

Characterization of Domino Tilings of Squares with Prescribed Number of Nonoverlapping 2×2 Squares

Evangelos Kranakis^{*†}
(kranakis@scs.carleton.ca)

Abstract

For $k = 1, 2, 3$ we characterize the domino tilings of $n \times n$ squares which have exactly k nonoverlapping 2×2 squares.

1980 Mathematics Subject Classification: 68R05

CR Categories: G.2.1

Key Words and Phrases: Domino tilings, flipping, nonoverlapping, 2×2 squares,

SCS Technical Report: TR-96-24

1 Introduction

Polyominoes ever since their introduction by S. W. Golomb [1] have generated numerous interesting results in combinatorial geometry [2]. Polyominoes are shapes formed of equal sized squares (e.g. unit squares). Dominoes are the simplest nontrivial polyominoes and consist of two unit squares adjacent along an edge.

This paper is concerned with domino tilings of rectangles and squares (also known as checkerboards). Every domino tiling of a nontrivial rectangle has two dominoes forming a 2×2 square of the form $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ or $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$.

Next we are interested in the problem of characterizing domino tilings of $n \times n$ squares with exactly k nonoverlapping 2×2 squares. We give such characterizations when $k < 4$: if a domino tiling of an $n \times n$ square has exactly k nonoverlapping squares then these squares must lie with a bounded subtiling located at the center of the square. In particular, there is a unique (up to rotation) domino tiling of an $n \times n$ square which has exactly one (respectively, two nonoverlapping) 2×2 square(s). However no similar characterization is known when $k = 4$.

^{*}Carleton University, School of Computer Science, Ottawa, Ontario, K1S 5B6, Canada

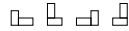
[†]Research supported in part by NSERC (Natural Sciences and Engineering Research Council of Canada) grant.

A related study on polyomino tilings is in [3]. Combinatorial studies on domino tilings can be found in the book [4]. Extensive studies and literature on tilings can be found in [5]. Throughout this paper we assume that the dominoes are located at integer lattice points.

2 Existence of 2×2 Squares

In this section we concentrate on proving that every domino tiling of a rectangle with both sides > 1 must have a (subtiling consisting of a) 2×2 square. All definitions below refer to a domino tiling of a given rectangle.

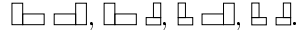
DEFINITION 1 *An L -domino (respectively, R -domino) configuration is a vertical domino with a horizontal domino attached either to its right (respectively, left) or below. The base of an L -domino (respectively, R -domino) configuration is the y -coordinate at the base of the horizontal domino attached to it. Thus in the picture*



the two domino configurations to the left are L -dominoes while the two domino configurations to the right are R -dominoes.

DEFINITION 2 *An (L, R) -domino pair is a pair consisting of an L -domino configuration and an R -domino configuration such that both configurations have identical bases. Moreover the configurations are nonoverlapping and the L -domino configuration lies to the left of the R -domino configuration.*

This gives rise to the following four (L, R) -domino pairs



By rotating the configurations 90, 180, 270 degrees it is clear that we may well define four types of (L, R) -domino pairs. Results below are valid for all four types but for convenience we mention the proofs only for the (L, R) -domino pair corresponding to the horizontal.

DEFINITION 3 *For a given (L, R) -domino pair whose vertical dominoes have x -coordinates $0, a$ respectively, the pyramid polygon is the polygonal line delimited by the horizontal straight line segment determined by the points $(0, 0), (a, 0)$, the points $(\lfloor a/2 \rfloor - 1, \lfloor a/2 \rfloor + 2), (\lfloor a/2 \rfloor + 1, \lfloor a/2 \rfloor + 2)$ and the line segments determined by tracing the points*

$$(0, 0), (0, 3), (1, 3), (1, 4), \dots, (\lfloor a/2 \rfloor - 1, \lfloor a/2 \rfloor + 2)$$

and

$$(a, 0), (a, 3), (a-1, 3), (a-1, 4), \dots, (\lfloor a/2 \rfloor + 1, \lfloor a/2 \rfloor + 2)$$

(see Figure 1.)

The definition above can be easily adapted to any (L, R) -domino pair of the form

$$\begin{array}{c} \square \square \\ \square \square \end{array}, \begin{array}{c} \square \square \\ \square \square \end{array}, \begin{array}{c} \square \square \\ \square \square \end{array}, \begin{array}{c} \square \square \\ \square \square \end{array}.$$

The following Lemma will be very useful for all our subsequent considerations.

