AN IMPROVED ALGORITHM FOR BOOLEAN MATRIX MULTIPLICATION

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1. INTRODUCTION

In parallel with the analysis of asymptotically fast methods, the research on Boolean matrix multiplication has also focused on the determination of 'efficient' bounds; that is, finding non-optimal but practical algorithms that outperform the asymptotic algorithms for a bounded matrix size or for special classes of matrices. Example of these results are the $O(N^2)$ algorithms for multiplying $N \times N$ sparse or dense Boolean matrices [2], and the $O(N^3/\log N)$ algorithm for multiplying arbitrary $N \times N$ Boolean matrices [1]. The latter algorithm, known as the "Four Russians' Method", unfortunately requires $O(N^3/\log N)$ additional bits to store the $O(N/\log N)$ sets, each containing N rows of N bits each.

In this paper, a new algorithm for Boolean matrix multiplication is presented; this algorithm is based on some properties of the product matrix, and it is shown to require $O(N^3/\log N)$ bit operations (thus, achieving the Four Russian' bound) but only $O(N \log N)$ bits of additional storage.

The paper is organized as follows. In the next section, some properties of the product matrix are discussed; based on these properties, in Section 3, an algorithm is presented which multiplies a $p \times N$ matrix by a $N \times N$ matrix in $O(N^2)$ bit operations using $O(N \log N)$ bits of additional storage where $p \leq [\log_2 N]$; finally, in Section 4, this algorithm is employed to obtain the claimed result. In the following, all logarithms are in base two and (P,Q,R) denotes the problems of multiplying a $P \times Q$ by a $Q \times R$ Boolean matrix.

2. PROPERTIES OF THE PRODUCT MATRIX

Consider the product $C = A \times B$ where A is a $p \times N$ Boolean matrix, $p \leq [\log N]$, and B is a $N \times N$ Boolean matrix.

Consider the sets

$$Z^{i+1}_{2k-1} = \{x \in Z^i_k : A[i+1, x] = 1\} \tag{1a}$$

and

$$Z^{i+1}_{2k} = \{x \in Z^i_k : A[i+1, x] = 0\} \tag{1b}$$

$(0 \leq i < p,\ 1 \leq k \leq 2^i)$, where

$$Z^0_1 = \{1,2,\ldots, N\}.$$

Obviously, $Z^{i+1}_{2k-1} \cap Z^{i+1}_{2k} = \emptyset$ and $Z^{i+1}_{2k} \cup Z^{i+1}_{2k} = Z^i_k$ $(0 \leq i < p,\ 1 \leq k \leq 2^i)$

Example 1 Consider the matrix

$$A = \begin{matrix} 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1 \\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0 \\ 1\ 1\ 0.0\ 1\ 0\ 0\ 1 \end{matrix}$$

In this case, the sets Z^i_k's are as follows

$Z^0_1 = \{1,\ldots,8\}$;

$Z^1_1 = \{2,3,6,8\}$, $Z^1_2 = \{1,4,5,7\}$;

$Z^2_1 = \{2,3\}$, $Z^2_2 = \{6,8\}$, $Z^2_3 = \{4,5\}$, $Z^2_4 = \{1,7\}$;

$Z^3_1 = \{2\}$, $Z^3_2 = \{3\}$, $Z^3_3 = \{8\}$, $Z^3_4 = \{6\}$, $Z^3_5 = \{5\}$, $Z^3_6 = \{4\}$,

$Z^3_7 = \{1\}$, $Z^3_8 = \{7\}$.

Let $f^i = \{1,2,\ldots,N\} \to \{1,2,\ldots,N\}$ $(0 \leq i \leq p)$ be the mapping

$$f^i(x) = k \text{ iff } x \in Z^i_k \tag{2}$$

A bijection $\pi:\{1,2,\ldots,N\} \to \{1,2,\ldots,N\}$ is said to be

<u>i-canonical</u> $(0 \leq i \leq p)$ if for all $x,y \in \{1,2,\ldots,N\}$

$$\pi^{-1}(x) < \pi^{-1}(y) \quad \text{if} \quad f^i(x) < f^i(y) \tag{3}$$

Obviously, the identity function $\mathcal{I}$ is 0-canonical.

