Complexity of Boolean Routing
(Extended Abstract)

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Abstract

We use the technique of Kolmogorov complexity to obtain lower bounds for boolean routing. For any integers n, d, we construct networks G on n vertices and degree $O(d)$, which require $\Omega(nd \log d/ \log n)$ memory bits per router on $n \log d/ \log n$ routers and hence a total of

$$\Omega(n^2 d \log^2 d/ \log^2 n)$$

memory bits for any full information routing scheme on the graph. The lower bound is tight when $\log d/ \log n$ is constant.
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1 Introduction

Routing schemes can be divided into two categories: partial- and full-information. In the former, each router is supplied with routing information representing only some (not necessarily all) shortest paths, to any other node at some link of the router. Such schemes include interval routing, prefix routing, etc, [10, 11]. In the latter, each link of a router must be supplied with routing information representing all shortest paths to any other node at all applicable links of the router. Such schemes include the recently introduced boolean routing scheme [1].

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Formally, in boolean routing we are looking for the boolean function of minimum complexity such that for any two vertices u, v and any edge e adjacent to u, $f(u, v, e) = 0$ if and only if there is a shortest path from u to v that uses edge e. Boolean routing has been shown [1] to perform extremely well in many standard networks, like rings, tori, hypercubes, product graphs, etc., as well as in some other networks on which other forms of compact routing (like, interval routing) perform poorly.

EXAMPLE **1** *Consider the n-vertex globe graphs described in [7]. It has been shown that for every linear interval routing scheme there exists a vertex which requires at least $\Omega(n^{1/3})$ intervals at one of its links. However, it is easy to give a boolean routing algorithm in these graphs by specifying the coordinates of the desination vertex. This requires a total of $O(\log n)$ bits per processor.*

In general, it is easy to see that boolean routing needs $O(n^2 d)$ memory bits on an arbitrary n vertex graph of maximal degree d. In this paper, we use the technique of Kolmogorov complexity in order to prove lower bounds for boolean routing. More precisely, we prove the following theorem.

THEOREM **2** *For any integers n, d, we construct a network G on n vertices and degree $O(d)$ which requires $\Omega(nd \log d / \log n)$ memory bits per router on $n \log d / \log n$ routers, and hence a total of $\Omega(n^2 d \log^2 d / \log^2 n)$ memory bits for boolean routing.*

Note that the lower bound is tight when $\log d / \log n$ is constant.

1.1 Outline of the paper

Let d, n be given parameters. The number n represents the number of vertices, while the maximal degree of the graph to be constructed will be $O(d)$. The construction is in two parts. First we consider the case of high degree graphs, i.e., $d \geq \sqrt{n}$, and second the case of low degree graphs, i.e., $d \leq \sqrt{n}$.

2 High degree graphs

The graph $G = (V, E)$ is the parallel composition (modulo a Kolmogorov random sequence) of a collection of cliques. Consider a clique on a set S of size n/d. For each $1 \leq a \leq d$ consider the set $S_a = S \times \{a\}$. Thus for each a we have a copy (S_a, E_a) of the clique S. For simplicity we denote the pair (u, a), where $u \in S$ and $1 \leq a \leq d$ by u_a (see Figure 1).

Now select a Kolmogorov random sequence s of length $\frac{n}{d}\frac{d(d-1)}{2} = \frac{n(d-1)}{2}$. Enumerate the sequence s in some canonical manner using triples of indices. I.e., divide up s into segments s^u each of size $\frac{d(d-1)}{2}$. It would be convenient to think of s^u as a $d \times d$ symmetric matrix of bits without its main diagonal. Hence, if a, b are distinct indices in the range $\{1, 2, \ldots, d\}$ then $s^u_{a,b}$ denotes the entry of the matrix in the ath row and bth column.