LEMMA 4 (The Pyramid Lemma) *Suppose that a domino tiling of a rectangle and an (L, R) -domino pair is given. Then there is a 2×2 square within the pyramid polygon formed by an (L, R) -domino pair.*

PROOF. There are two L -domino and two R -domino configurations. This give rise to four possible (L, R) -domino pairs. It is easy to see that it is enough to consider only one such domino pair, namely $\begin{array}{c} \square \square \\ \square \square \end{array}$. (We leave it to the reader to find the elementary transformations that reduce to this case.)

Starting with the dominoes L, R we search for a 2×2 square in the domino tiling by alternating between left and right side of the pyramid polygon determined by the given (L, R) -domino pair. Look at the “vacant” square position immediately to the right of the vertical domino in the L -domino configuration. If this is occupied by a horizontal domino then the 2×2 square has been found and the proof is complete. Otherwise this square must be occupied by a vertical domino. If the position which is below and to the right of this domino is occupied by a vertical domino then again the 2×2 square has been found and the proof is complete. Otherwise this square must be occupied by a horizontal domino. Next imitate this argument with domino R . Look at the “vacant” square position immediately to the left of the vertical domino of the R -configuration. If this is occupied by a horizontal domino then the 2×2 square has been found and the proof is complete. Otherwise this square must be occupied by a vertical domino. If the position which is below and to the left of this domino is occupied by a vertical domino then again the 2×2 square has been found and the proof is complete. Otherwise this square must be occupied by a horizontal domino. The idea of the previous argument is depicted in Figure

$$\begin{array}{c} \square \square \square \square \\ \square \square \square \square \end{array} \rightarrow \begin{array}{c} \square \square \square \square \\ \square \square \square \square \end{array},$$

which also illustrates the step-by-step formation of the pyramid.

This gives rise to a new (L, R) -domino pair. Now we iterate this procedure by alternating left-to-right forming a puramide like structure within which a 2×2 square is guaranteed to exist. The pyramid structure formed is depicted in Figure 1. The proof of the Lemma is now complete. ■

There are several possible extensions of the proof of the pyramid Lemma. For eexample, starting with any of the configurations $\begin{array}{c} \square \square \\ \square \square \end{array}$ alone we can iterate the above idea of “filling spaces” in order to guarantee the existence of a 2×2 square in the space delimited by the domino configuration and the perimeter of the rectangle. Details of this are left to the reader. An interesting application is the following Theorem.

THEOREM 5 *Domino tilings of rectangles with both sides of length > 1 must always have a 2×2 square.*

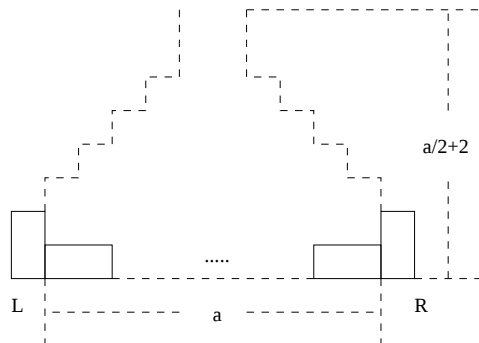


Figure 1: The pyramid of a tiling determined by an (L, R) -domino pair. If a is the distance of the vertical dominoes of the (L, R) -domino pair then a 2×2 square is guaranteed to exist within the pyramid depicted. The height of the pyramid is at most $\lfloor a/2 \rfloor + 2$.

PROOF. Let the given rectangle be of dimension $a \times b$. The proof is by induction on the area ab of the rectangle. The result is easy to prove by inspection if either a or b is equal to 2. Hence we may assume without loss of generality that both $a, b > 2$. A simple area argument shows that either a or b is even. Without loss of generality we may assume the horizontal dimension a is even. If there is no horizontal domino lying along the horizontal touching the perimeter then we can peel off a layer from the rectangle and reduce to a rectangle of dimension $a \times (b - 1)$. Since the resulting rectangle has area $a(b - 1) < ab$, the theorem follows by induction. Hence without loss of generality we may assume there is a side with at least two vertical dominoes. Choose two such adjacent dominoes. Then the dominoes between them and adjacent to the perimeter must be all horizontal. Hence Lemma 4 implies the desired result, and proves the theorem. ■

3 Tilings of Squares

In the sequel we consider domino tilings of $n \times n$ squares which have a given fixed number of 2×2 squares. Our main theorem gives precise characterizations for such tilings when the number of nonoverlapping 2×2 squares is either 1 or 2 or 3. A feature of our result is the existence of a “bounded” (i.e. independent of n) subtiling of the original tiling with the specified number of 2×2 squares. In particular we prove the following theorem.