Given a bijection $\pi: \{1,\ldots,N\} \to \{1,\ldots,N\}$, let

$$\pi[i,k] = \{\pi^{-1}(x) : x \in Z^i_k\}$$

$$\ell_\pi[i,k] = \min\{a \in \pi[i,k]\}$$

$$h_\pi[i,k] = \max\{a \in \pi[i,k]\}.$$

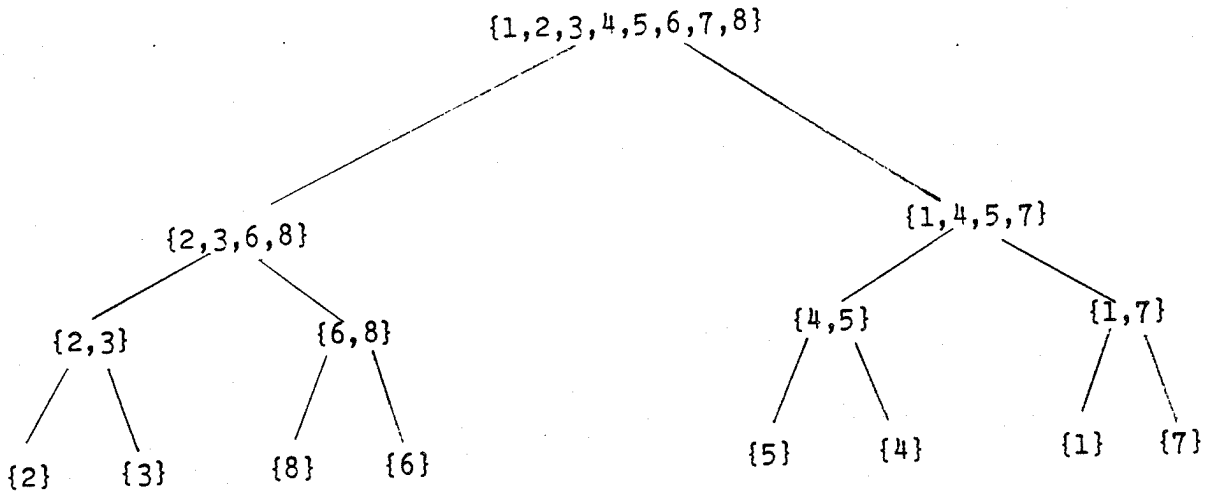
<u>Lemma 1</u> Let π be i-canonical. Then, for all non-empty Z^i_k $(1 \leq k \leq 2^i)$

$$Z^i_k = \{\pi(\ell_\pi[i,k]),\ \pi(\ell_\pi[i,k]+1),\ \ldots,\ \pi(h_\pi[i,k])\}$$

Proof. By definition of i-canonical mapping, each $x \in Z^i_k$ is such that $\pi^{-1}(w) < \pi^{-1}(x) < \pi^{-1}(z)$ for all $w \in Z^i_{k'}$, $z \in Z^i_{k''}$ where $k' < k < k''$. Hence, $\pi[i,k]$ is a sequence of consecutive integers. By definition of $\ell_\pi[i,k]$ and $h_\pi[i,k]$, and since π is a bijection the Lemma follows. [ ]

The sets Z^i_k may conveniently be imagined as lying in a binary tree, each Z^i_k having Z^{i+1}_{2k-1} and Z^{i+1}_{2k} as left and right children. The i-th level of the tree then consists of the sets Z^i_k $(k = 1,2,\ldots,2^i)$ which form a partition (possibly with some empty parts) of $\{1,2,\ldots,N\}$; the function f^i is the characteristic function of the partition. If the nodes at the i-th level, $Z^i_1, Z^i_2, \ldots$, are sets of size $m^i_1, m^i_2, \ldots$, an i-canonical bijection is one that maps the first m^i_1 positive integers onto Z^i_1, the next m^i_2 integers onto Z^i_2, and so on.

