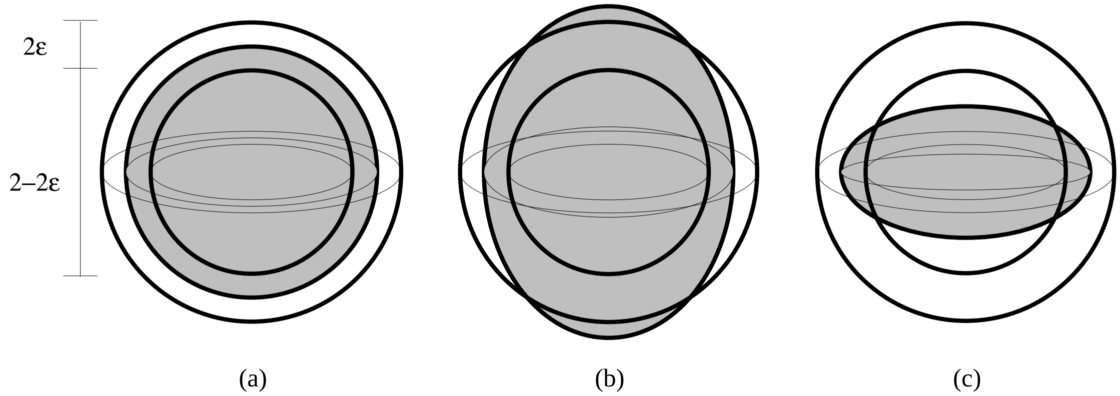

Figure 1: Examples of (a) good and (b,c) bad objects.

In this paper we describe strategies for testing the roundness of manufactured balls. (A ball is a solid object whose surface is a sphere.) We use the *finger probing* model of Cole and Yap [3]. In this model, the measurement device can identify a point in the interior of I and can probe along any ray, i.e., determine the first point on the ray which intersects the boundary of I. The finger probing model is a reasonable abstract model of a coordinate measuring machine [15].

We describe a procedure for testing the roundness of a manufactured ball I using $O(1/\mathrm{qual}(I)^2)$ finger probes. Here $|\mathrm{qual}(I)|$ measures how far the object I is from the boundary between good and bad. When the center of I is known in advance, the procedure requires $O(1/\mathrm{qual}(I)^2)$ computation time. When a center of I is not known, the procedure requires $O(1/\mathrm{qual}(I)^4)$ computation time. As part of this procedure, we describe a technique for finding a near-center of a 3-dimensional ball that requires only 10 probes, which is two less than the number required by the procedure of Fu and Yap [7]. We also give a lower bound which shows that our procedures are optimal, up to constant factors, in terms of the number of probes used.

The remainder of the paper is organized as follows: Section 2 gives definitions and notation used throughout the remainder of the paper. Section 3 describes a procedure for find a point near the center of an object. Section 4 discusses procedures for testing the roundness of of an object. Section 5 gives a lower bound on the number of probes needed for this problem. Section 6 summarizes and suggests directions for future work.

2 Definitions, Notation, and Assumptions

In this section, we introduce definitions and notation used throughout the remainder of this paper, and state the assumptions we make on the object being tested. For the most part, notation and definitions are consistent with, or analagous to, [2, 10].

For a point p, we use the notation $\mathrm{x}(p)$, $\mathrm{y}(p)$, and $\mathrm{z}(p)$ to denote the x, y, and z coordinates of p, respectively. The letter O is used to denote the origin of the coordinate system. We use the notation $\text{dist}(a, b)$ to denote Euclidean distance between two objects. When a and b are not points, $\text{dist}(a, b)$ is the minimum distance between all pairs of points in a and b. The angle formed by three points a, b, and c, is denoted by $\angle abc$, and we always mean the smaller angle unless stated otherwise.

A *sphere* (*ball*) of radius r centered at a point c is the set of all points p such that $\text{dist}(p, c) = r$ ($\text{dist}(p, c) \leq r$). A sphere (ball) of with radius $r = 1$ is called a *unit sphere* (*unit ball*). Two spheres or balls are said to be *concentric* if they are centered at the same point.

An *object* I is defined to be any compact simply connected subset of 3-space, with boundary denoted by $\text{bd}(I)$. For a point p, we use $R(p, I)$ and $r(p, I)$ to denote the maximal and minimal distance, respectively, from p to a point in $\text{bd}(I)$. I.e.,

$$\begin{aligned} R(p, I) &= \max\{\text{dist}(p, p') : p' \in \text{bd}(I)\} && (1) \\ r(p, I) &= \min\{\text{dist}(p, p') : p' \in \text{bd}(I)\} && (2) \end{aligned}$$

For a point p, let

$$\text{qual}(p, I) = \min\{r(p, I) - (1 - \epsilon), (1 + \epsilon) - R(p, I)\} \tag{3}$$

and let

$$\text{qual}(I) = \max_{p \in I} \text{qual}(p, I). \tag{4}$$

Any point c_I with $\text{qual}(c_I, I) = \text{qual}(I)$ is called a *center* of I. Note that there may be more than one point with this property, i.e., the center of I is not necessarily unique.