Now define the edges of the graph as follows. We take all the edges $\bigcup_{1 \leq a \leq d} E_a$ of all the cliques plus the following "parallel" edges for all $u \in S$:

$$\{u_a, u_b\} \text{ is an edge } \Leftrightarrow s^u_{a,b} = 1, \text{ where } 1 \leq a, b \leq d.$$

LEMMA **3** *The maximal degree of the graph G is $\Theta(d)$.*

PROOF The graphs G_a are cliques on $\frac{n}{d}$ vertices. Hence the maximal degree of G is $\frac{n}{d} + d = O(d)$, since $d \geq \sqrt{n}$. ∎

LEMMA **4** *For each $u \in S$ and any a, b there exists a c such that $\{u_a, u_c\}$ and $\{u_b, u_c\}$ are edges of the graph G.*

PROOF (OUTLINE) Let u be an arbitrary but fixed element of S, and let b, c be given. Assume on the contrary that for all c either $\{u_a, u_c\}$ or $\{u_a, u_c\}$ is not an edge of the graph. This implies that for all c, either $s^u_{a,c} \neq s^u_{b,c}$ or $s^u_{a,c} = s^u_{b,c} = 0$. Consider the sequences of bits

$$\{s^u_{a,c} : c \neq a\}, \{s^u_{b,c} : c \neq b\}. \tag{1}$$

Since s is Kolmogorov random so is the concatenation of the two sequences in (1). Now we can encode the concatenation of the two sequences in (1) as follows:

- the sequence $\{s^u_{a,c} : c \neq a\}$,
- the subsequence of $\{s^u_{b,c} : c \neq b\}$ consisting of the bits $s^u_{b,i}$ such that $s^u_{a,i} = 1$, preserving the order they occur in $\{s^u_{b,c} : c \neq b\}$.

By [8][Theorem 2.15, page 131] the number of 1s of a Kolmogorov random sequence of length d is at least $d/2 - \Theta(\sqrt{d})$. Hence the above information encodes a Kolmogorov random sequence, of length $2d$ with $3d/2 + \Theta(\sqrt{d})$ bits, which is a contradiction. ∎

LEMMA **5** *The diameter of the graph G is 3. Moreover, any two nonadjacent vertices can be connected with at least two paths of length 3.*

PROOF (OUTLINE) We prove only the claim on the diameter of the graph. Using this the second part follows easily. Take any two vertices of G, say u_a and v_b. If $u = v$ then it follows easily from 4 that $d(u_a, u_b) \leq 2$. If $a = b$ then u_a, v_b are in the same clique so the Lemma is trivial. Hence without loss of generality we assume that $u \neq v$ and $a \neq b$. If $\{u_a, v_a\}$ is an edge then $u_a v_a v_b$ is a path of length 2 and $d(u_a, v_b) = 2$. Similarly, if u_b, v_b is an edge then $d(u_a, v_b) = 2$. Therefore it remains to consider the case where neither $\{u_a, v_a\}$ nor $\{u_a, v_a\}$ is an edge. In this case arguing as in Lemma 4 and using the incompressibility of Kolmogorov random strings we can find a c such that both $\{u_a, u_c\}$ and $\{v_b, v_c\}$ are edges of the graph. It follows that $u_a u_c v_c v_b$ is a path of length 3 connecting u_a to v_b. ■

2.1 Lower bound

Now we are in a position to prove the main result of this section.

THEOREM **6** *Assume that $d \geq \sqrt{n}$. Any full information shortest path routing scheme on the graph G requires $\Omega(nd)$ memory bits at each router. In particular, any such scheme requires a total of $\Theta(n^2 d)$ memory bits.*

PROOF (OUTLINE) It is enough to show that any shortest path routing scheme requires at least $nd/9$ memory bits at each node. Assume on the contrary there is a permutation, say π, of the vertices and a vertex, say $u_a := \pi(1)$, which requires $\leq nd/9$ bits at this node. By [8][Theorem 2.15, page 131] the vertex u_a has at least $d/2 - \Theta(\sqrt{d})$ links (see Figure 1). Each of these links has a set which contains the vertices that have shortest paths to u_a through the link. Our shortest path routing scheme must be able to reconstruct each of these sets, say $S_1, S_2, \ldots, S_t$. By looking at these sets we can distinguish the vertices adjacent to u_a (by Lemma 5, these are the only vertices which occur in exactly one of these sets). Let u_b be the vertex adjacent to u_a along the given link and let M be the set on u_a's table representing the vertices to which there is a shortest path from u_a through the link. Suppose that v_b is any vertex occuring in M. We can prove the following lemma.