In the case of the example above, we have the tree



Given an i-canonical bijection π_i, consider the following construction

Construction 2.1

1. For all $k(1 \leq k \leq 2^i)$, if $Z_k^i \neq \emptyset$

   a) Partition $\pi[i,k]$ into two sets P_k^i and Q_k^i such that

   $P_k^i = \{a \in \pi[i,k] : A[i+1, \pi_i(a)] = 1\}$ and

   $Q_k^i = \{a \in \pi[i,k] : A[i+1, \pi_i(a)] = 0\}$

   b) Define $\psi_k : \pi[i,k] \to \pi[i,k]$ to be a bijection such that for all $a \in P_k^i$, $b \in Q_k^i$ $\psi_k(a) < \psi_k(b)$.

2. Define $\pi_{i+1} : \{1,2,\ldots, N\} \to \{1,2,\ldots, N\}$ to be the bijection defined by

$$\pi_{i+1}^{-1}(x) = \psi_k(\pi_i^{-1}(x)) \quad \underline{\text{iff}} \quad \pi_i^{-1}(x) \in \pi[i,k]$$

Example 2

It is easy to verify that the bijection π_2 defined as follows

$$1 \to 2,\ 2 \to 3,\ 3 \to 6,\ 4 \to 8,\ 5 \to 4,\ 6 \to 5,\ 7 \to 1,\ 8 \to 7$$

is 2-canonical for the matrix of Example 1. The sets $\pi_2[i,k]$ are as follows $\pi[2,1] = \{1,2\}$, $\pi[2,2] = \{3,4\}$, $\pi[2,3] = \{5,6\}$, $\pi[2,4] = \{7,8\}$;

the partitions P_k^i, Q_k^i obtained by applying Construction 2.1 are

$P_1^2 = \{1\}$, $P_2^2 = \{4\}$, $P_3^2 = \{6\}$, $P_4^2 = \{7\}$;

$Q_1^2 = \{2\}$, $Q_2^2 = \{3\}$, $Q_3^2 = \{5\}$, $Q_4^2 = \{8\}$;

and the bijections ψ_k are

$\psi_1 : 1 \to 1,\ 2 \to 2;\ \psi_2: 4 \to 3,\ 3 \to 4;\ \psi_3 \to 6 \to 5,\ 5 \to 6;\ \psi_4: 7 \to 7,\ 8 \to 8$

The bijection π_3 obtained by step 2 of Construction 2.1 is then

$\pi_3: 1 \to 2,\ 2 \to 3,\ 3 \to 8,\ 4 \to 6,\ 5 \to 5,\ 6 \to 4,\ 7 \to 1,\ 8 \to 7.$

Lemma 2 Let π_i be i-canonical, and let π_{i+1} be the bijection obtained by Construction 2.1. Then, π_{i+1} is (i+1)-canonical.

Proof. It is not difficult to see that if π_i is i-canonical, then π_{i+1} is i-canonical. Hence, it sufficies to prove that $\pi_{i+1}^{-1}(x) < \pi_{i+1}^{-1}(y)$ if $f^{i+1}(x) < f^{i+1}(y)$. Let $x \in Z_{k'}^{i+1}$, $y \in Z_{k''}^{i+1}$ with $k' < k''$. Two cases may arise. Case 1 ($k' = k''-1 = 2k-1$). In this case, $\pi_i^{-1}(x) \in P_k^i$ and $\pi_i^{-1}(y) \in Q_k^i$; by Step 1(b) of Construction 3.1 $\pi_{i+1}^{-1}(x) = \psi_k(\pi_i^{-1}(x)) < \psi_k(\pi_i^{-1}(y)) = \pi_{i+1}^{-1}(y)$. Case 2. Let $c' = [k'/2]$ and $c'' = [k''/2]$, obviously $c' < c''$. By definition (1), $x \in Z_{c'}^i$, and $y \in Z_{c''}^i$; since π_{i+1} is i-canonical, then $\pi_{i+1}^{-1}(x) < \pi_{i+1}^{-1}(y)$.[ ]

Let $\phi_k^i(j)$ be the Boolean function ($1 \le i < p$, $1 \le k \le 2^i$, $1 \le j \le N$)

$$\phi_k^i(j) = \begin{cases} \bigvee_{x \in Z_k^i} B[x,j] & \text{if } Z^i \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

where $\vee$ denotes the Boolean OR.