The value $\text{qual}(I)$ is called the *quality* of the object I, since it measure the maximum deviation of I from a ball of unit radius. An object I with $\text{qual}(I) > 0$ is *good* while an object I with $\text{qual}(I) < 0$ is *bad*. A procedure which determines whether an object is good or bad is called a *roundness classification procedure.*

In order to have a roundness classification procedure which is correct and which terminates, it is necessary to make some assumptions about the object I being tested. The following assumption is referred to as the *minimum quality assumption*, and refers to the fact that the manufacturing process can guarantee that manufactured objects have a minimum quality (although perhaps not enough to satisfy our roundness criteria).

Assumption 1. There exists two concentric balls, I_{in} and I_{out}, with radii $1 - \delta$ and $1 + \delta$, respectively, such that $I_{\text{in}} \subseteq I \subseteq I_{\text{out}}$, for any $\delta < 1/30$.

The minimum quality assumption alone is not sufficient. If the object under consideration contains oddly shaped recesses, then it may be the case that these recesses can not be found using finger probes. We say that an on object I is *star-shaped* if there exists a point $k \in I$ such that for any point $p \in I$, the line segment joining k and p is a subset of I. We call the set of all points with this property the *kernel* of I. There is a region about the center c_I of I which is of particular interest. The following assumption ensures that all points in $\text{bd}(I)$ can be probed by directing probes close to a center of I.

Assumption 2. Let c_I be any center of I. I is a star-shaped object, and its kernel contains all points p such that $\mathrm{dist}(c_I, p) \leq \alpha$, for some constant $1 - \delta > \alpha > 2\delta$.

3 Finding a Near Center

In this section we describe a procedure for finding a point close to the center, c_I, of I . A *near-center* of I is any point c_0 such that $\mathrm{dist}(c_0, c_I) \leq 2\delta$. Our procedure uses three simple subroutines $X(p)$, $Y(p)$ and $Z(p)$. These subroutines perform two probes directed at p. The two probes come from opposite directions, and are parallel to the x, y, and z axes, respectively. If the two probes contact I at points a and b, then the subroutines return $(a+b)/2$, i.e., the midpoint between a and b. If the probes do not contact I then the routines return the point p. Pseudocode is given in Procedure 1.

Procedure 1 Returns a near-center given a point $p_0 \in I$.

$p_1 \leftarrow X(p_0)$
$p_2 \leftarrow Y(p_1)$
$p_3 \leftarrow Z(p_2)$
$p_4 \leftarrow X(p_3)$
$p_5 \leftarrow Y(p_4)$
return p_5

Theorem 1. *Let I be an object with center c_I and satifying Assumption 1. Then 10 probes and constant computation time suffice to find a point c_0 such that* $\mathrm{dist}(c_I, c_0) \leq 2\delta$.

Proof. We will incrementally refine the bounds on the x, y, and z coordinates of p_1–p_5. Refer to Figure 2 for illustrations. When using the terms left, right, up, down, vertical and horizontal, we will do so with respect to the relevant part of Figure 2.

Assume wlog that $c_I = 0$. First we show that

$$|\mathrm{x}(p_1)| \leq 2\sqrt{\delta}. \tag{5}$$

Let L_1 be the line $y = \mathrm{y}(p_0)$, $z = \mathrm{z}(p_0)$, i.e., the line along which the first two probes are taken. Consider the intersection of the plane which contains both L_1 and c_I with I (Figure 2 (a)). We use the notation $\mathrm{y}'(p)$ to denote the distance of the point p from the x-axis. There are two cases to consider.

Case 1: $|\mathrm{y}'(p_0)| \leq 1 - \delta$. Let v_l and v_r be the contact points of the left and right probes, respectively. Since the situation is symmetric about the y'-axis, we bound $|\mathrm{x}(p_1)| = |(\mathrm{x}(v_l) + \mathrm{x}(v_r))/2|$ by maximizing both $\mathrm{x}(v_l)$ and $\mathrm{x}(v_r)$. Using the Pythagorean theorem, we obtain

$$\begin{aligned} \mathrm{x}(v_l) + \mathrm{x}(v_r) &= \left(\mathrm{dist}(v_r, c_I)^2 - \mathrm{y}'(p_0)^2\right)^{\frac{1}{2}} - \left(\mathrm{dist}(v_l, c_I)^2 - \mathrm{y}'(p_0)^2\right)^{\frac{1}{2}} \qquad (6) \\ &\leq \left((1+\delta)^2 - \mathrm{y}'(p_0)^2\right)^{\frac{1}{2}} - \left((1-\delta)^2 - \mathrm{y}'(p_0)^2\right)^{\frac{1}{2}} \qquad (7) \end{aligned}$$