LEMMA **7** *If $\{u_a, u_c\}$ and $\{v_a, v_c\}$ are not edges but $\{v_b, v_c\}$ is an edge of the graph G then $v_c \in M$.*

PROOF of Lemma 7. The situation of the Lemma is depicted in Figure 1. The proof is easy because the conditions of the Lemma imply that any shortest path to v_c from u_a must be of length 3. This completes the proof of Lemma 7. ■

Now we return to the proof of the Theorem. Clearly, Lemma 7 implies that from each vertex $u_b \in M$ we can extract roughly $\frac{1}{8}$th of its neighbors in the set $\{u_c : c \neq b\}$, i.e. (more precisely) a total of at least

$$\frac{d}{8} - \Theta(\sqrt{d}).$$

Using this idea for each link of u_a, it follows that from the routing scheme we can extract a total of at least

$$\frac{nd}{8} - \Theta(n\sqrt{d})$$

vertices. It follows that we can encode the sequence s as follows using

- the permutation π,
- the remaining bits not mentioned above, i.e. a total of at most

$$\frac{nd}{2} - \frac{nd}{8} = \frac{3nd}{8} + \Theta(n\sqrt{d})$$

  bits,
- the information associated with u_a in the assumed routing scheme.

The above information and the argument preceeding it can be used to encode the sequence s with a total of

$$O(n \log n) + \frac{3nd}{8} + O(d\sqrt{d}) + \frac{nd}{9} + O(\log n)$$

bits. However, this is a contradiction since s is a Kolmogorov incompressible sequence of length $n(d-1)/2$. The proof of the theorem is now complete. ■

3 Low degree graphs

Throughout this section we assume that $d = (n/k)^{1/k}$, where $k \geq 2$ is an integer. The general case will handled in the full paper.

As before the graph G consists of the following.

- A set $\{G_a = (V_a, E_a) : 1 \leq a \leq d\}$ of graphs each consisting of n/d vertices. V_a is the set of vertices and E_a the set of links of the graph.
- For each a, a distinguished set S_a which is a subset of V_a of size $\frac{n}{dk}$.

The graphs G_a are identical copies of a certain graph to be constructed in the sequel.

3.1 Constructing the components

We construct G_a as follows. Take a balanced tree T_a of height k and degree d. Such a tree has $\frac{d^k-1}{d-1}$ leaves. To each leaf append a chain of $k-1$ vertices. This gives rise to a new tree G_a whose total number of leaves is again $\frac{d^k-1}{d-1}$. This defines the graph G_a. The total number of vertices is

$$\frac{d^k-1}{d-1} + (k-1)\frac{d^k-1}{d-1} = k\frac{d^k-1}{d-1} \approx \frac{n}{d}.$$

The set S_a consists of the leaves of the tree G_a. It is clear that $|S_a|$ is roughly equal to $\frac{n}{dk}$.

LEMMA **8** *In the graph G_a any two vertices have distance at least $2k$ and at most $4k-2$.*

PROOF (OUTLINE) Immediate from the construction of the graph G_a. ■

3.2 Constructing the graph

We now complete the construction of the graph G with the aid of the Kolmogorov random sequence. In this case we select a Kolmogorov random sequence s of length $\frac{n}{dk}\frac{d(d-1)}{2} = \frac{n(d-1)}{2k}$.

As in section 2, enumerate the sequence s in some canonical manner using triples of indices. I.e., regardless of the size of d, divide up s into segments s^u each of size $\frac{d(d-1)}{2}$. We think of s^u as a $d \times d$ symmetric matrix of bits with its main diagonal missing. Hence, if a, b are distinct indices in the range $\{1, 2, \ldots, d\}$ then $s^u_{a,b}$ denotes the entry of the matrix in the ath row and bth column. Now consider the sets $\{S_a : 1 \leq a \leq d\}$ and define the set E of edges of G as $\bigcup_{1 \leq d \leq d} E_a$ plus the following edges:

$$\{u_a, u_b\} \text{ is an edge } \Leftrightarrow s^u_{a,b} = 1, \text{ where } u \in S, 1 \leq a, b \leq d.$$

Observe above that we use the Kolmogorov random sequence s in order to add new edges in parallel for the vertices in S_a while at the same time the vertices in $V_a \setminus S_a$ are not affected in any way. The following lemmas can be proved. Details will appear in the full paper.