Lemma 3 Let π be p-canonical. Then

$$\phi_k^p(j) = \begin{cases} \bigvee_{s=\ell_\pi[p,k]}^{h_\pi[p,k]} B[\pi(s), j] & \text{if } Z_k^p \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq k \leq 2^i$, $1 \leq j \leq N$.

Proof. By Lemma 1.

Lemma 4 For $1 \leq i \leq p$, $1 \leq k \leq 2^i$, $1 \leq j \leq N$

$$\phi_k^i(j) = (\phi_{2k-1}^{i+1}(j) \vee \phi_{2k}^{i+1}(j))$$

Proof. By observing that $Z_k^i = Z_{2k-1}^{i+1} \nu Z_{2k}^{i+1}$. [ ]

Theorem 1 $C[i,j] = 1$ iff $\bigvee_{r=1}^{2^{i-1}} \phi_{2r-1}^i(j) = 1$ $(1 \leq i \leq p,\ 1 \leq j \leq N)$

Proof. By (1), it follows that $A[i,\ell] = 1$ iff $r \in \{1,\ldots,2^{i-1}\}: \ell \in Z_{2r-1}^i$.

Then, by (4) it follows that

$C[i,j] = 1$ iff $\exists\, \ell \in \{1,\ldots,N : A[i,\ell] = B[\ell,j] = 1$ iff

$\exists\, r \in \{1,\ldots,2^{i-1}\}, \ell \in \{1,\ldots,N\} : \ell \in Z_{2r-1}^i \wedge B[\ell,j] = 1$ iff

$\exists\, r \in \{1,\ldots,2^{i-1}\} : \bigvee_{x \in Z_{2r-1}} B[x,j] = 1$ iff

$\exists\, r \in \{1,\ldots,2^{i-1}\} : \phi_{2r-1}^i(j) = 1$ iff

$\bigvee_{r=1}^{2^{i-1}} \phi_{2r-1}^i(j) = 1$.

3. AN ALGORITHM FOR (p,N,N) WITH $p \leq \lceil \log N \rceil$

Theorem 1 shows that to compute the entries of the j-th column of the product matrix $C = A \times B$ it is sufficient to compute the OR of the Boolean functions $\phi_k^i(j)$'s over the appropriate indices. Lemma 4 gives a method for computing the $\phi_k^i(j)$'s once the $\phi_k^{i+1}(j)$'s have been computed; Lemma 3 shows how to compute the starting values $\phi_k^p(j)$'s once a p-canonical bijection π and the values $h_\pi[p,k]$'s and $\ell_\pi[p,k]$'s are known. Finally, Construction 2.1 provides a method for determining a $(i+1)$-canonical bijection once a i-canonical bijection is known. Observe that, by Lemma 1, each set $\pi_i[i,k]$ is uniquely defined by the values $\ell_{\pi_i}[i,k]$ and $h_{\pi_i}[i,k]$ $(1 \leq k \leq 2^i)$.

The above considerations lead to the following algorithm for computing the product $C = A \times B$.

Algorithm 3.1

Step 1. (Computation of a p-canonical bijection: Initialization)

Set π_o to be the identity function on $\{1,\ldots,N\}$; set $\ell_{\pi_o}[0,1] := 1$, $h_{\pi_o}[0,1] := N$, $i := 1$.

Step 2. (Computation of a p-canonical bijection: Iteration)

a) Compute an i-canonical bijection π_i from π_{i-1} using Construction 2.1. (Note: the set $\pi_i[i-1,k]$ is uniquely defined by the values $\ell_{\pi_{i-1}}[i-1,k]$ and $h_{\pi_{i-1}}[i-1,k]$, $1 \leq k \leq 2^{i-1}$);

b) Compute the values $\ell_{\pi_i}[i,k]$ and $h_{\pi_i}[i,k]$ $(1 \leq k \leq 2^i)$;

c) Set $i := i + 1$; if $i \leq p$, repeat Step 2.

Step 3. (Computation of product matrix: Initialization)

Set j: = 1

Step 4. (Computation of j-th column of product matrix: Initialization)

a) Set i: = p;

b) Compute $\phi_k^p(j)$ $(1 \leq k \leq 2^p)$ using Lemma 3.