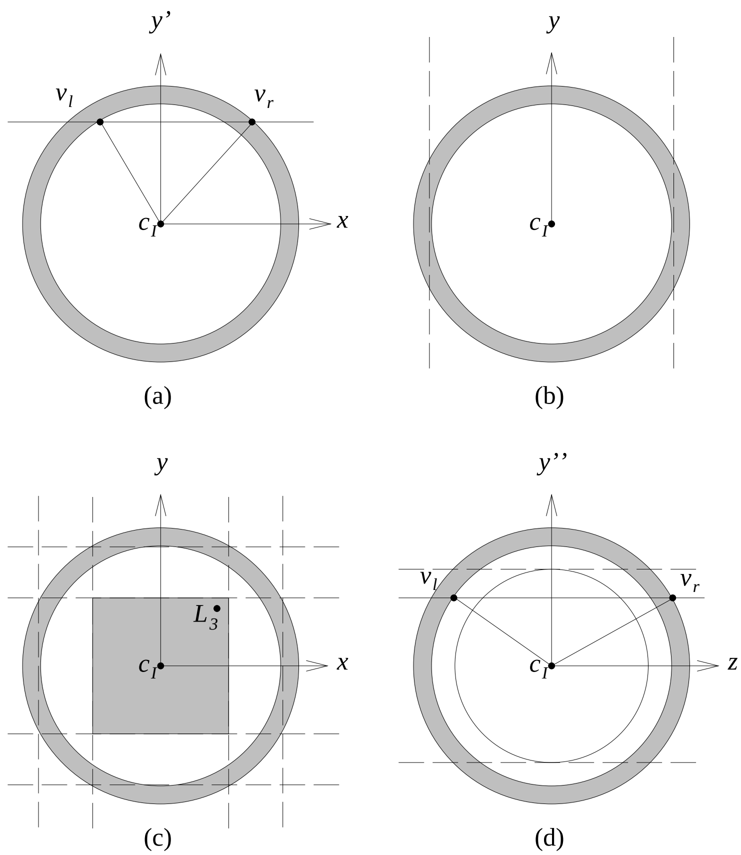

Figure 2: The proof of Theorem 1. The shaded annulus has inner radius $1 - \delta$ and outer radius $1 + \delta$.

$$
\begin{aligned}
&= \frac{(1+\delta)^2 - \mathrm{y}'(p_0)^2 - ((1-\delta)^2 - \mathrm{y}'(p_0)^2)}{((1+\delta)^2 - \mathrm{y}'(p_0)^2)^{\frac{1}{2}} + ((1-\delta)^2 - \mathrm{y}'(p_0)^2)^{\frac{1}{2}}} && (8)\\
&= \frac{4\delta}{((1+\delta)^2 - \mathrm{y}'(p_0)^2)^{\frac{1}{2}} + ((1-\delta)^2 - \mathrm{y}'(p_0)^2)^{\frac{1}{2}}} && (9)\\
&\leq \frac{4\delta}{((1+\delta)^2 - (1-\delta)^2)^{\frac{1}{2}} + ((1-\delta)^2 - (1-\delta)^2)^{\frac{1}{2}}} && (10)\\
&= \frac{4\delta}{\sqrt{4\delta}} && (11)\\
&= 2\sqrt{\delta}. && (12)
\end{aligned}
$$

Therefore, $|(\mathrm{x}(v_l) + \mathrm{x}(v_r))/2| \leq \sqrt{\delta}$, implying (5).

Case 2: $|\mathrm{y}'(p_0)| > 1 - \delta$. In this case, we can define v_l and v_r as above. Note that $|(\mathrm{x}(v_l) + \mathrm{x}(v_r))/2| \leq \max\{|\mathrm{x}(v_l)|, |\mathrm{x}(v_r)|\}$. By the Pythagorean theorem, we have

$$
\begin{aligned}
\max\{|\mathrm{x}(v_l)|, |\mathrm{x}(v_r)|\} &\leq \left((1+\delta)^2 - \mathrm{y}'(p_0)^2\right)^{\frac{1}{2}} && (13)\\
&\leq \left((1+\delta)^2 - (1-\delta)^2\right)^{\frac{1}{2}} && (14)\\
&= 2\sqrt{\delta} && (15)
\end{aligned}
$$

which is the desired result.

Next we show that

$$|\mathrm{y}(p_2)| \leq 2\sqrt{\delta}. \tag{16}$$

There are two cases to consider:

Case 1: The two probes contact I. In this case the same analysis used to show (5) yields the desired result.

Case 2: The two probes do not contact I. Let L_2 be the line defined by $x = \mathrm{x}(p_1), y = \mathrm{y}(p_1)$, i.e., the line along which the second set of probes are taken. Consider the intersection the plane containing L_2 and c_I with I (Figure 2 (b)). The point p_2 cannot be contained in the ball of radius $1 - \delta$ centered at c_I, or else the probes would have contacted I. Therefore the probes must be to the left of the left dashed line or to the right of the right dashed line. Note however that p_1 must be contained in the sphere of radius $1 + \delta$ centered at c_I. Thus, as in Case 2 of (5), the Pythagorean theorem provides the desired bound.