LEMMA **9** *The maximal degree of the graph G is $\Theta(d)$.* ■

LEMMA **10** *For each $u \in S$ and any a, b there exists a c such that $\{u_a, u_c\}$ and $\{u_b, u_c\}$ are edges of the graph G.* ■

An immediate consequence of Lemma 8 is the following result.

LEMMA **11** *The diameter of the graph G is $\leq 8k-2$. In additionj if the vertices u_a and v_b (where $u, v \in S$) are not adjacent then they can be connected with at least two paths of length at most $8k-2$.* ■

3.3 Lower bound

In this section we prove that the graph G requires total memory $\Omega(n^2 d/k^2)$ for boolean routing. We use the notation of Section 2. As in Theorem 6 we can prove the following result.

THEOREM **12** *Any full information shortest path routing scheme on the graph G requires $\Omega(nd/k)$ memory bits at each router. In particular, any such scheme requires a total of $\Omega(n^2 d/k)$ memory bits.*

PROOF (OUTLINE) It is enough to show that any shortest path routing scheme requires at least $nd/9k$ memory bits at each node among $\bigcup_{1 \leq a \leq d} S_a$. Assume on the contrary there is a permutation, say π, of the vertices and a vertex, say $u_a := \pi(1)$, which requires $\leq nd/9k$ bits at this node. By [8][Theorem 2.15, page 131] the vertex u_a has at least $d/2 - \Theta(\sqrt{d})$ links (see Figure 1). Each of these links has a set which contains the vertices that have shortest paths to u_a through the link.

The rest of the proof is similar to the proof of Theorem 6. We use the same notation. The main difference lies in the proof of Lemma 7. The situation is depicted in Figure 1 however now the blocks are not cliques any longer but rather the graphs depicted in Figure 2. The result of Lemma 7 is still valid, however the reason is due to Lemma 8. Indeed, if the vertices $u_a, u_b, u_c, v_a, v_b, v_c$ are as in Lemma 7 then it is immediate from the construction of the component graphs that $d(u_a, u_c) = k$. It follows that $u_c \in M$, as desired.

As before, from each vertex $u_b \in M$ we can extract roughly $\frac{1}{8}$th of its neighbors in the set $\{u_c : c \neq b\}$, i.e. (more precisely) a total of at least

$$\frac{d}{8} - \Theta(\sqrt{d}).$$

Using this idea for each link of u_a, it follows that from the routing scheme we can extract a total of at least

$$\frac{nd}{8k} - \Theta\left(\frac{n}{8k}\sqrt{d}\right)$$

vertices. Hence we can encode the sequence s as follows using

- the permutation π,
- the remaining bits not mentioned above, i.e. a total of at most

$$\frac{nd}{2k} - \frac{nd}{8k} = \frac{3nd}{8k} + \Theta(n\sqrt{d})$$

  bits,
- the information associated with u_a in the assumed routing scheme.

The above information and the argument preceeding it can be used to encode the sequence s with a total of

$$O(n \log n) + \frac{3nd}{8k} + O(d\sqrt{d}) + \frac{nd}{9k} + O(\log n)$$

bits. However, this is a contradiction since s is a Kolmogorov incompressible sequence of length $n(d-1)/2k$. The proof of the theorem is now complete. ∎

Conclusion

The main contribution of the present paper is to obtain optimal worst-case bounds for boolean routing. Some of the interesting remaining problems include: (1) tightening the bounds for bounded degree networks, (2) and studying the case of planar and Cayley networks.

References

[1] M. Flammini, G. Gambosi and S. Salamone, "Boolean Routing", in WDAG'93, SVLNCS Vol. 725, pp. 219 - 233, 1993.

[2] M. Flammini, J. van Leeuwen, and A. Marchetti-Spaccamela, "The Complexity of Interval Routing on Random Graphs", In proceedings of MFCS, SVLNCS, 1995.