Step 5. (Computation of j-th column of product matrix: Iteration)

a) Compute C[i,j] using Theorem 1;

b) Compute $\phi_k^{i-1}(j)$ $(1 \leq k \leq 2^{i-1})$ using Lemma 4;

c) Set i: = i-1; if $i \geq 1$, repeat this step.

Step 6. (Computation of product matrix: Iteration)

Set j: = j+1; if $j \leq N$, goto Step 4.

Theorem 2 Algorithm 3.1 correctly computes the product C = A x B within finite time.

Proof. Finiteness is obvious. Correctness follows from Theorem 1 and Lemmas 2-4.[ ]

Lemma 5 Given an i-canonical bijection π_i, and the values $\ell_{\pi_i}[i,k]$ and $h_{\pi_i}[i,k]$ for each non empty Z_k^i $(1 \leq k \leq 2^i)$, an (i+1)-canonical bijection can be computed by Construction 2.1 using O(N log N) bit operations.

Proof: Each non-empty $\pi[i,k]$ is composed (see Lemma 1) of the consecutive integers between $\ell_{\pi_i}[i,k]$ and $b_{\pi_i}[i,k]$. To construct P_k^i and Q_k^i, it is sufficient to test each entry $A[i+1,\pi_i(x)]$ $(\ell_{\pi_i}[i,k] \leq x \leq h_{\pi_i}[i,k])$; if the entry is zero, then x is added to Q_k^i; otherwise, it is added to P_k^i. Note that since $x \leq h_{\pi_i}[i,k] \leq N$, the addition of x to either P_k^i or Q_k^i would require $\lceil \log N \rceil$ bit operations. Hence, the construction of P_k^i and Q_k^i requires a total of $|\pi_i[i,k]|(\lceil \log N \rceil + 1)$ bit operations; since ϕ_k can be

constructed by simply examining the sets P_k^i and Q_k^i (in that order), and assigning consecutive integers between $\ell_{\pi_i}[i,k]$ and $h_{\pi_i}[i,k]$, an additional $|\pi_i[i,k]|\lceil \log N \rceil$ bit operations suffice. Assuming that testing on whether $\pi_i[i,k]$ is empty can be done in $\lceil \log N \rceil$ bit operations (note: this can be obviously achieved by setting $\ell_{\pi_i}[i,k] = h_{\pi_i}[i,k] = 0$ for empty $\pi_i[i,k]$), the execution of Step 2 requires at most

$$\sum_{k=1}^{2^i} |\pi_i[i,k]|(2\lceil \log N \rceil + 1) + \lceil \log N \rceil$$

bit operations. Since the $\pi_i[i,k]$'s are all disjoint and $\bigcup_k \pi_i[i,k] = \{1,\ldots,N\}$, then $O(N \log N)$ bit operations are required in total by Step 1. Step 2 can be obviously performed in an additional $O(N \log N)$ bit operations; hence, the Lemma holds. [ ]

Lemma 6 The total execution of Step 2 of Algorithm 3.1 requires $O(p\ N \log N)$ bit operations.

Proof: By Lemma 5, the i-th iteration of Step 2(a) requires $O(N \log N)$ bit operations. Since the values $\ell_{\pi_i}[i,k]$ and $h_{\pi_i}[i,k]$ (in Step 2(b)) can be computed from the sets P_k^{i-1} and Q_k^{i-1} in $|\pi_{i-1}[i-1,k]| \log N$ bit operations, and since $\sum_k |\pi_{i-1}[i-1,k]| = N$ it follows that the i-th iteration of Step 2(b) requires $O(N \log N)$ bit operations. Since Step 2(a) and 2(b) are performed P times, the Lemma holds. [ ]

Lemma 7 For a given j, $1 \leq j \leq N$, the total execution of Step 5 of Algorithm 3.1 requires $O(N)$ bit operations.