Next we will bound $\mathrm{dist}(c_I, c_0)$ by giving bounds on $|\mathrm{z}(p_3)|$, $|\mathrm{x}(p_4)|$ and $|\mathrm{y}(p_5)|$. Let L_3 be the line defined by $x = \mathrm{x}(p_2), y = \mathrm{y}(p_2)$, i.e., the line along which the third pair of probes is taken. From (5) and (16) it follows that $\mathrm{dist}(L_3, c_I) \leq \sqrt{8\delta}$ (Figure 2 (c)). Note that since $\delta \leq 1/30$, $\sqrt{8\delta} \leq 1 - \delta$, and therefore, by Assumption 1 the third pair of probes must contact the object I.

Now consider the intersection of the plane which contains L_3 and O with the object I (Figure 2 (d)). We will use the notation $\mathrm{y}''(p)$ to denote the distance of the point p from the z-axis. Define $d_3 = \sqrt{8\delta}$, and a computation similar to the one used to obtain (5) (Case 1) yields

$$|\mathrm{z}(p_3)| \leq \frac{2\delta}{((1+\delta)^2 - (d_3)^2)^{\frac{1}{2}} + ((1-\delta)^2 - (d_3)^2)^{\frac{1}{2}}}. \tag{17}$$

By the same argument, define $d_4 = \left(2\sqrt{\delta} + \mathrm{z}(p_3)^2\right)^{\frac{1}{2}}$ and $d_5 = (\mathrm{z}(p_3)^2 + \mathrm{x}(p_4)^2)^{\frac{1}{2}}$, and we obtain

$$|\mathrm{x}(p_4)| \leq \frac{2\delta}{((1+\delta)^2 - (d_4)^2)^{\frac{1}{2}} + ((1-\delta)^2 - (d_4)^2)^{\frac{1}{2}}} \tag{18}$$

$$|\mathrm{y}(p_5)| \leq \frac{2\delta}{((1+\delta)^2 - (d_5)^2)^{\frac{1}{2}} + ((1-\delta)^2 - (d_5)^2)^{\frac{1}{2}}}. \tag{19}$$

Note that $\mathrm{z}(c_0) = \mathrm{z}(p_3)$, $\mathrm{x}(c_0) = \mathrm{x}(p_4)$, and $\mathrm{y}(c_0) = \mathrm{y}(p_5)$, so

$$\mathrm{dist}(c_I, c_0) = \left(\mathrm{x}(p_4)^2 + \mathrm{y}(p_5)^2 + \mathrm{z}(p_3)^2\right)^{\frac{1}{2}}. \tag{20}$$

By expanding (20) and substituting $\delta \leq 1/30$ in the denominators of (17), (18), and (19), it is straightforward, but tedious, to verify that $\mathrm{dist}(c_I, c_0) \leq 2\delta$. (We used Maple.) $\square$

4 Testing Quality

4.1 The Probing Strategy

In this section, we describe a probing strategy for taking $\Theta(n^2)$ probes directed at a point p, where n is an even positive integer. The strategy is designed so that for any direction d, there is a probe which originated "not too far" from d. Refer to Figure 3 for an illustration of what follows.

Consider the spherical coordinates (ϕ, ρ) of the unit sphere centered at p, where angles ϕ and ρ are in the set $[0, 2\pi)$. We first divide the sphere into n parallel *slices*, $s_0, \ldots, s_{n-1}$, such that slice s_i contains all points where $\rho \in [i\pi/n, (i+1)\pi/n]$. Each slice s_i is further subdivided into $m_i = \lceil 2n \max\{\sin(i\pi/n), \sin((i+1)\pi/n)\}\rceil$ similar *pieces*, $s_{i0}, \ldots, s_{i(m_i-1)}$, such that piece s_{ij} contains all points in s_i where $\phi \in [j\pi/m_i, (j+1)\pi/m_i]$. We define the *center* of a piece s_{ij} as the point with spherical coordinates $((2i+1)\pi/2n, (2j+1)\pi/2m_i)$.

Lemma 1. *Let a be any point in s_{ij}, and let b be the center of s_{ij}. Then $\angle apb \leq \pi/n$.*

Proof. We begin by observing that the angle $\angle apb$ is the length of the shortest path between a and b which remains on the surface of the unit sphere centered at p. We will proceed by constructing a two step path with the desired length.

The first step in our path is from a to $(\phi(a), \rho(b))$. By construction of the slice s_i, we have $|\rho(a) - \rho(b)| \leq \pi/2n$. Hence the first step in the path is of length at most $\pi/2n$.