[3] G. N. Frederickson and R. Janardan, "Designing Networks with Compact Routing Tables", *Algorithmica*, vol. 3, 1988, pp. 171-190.

[4] G. N. Frederickson and R. Janardan, "Efficient Message Routing in Planar Networks", *SIAM J. Comput.*, vol. 18, no. 4, Aug. 1989, pp. 843-857.

[5] G. N. Frederickson and R. Janardan, "Space-Efficient Message Routing in c-Decomposable Networks", *SIAM J. Comput.*, vol. 19, no. 1, Feb. 1990, pp. 164-181.

[6] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, MA, 1969.

[7] E. Kranakis, D. Krizanc and S. S. Ravi, "On Multiple Linear Interval Routing Schemes", in proceedings of WG'93 (Workshop on Graph Theoretic Concepts in Computer Science), Vol. 790, Springer Verlag LNCS.

[8] M. Li and P. Vitanyi, "Introduction to Kolmogorov Complexity and its Applications" Springer Verlag, 1993.

[9] D. Peleg and E. Upfal "A Tradeoff between Space and Efficiency for Routing Tables", in ACM STOC 1988, pages 43 - 52 (also in Journal of ACM, Vol. 36, pages 510 - 530, 1989).

[10] N. Santoro and R. Khatib, "Labelling and Implicit Routing in Networks", The Computer Journal, vol. 28, no. 1, 1985, pp. 5-8.

[11] J. van Leeuwen and R. B. Tan, "Interval Routing", *The Computer Journal*, vol. 30, no. 4, 1987, pp. 298-307.

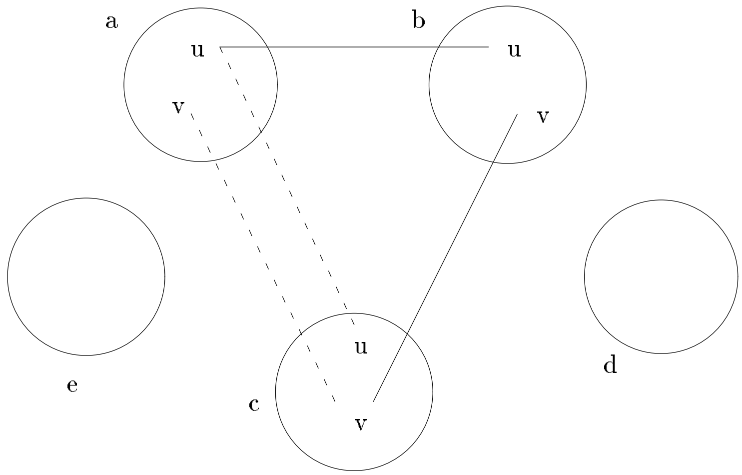

Figure 1: The graph G formed by parallel composition. Solid lines are edges but the dashed lines are not.

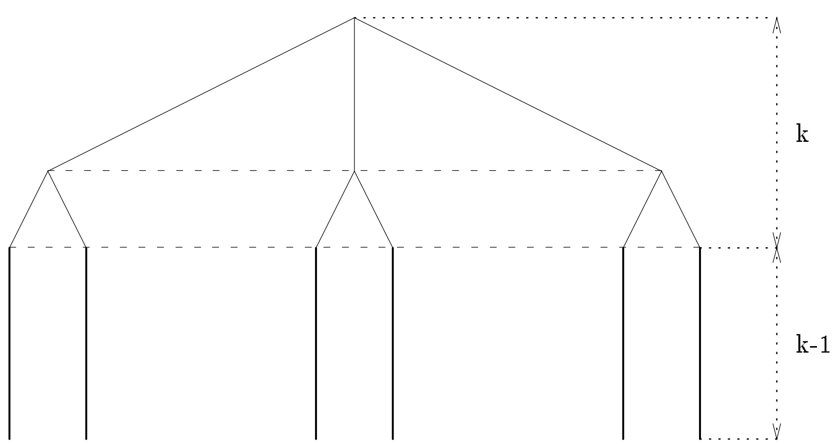

Figure 2: The components G_a of the graph G.