Proof: The i-th execution of Step 5(a) (for a fixed j) requires at most $2^{i-1}-1$ bit operations (see Theorem 1); Step 5(b) requires 2^{i-1} bit operations, one for each $\phi_k^{i-1}(j)$ (see Lemma 4). Since Steps 5(a) and 5(b) are executed for $i\in\{0,\ldots,p\}$, $\sum_{i=0}^{p} (2^{i-1}-1 + 2^{i-1}) = 2^{p+1}-p-1 = 0(N)$ bit operations are required in total. [ ]

Theorem 3 Algorithm 3.1 computes the product $C = A \times B$ using at most $0(N^2)$ bit operations.

Proof: By Lemmas 6 and 7, and by observing that each execution of Step 4 requires $0(N)$ bit operations (see Lemma 4), and that Steps 4 and 5 are executed N times while Step 2 is executed only once. [ ]

Algorithm 2.1 can be implemented so to employ only $0(N \log N)$ bits of additional storage. The basic ideas of this implementation are the following.

An i-canonical bijection π_i can be obviously stored in an integer array of N elements, where the j-th entry contains $\pi_i(j)$. Since $\pi_{i+1}(j)$ is constructed only afterwards, the storage area for $\pi_i(j)$ can be reused for $\pi_{i+1}(j)$. Since $P_k^i \vee Q_k^i \subseteq \{1,\ldots,N\}$ and $P_k^i \cap Q_k^i = \emptyset$, each of the auxiliary sets P_k^i and Q_k^i can obviously be stored in an integer array of N elements; furthermore, the same storage area can be used to store P_{k+1}^i and Q_{k+1}^i once

ψ_k has been computed. Since the mapping ψ_k is only a 'fragment' of π_{i+1} (see Step 2 of Construction 2.1), it is implicitly contained in π_{i+1}. To store all values $\ell_{\pi_i}[i,k]$ and $h_{\pi_i}[i,k]$, two integer arrays of size $2N$ each suffice (recall, there are $2^{p+1}-1 \leq 2N-1$ possible ℓ's, and the same number of h's). Since all these integers are in $\{1,\ldots,N\}$, $O(N \log N)$ bits of additional storage in total suffice to implement Steps 1 and 2 of Algorithm 2.1.

For a fixed j, $1 \leq j \leq N$, all the $2^{p+1}-1$ values $\phi_k^i(j)$'s can be stored in a Boolean array of $2N$ elements (recall, $2^{p+1}-1 \leq 2N-1$); the same array can obviously be employed for successive j's. Hence, $O(N)$ bits of additional storage in total are sufficient to implement the remaining Steps 3-6 of Algorithm 2.1. Therefore

Theorem 4 Algorithm 2.1 can be implemented so to use at most $O(N \log N)$ bits of additional space.

3. AN ALGORITHM FOR (N,N,N)

Consider now the product of two $N \times N$ Boolean matrices, A and B. Let $m = \lfloor \log N \rfloor$ and $k = \lceil N/m \rceil$. Partition matrix A into matrices $A_1, A_2, \ldots, A_k$, where matrix A_i $(1 \leq i < k)$ consists of rows $m(i-1)+1,\ldots,mi$ of A, and A_k consists of the remaining rows. The product $C_i = A_i \times B$ $(1 \leq i < k)$ is a $m \times N$ matrix which consists of the rows $m(i-1)+1,\ldots,mi$ of the product matrix $C = A \times B$; $C_k = A_k \times B$ consists of the remaining rows. Therefore, to compute C, it sufficies to compute the products $A_i \times B$ $(1 \leq i \leq k)$ using the matrix multiplication algorithm proposed in Section 2. Since $m = \lfloor \log N \rfloor$, the computation of each product will take $O(N^2)$ bit operations; hence $O(N^3/\log N)$ bit operations will be performed in total. The additional storage required when multiplying $A_i \times B$ can obviously be employed in the computation of $A_{i+1} \times B$;

hence, by Theorem 4, $O(N \log N)$ bits of additional storage suffice to compute the product matrix C.

4. CONCLUSIONS

A new algorithm for computing the product of two arbitrary $N \times N$ Boolean matrices has been presented. It has been shown that the proposed algorithm requires $O(N^3/\log N)$ bit operations (thus, achieving the Four Russians' bound) but only $O(N \log N)$ bits of additional storage. It should be pointed out that unlike the Four Russians' Method, the proposed algorithm cannot be directly executed on a vector computer.

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