Now note that, since n is even, the slice s_i is the union of circles with radii in the range $[\sin(i\pi/n), \sin((i+1)\pi/n)]$, and that one of these circles, call it c, contains both $(\phi(a), \rho(b))$

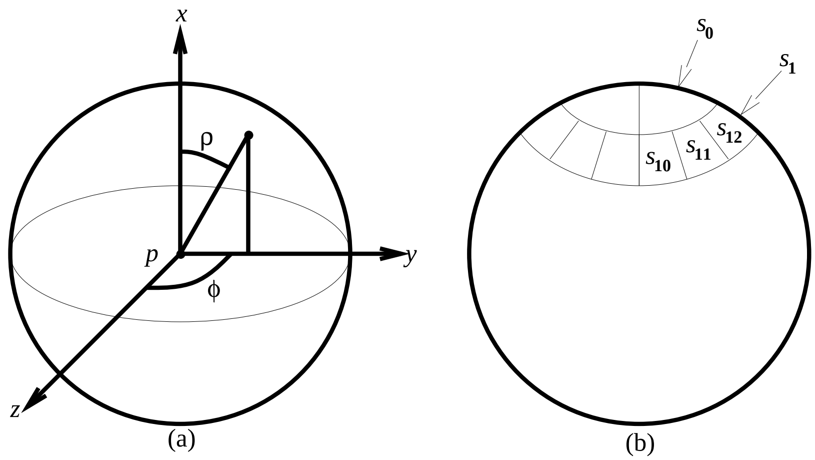

Figure 3: Illustration of (a) sperical coordinates and (b) partioning the sphere into slices and pieces.

and b. The radius $r(c)$ of c is at most $\max\{\sin(i\pi/n, \sin(i+1)\pi/n)\}$, therefore the piece s_{ij} contains an arc of c of length at most

$$2\pi r(c)/m_i \leq \frac{2\pi \max\{\sin(i\pi/n), \sin((i+1)\pi/n)\}}{\lceil 2n \max\{\sin(i\pi/n), \sin((i+1)\pi/n)\}\rceil} \quad (21)$$

$$\leq \pi/n. \quad (22)$$

Since b is in the center of this arc, the distance from $(\phi(a), \rho(b))$ to b on c is at most $\pi/2n$.

Our path from a to b stays on the surface of the unit sphere, and each of the two steps of our path is of length at most $\pi/2n$. This concludes the proof. □

Lemma 2. $\sum_{i=0}^{n-1} m_i \in \Theta(n^2)$, *i.e., the partitioning of the sphere described above contains* $\Theta(n^2)$ *pieces.*

Proof. That the number of pieces is $O(n^2)$ follows from the inequality $\sin(\tau) \leq 1$. That the number of pieces is $\Omega(n^2)$ follows from the inequality $\sin(\tau) \geq 2\tau/\pi$, for $\tau \in [0, \pi/2]$. □

Our probing strategy involves directing probes along each of the half lines with an endpoint at p and passing through the center of each piece. In the remainder of the paper, we will use the notation probe(n, p) to denote the set of probes obtained when using this strategy.

4.2 The Simplified Procedure

In this section we describe a simplified roundness classification procedure which assumes that we know the object being tested is centered at the origin, O. Our roundness classification procedure (Procedure 2) tests the roundness of an object I by taking a set S of probes in the manner described in the previous section. The procedure repeatedly doubles the number of probes until either (1) a set of sample points is found which proves that I is a bad object, in which case I is rejected, or (2) the quality of the set of sample points is "significantly larger" than 0, in which case we can prove that $\mathrm{qual}(I) > 0$.

Procedure 2 Tests the roundness of the object I centered at the origin.

$r \leftarrow 1$
$R \leftarrow 1$
$n \leftarrow n_0$
$\Delta \leftarrow f(n) \in O(1/n)$
repeat
  $S \leftarrow \mathrm{probe}(n, O)$
  if $\exists p \in S : \mathrm{dist}(p, O) > 1 + \epsilon$ or $\mathrm{dist}(p, O) < 1 - \epsilon$ **then**
    return REJECT
  end if
  $r \leftarrow 1 - \epsilon + \Delta$
  $R \leftarrow 1 + \epsilon - \Delta$
  $n \leftarrow 2n$
  $\Delta \leftarrow f(n)$
until $\forall p \in S : \mathrm{dist}(p, O) < R$ and $\mathrm{dist}(p, O) > r$
return ACCEPT

The function $f(n)$ which appears in the procedure is defined as

$$f(n) = \frac{1}{n} \left(\frac{(1+\delta)^2(\alpha^2 + 1 + 2\delta + \delta^2)}{\alpha^2} \right)^{\frac{1}{2}} \tag{23}$$

and the constant n_0 is defined as

$$n_0 = \lceil \pi / \arctan(\alpha/(1+\delta)) \rceil \ . \tag{24}$$

Lemma 3. *Let I be an object with center $c_I = O$. Let $S = \mathrm{probe}(n, c_I)$, for any $n \geq n_0$. Then for any point $p \in \mathrm{bd}(I)$, there exists a point $p' \in S$ such that $\mathrm{dist}(p, p') \leq f(n)$.*

Proof. We will bound $|\mathrm{x}(p) - \mathrm{x}(p')|$, $|\mathrm{y}(p) - \mathrm{y}(p')|$, and $|\mathrm{z}(p) - \mathrm{z}(p')|$. Refer to Figure 4 for an illustration of what follows.

By Lemma 1, there exists a point $p' \in S$ such that

$$\angle p c_I p' \leq \pi/n \ . \tag{25}$$

By orienting the coordinate system so that the plane $z = 0$ passes through p, p' and c_I, we can assume, wlog, that

$$|\mathrm{z}(p) - \mathrm{z}(p')| = 0 \ . \tag{26}$$

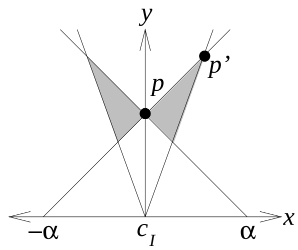

Figure 4: Constraints on the position of p'. The point p' must be in the shaded region, and $\mathrm{dist}(p, p')$ is maximized when p' is placed as shown.

Next note that we can rotate the coordinate system about the z-axis so that we can assume, wlog, that $\mathrm{x}(p) = 0$, and $1 - \delta \leq \mathrm{y}(p) \leq 1 + \delta$. Assumption 1 ensures that $\mathrm{dist}(O, p') \leq 1 + \delta$. So, by (25), an upper bound on $|\mathrm{x}(p) - \mathrm{x}(p')|$ is

$$\begin{aligned} |\mathrm{x}(p) - \mathrm{x}(p')| = |\mathrm{x}(p')| &\leq (1+\delta)\sin(\pi/n) && (27) \\ &\leq (1+\delta)\pi/n \ . && (28) \end{aligned}$$

Since $\angle p c_I p' \leq \pi/n$, the point p' must lie in the cone defined by the inequality

$$\mathrm{y}(p') \geq |\mathrm{x}(p')| \left(\frac{\cos(\pi/n)}{\sin(\pi/n)} \right) . \tag{29}$$

Next we note that the slope of the line through p' and p must be in the range $[-\mathrm{y}(p)/\alpha, \mathrm{y}(p)/\alpha]$, otherwise Assumption 2 is violated. If $n \geq n_0 = \pi / \arctan(\alpha/\mathrm{y}(p))$, the region in which p' can be placed is bounded, and $|\mathrm{y}(p) - \mathrm{y}(p')|$ is maximized when p' lies on one of the bounding lines

$$\begin{aligned} f_l(x) &= x\mathrm{y}(p)/\alpha + \mathrm{y}(p) && (30) \\ f_r(x) &= -x\mathrm{y}(p)/\alpha + \mathrm{y}(p) && (31) \end{aligned}$$

Since both lines are symmetric about $x = 0$ we can assume that $\mathrm{x}(p')$ lies on f_l, giving us

$$\begin{aligned} |\mathrm{y}(p) - \mathrm{y}(p')| &\leq |\mathrm{y}(p) - f_l(\mathrm{x}(p'))| && (32) \\ &= |\mathrm{x}(p')\mathrm{y}(p)/\alpha| && (33) \\ &\leq |\mathrm{x}(p')(1+\delta)/\alpha| && (34) \\ &= (1+\delta)^2\pi/\alpha n && (35) \end{aligned}$$

Plugging (28), (35), and (26) into the Euclidean distance formula and simplifying yields the desired result. □

Theorem 2. *There exists a roundness classification procedure that can correctly classify any object I with center $c_I = O$ and satisfying Assumptions 1 and 2 using $O(1/\mathrm{qual}(I)^2)$ probes and $O(1/\mathrm{qual}(I)^2)$ computation time.*

Proof. We begin by showing that the Procedure 2 is correct. We need to show that the procedure never rejects a good object and never accepts a bad object. The former follows from the fact that the procedure only ever rejects an object when it finds a point on the object's boundary whose distance is less than $1 - epsilon$ or greater than $1 + \epsilon$ from c_I. To show the latter, we note that Lemma 3 implies that there is no point in $\mathrm{bd}(I)$ which is of distance greater than $f(n)$ from all points in S. The procedure only accepts I when the distance of all points in S from c_I are in the range $[1-\epsilon+f(n), 1+\delta-f(n)]$. Therefore, if the procedure accepts I, the distance of all points in $\mathrm{bd}(I)$ from c_I is in the range $[1-\epsilon, 1+\delta]$, i.e., the object is good.

Next we prove that the running time is $O(1/\mathrm{qual}(I)^2)$. First we observe that $f(n) \in O(1/n)$. Next, note that the computation time and number of probes used during each iteration is linear with respect to the value of n, and the value of n doubles after each iteration. Thus, asymptotically, the computation time and number of probes used are dominated by the value of n^2 during the last iteration. There are two cases to consider.

Case 1: Procedure 2 accepts I. In this case, the procedure will certainly terminate once $\Delta \leq \mathrm{qual}(I)$. This takes $O(\log(1/\mathrm{qual}(I)))$ iterations. During the final iteration, $n \in O(1/\mathrm{qual}(I))$.

Case 2: Procedure 2 rejects I. In this case, there is a point on $\mathrm{bd}(I)$ at distance $\mathrm{qual}(I)$ outside the circle with radius $1+\epsilon$ centered at O, or there is a point in $\mathrm{bd}(I)$ at distance $\mathrm{qual}(I)$ inside of the circle with radius $1-\epsilon$ centered at O. In either case, Lemma 3 ensures that the procedure will find a bad point within $O(\log|1/\mathrm{qual}(I)|)$ iterations. During the final iteration, $n \in O(|1/\mathrm{qual}(I)|)$. □

4.3 The Full Procedure

In the more general (and realistic) version of the roundness classification problem, we do not know the center of the object being tested. However, Theorem 1 allows us to use this procedure anyhow. The significance of Theorem 1 is that it allows us to find a near-center, c_0, of I. As the following lemma shows, knowing a near center is almost as useful as knowing the true center. Before we state the lemma, we need the following definitions.

$$f'(n) = \frac{1}{n}\left((1+3\delta)^2\pi^2 + \frac{(1+3\delta)^4\pi^2}{(\alpha-2\delta)^2}\right)^{\frac{1}{2}} \tag{36}$$

$$n'_0 = \lceil \pi/\arctan(\alpha/(1+3\delta)) \rceil \tag{37}$$

Lemma 4. *Let I be an object with center c_I and near center c_0, and satisfying Assumptions 1 and 2. Let $S = \mathrm{probe}(n, c_0)$, for any $n \geq n'_0$, where $\mathrm{dist}(c_0, c_I) \leq 2\delta$. Then for any point $p \in \mathrm{bd}(I)$, there exists a point $p' \in S$ such that $\mathrm{dist}(p, p') \leq f'(n)$.*

Proof. The proof is almost a verbatim translation of the proof of Lemma 3, except that we assume, wlog, that $c_0 = O$. With this assumption we derive the bounds

$$|\mathrm{x}(p) - \mathrm{x}(p')| \leq (1+3\delta)(\pi/n) \tag{38}$$

$$|\mathrm{y}(p) - \mathrm{y}(p')| \leq (1+3\delta)^2\pi/n(\alpha-2\delta)| \tag{39}$$

Substituting these values into the formula for the Euclidean distance and simplifying yields the desired result. ☐

Lemma 5. *Let I be an object with center c_I and near-center c_0 and satisfying Assumptions 1 and 2. Let $S = \mathrm{probe}(n, c_0)$, for any $n \geq n'_0$, and let c_S be a center of S. Then*

$$\begin{array}{rcccl} R(c_S,S) & \leq & R(c_S,I) & \leq & R(c_S,S)+f'(n) \\ r(c_S,S)-f'(n) & \leq & r(c_S,I) & \leq & r(c_S,S) \ . \end{array}$$

Proof. We prove only the bounds on the $R(c_S, I)$ as the proof of the bounds on $r(c_S, I)$ are symmetric. The lower bound on $R(c_S, I)$ is immediate, since $S \subset \mathrm{bd}(I)$. To see the upper bound, choose any point $p \in \mathrm{bd}(I)$ such that $\mathrm{dist}(c_S, p) = R(c_S, I)$. By Lemma 4 there exists $p' \in S$ such that $\mathrm{dist}(p', p) \leq f'(n)$. Therefore $\mathrm{dist}(c_S, p) \leq \mathrm{dist}(c_S, p') + f'(n)$, which implies that $R(c_S, I) \leq R(c_S, S) + f'(n)$. ☐

Lemma 6. *Let I be an object with center c_I and near-center c_0 and satisfying Assumptions 1 and 2. Let $S = \mathrm{probe}(n, c_0)$, for any $n \geq n'_0$. Then* $\mathrm{qual}(S) - f'(n) \leq \mathrm{qual}(I) \leq \mathrm{qual}(S)$

Proof. $\mathrm{qual}(I) \leq \mathrm{qual}(S)$ follows immediately from the fact that $S \subset \mathrm{bd}(I)$. For the lower bound, observe,

$$\begin{aligned} \mathrm{qual}(I) &= \max_{p\in I} \mathrm{qual}(p, I) && (40) \\ &\geq \mathrm{qual}(c_S, I) && (41) \\ &= \min(r(c_S, I) - (1-\epsilon), && (42) \\ &\quad (1+\epsilon) - R(c_S, I)) && (43) \\ &= \min(r(c_S, S) - O(1/n) - (1-\epsilon), && (44) \\ &\quad (1+\epsilon) - (R(c_S, S) + O(1/n)) && (45) \\ &= \min(r(c_S, S) - (1-\epsilon), && (46) \\ &\quad (1+\epsilon) - R(c_S, S)) - O(1/n) && (47) \\ &= \mathrm{qual}(S) - O(1/n). && (48) \end{aligned}$$

☐

In [5], an algorithm is described which determines, given a set S of points in 3-space, whether there exists two concentric closed balls I_{in} and I_{out}, with radii $1-\epsilon$ and $1+\epsilon$, respectively, such that $S \cap I_{\mathrm{in}} = \emptyset$ and $S \cap I_{\mathrm{out}} = S$. The running time of the algorithm is $O(|S|^2)$. Combining Lemma 4 with this algorithm, we obtain the following result.

Theorem 3. *There exists a roundness classification procedure that can correctly classify any object I satisfying Assumptions 1 and 2 using $O(1/\mathrm{qual}(I)^2)$ probes and $O(/\mathrm{qual}(I)^4)$ computation time.*

Proof. We make the following modifications to Procedure 2. In Line 3, we set the value of n to n'_0. In Lines 4 and 13, we replace $f(n)$ with $f'(n)$. In Line 6 we directed our probes at

c_0 rather than O. In Lines 7 and 14, we replace the simple test with a call to the algorithm of [5].

Lemma 6 ensures that the procedure never accepts a bad object and never rejects a good object. i.e., the procedure is correct. The procedure terminates once $f'(n) < |\text{qual}(I)|$. This happens after $O(\log|1/qual(I)|)$ iterations, at which point $n \in O(|1/\text{qual}(I)|$. $\square$

5 Lower Bounds

In this section, we give a lower bound which shows that any correct roundness classification procedure for spheres requires, in the worst case, $\Omega(1/\text{qual}(I)^2)$ probes to determine if I is good or bad. The lower bound uses an adversary argument to show that if a procedure uses $o(1/\text{qual}(I)^2)$ probes, then an adversary can orient a bad object so that its defects are "hidden" from all the probes, making the bad object indistigusihable from a similar good object.

Lemma 7. *Let S be a set of n^2 points on the unit sphere. Then, there exists a spherical cap c with radius $1/n$ such that c contains no points of S*

Proof. We will assume that the points of S do not lie in a single hemisphere, since if they do, then there exists an empty spherical cap of radius 1, and we are done.

Consider the spherical Delaunay triangulation of S, which has $2n^2 - 4$ faces. In total, the faces have surface area 4π. By the pigeonhole principle, some face f must have surface area at least $2\pi/n^2$. The plane which passes through the three vertices of f cuts off a spherical cap c with at least the same surface area, and since this face is part of the Delaunay triangulation, c contains no points of S. The surface area of c obeys the inequality $\text{sa}(c) \leq 2\pi r^2$, where r is the radius of c. Thus, we have the inequalities

$$2\pi r^2 \geq \text{sa}(c) \geq 2\pi/n^2 \quad , \tag{49}$$

yielding $r \geq 1/n$. $\square$

Theorem 4. *Any roundness classification procedure that is always correct requires, in the worst case, $\Omega(|1/\text{qual}(I)^2|)$ probes to classify a object I with center $c_I = O$ and satisfying Assumptions 1 and 2.*

Proof. We prove the theorem by exhibiting two objects I and I' with $\text{qual}(I) = \psi = -\text{qual}(I')$, for any $0 \leq \psi \leq \epsilon$, such that I and I' cannot be distinguished by any algorithm which uses $o(|1/\text{qual}(I)|)$ probes.

The object I is a perfect circle with radius $1 - \epsilon + \psi$. The object I' is similar to I, except that it contains a conic recess of depth 4ψ which removes a circle of diameter $\alpha 8\psi$ from the surface of I (see Figure 5). Note that $\text{qual}(I) = \psi$ and $\text{qual}(I') = -\psi$, and that for $\alpha = 1/9$, $\delta \leq 1/21$, and $\psi \leq \epsilon \leq \delta$, I and I' satisfy Assumptions 1 and 2.

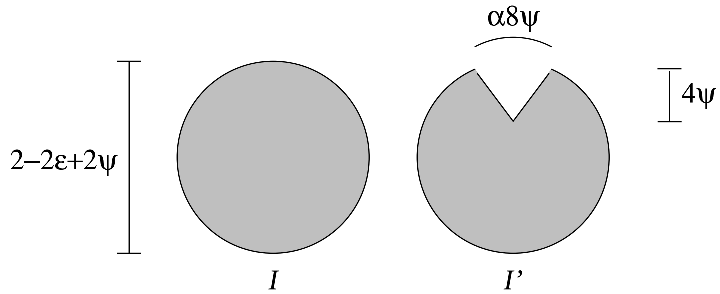

Figure 5: An example of two objects, I and I' which cannot be distinguished using $o(1/\mathrm{qual}(I)^2)$ probes.

Assume by way of contradiction that there exists a roundness classification procedure $\mathcal{P}$ that always accepts I and always rejects I' using $o(1/\psi^2)$ probes. Let S be the set of probes made by $\mathcal{P}$ in classifying I. By Lemma 7, there exists a spherical cap c on the surface of I with radius $\omega(\psi)$ such that c contains no point of S.[1] Therefore, for sufficiently small ψ, c has diameter larger than $\alpha 8\psi$ and an adversary can orient I' so that $\mathcal{P}$ does not direct any probes at the conic recess in I'. The results of probes performed by, and therefore the actions of $\mathcal{P}$, would be the same for I and I'. But this is a contradiction, since we assumed that $\mathcal{P}$ always correctly classifies both I and I'. □

6 Conclusions

We have described the first roundness classification procedure for balls and given lower bounds which show that the procedure is optimal in terms of the number of probes used. In the case when the center of the object is known in advance, the procedure is also optimal in terms of computation time.

When the center of the object is not known, our procedure would benefit significantly from an improved algorithm for testing the roundness of a 3-dimensional point set. The algorithm in [5], which we rely on, solves the problem by first constructing the Voronoi diagram of the point set, which can have quadratic complexity in the worst case. A subquadratic time algorithm is still an important open problem.

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