THEOREM 6 *Let T be a domino tiling of an $n \times n$ square which has exactly k nonoverlapping 2×2 squares. For each $k \leq 2$ there is a subtiling of T located*

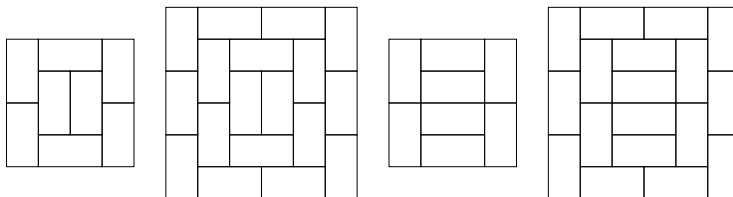


Figure 2: The two leftmost squares depict a 4×4 and a 6×6 tiling with a unique 2×2 square and the two rightmost a 4×4 and a 6×6 tiling with exactly two nonoverlapping 2×2 squares. These figures can be used to generate for each even n an $n \times n$ square with a unique (respectively, exactly two nonoverlapping) 2×2 square(s) by “surrounding” the 4×4 squares with a layer of dominoes, and so on recursively.

at the center of the $n \times n$ square and forming a $2k \times 2k$ square and which has exactly k nonoverlapping 2×2 squares.

Before proceeding to the proof of the theorem we note the following immediate corollaries.

COROLLARY 7 *For each n even there is a unique (up to rotation) domino tiling of an $n \times n$ which has a unique 2×2 square.*

COROLLARY 8 *For each n even there is a unique (up to rotation) domino tiling of an $n \times n$ square which has exactly two nonoverlapping 2×2 squares.*

The two unique domino tilings for the cases $k = 1, 2$ are depicted in Figure 2. For $k = 3$ nonoverlapping 2×2 squares we have.

THEOREM 9 *Let T be a domino tiling of an $n \times n$ square which has exactly 3 nonoverlapping 2×2 squares. Then there is a subtiling of T located at the center of the $n \times n$ square and forming a 8×8 square and which has exactly 3 nonoverlapping 2×2 squares.*

In the sequel we give the proof of the theorems by considering each of the three cases.

Tilings with exactly one 2×2 square

Let us suppose we are given a tiling T which has a unique 2×2 square. Following the details of the proof it will be shown that T must have a certain canonical representation which is unique.

The case $n = 2$ is trivial. So without loss of generality we may assume that n is an even integer > 2 . The tiling covers the entire square. Moreover it is easy to see that there must exist a side which has dominoes adjacent and perpendicular

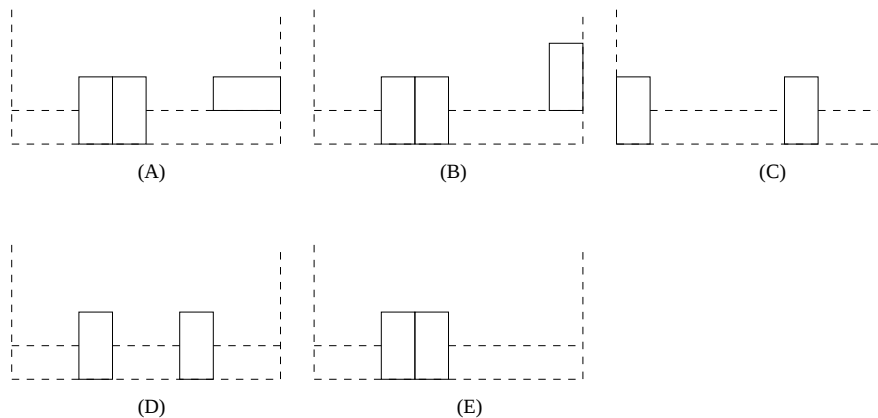


Figure 3: The characterization of domino tilings with one or two nonoverlapping 2×2 squares.

to it. Also n is even, hence the number of such dominoes must also be even. Consider a side of the square with exactly two dominoes perpendicular to it. (See Figure 3, where we are assuming, without loss of generality, that the side under consideration is horizontal.) First of all we prove that these two dominoes cannot be adjacent. If they were then we would have two possibilities as depicted in parts (A) and (B) of Figure 3. All the dominoes, but two, adjacent to this side must be horizontal. Consider the unit square above the rightmost square position. This is covered either by a horizontal domino (resulting in part (A) of the Figure) or by a vertical domino (resulting in part (B) of the Figure). In part (A) this gives rise to two 2×2 squares, which is contradiction. In part (B) we can use the pyramid Lemma to produce a second 2×2 square which gives a contradiction. It follows that the two dominoes cannot be adjacent.

We claim that both vertical dominoes must be adjacent to corners of the square. If not then at least one of the squares is not adjacent. This gives rise to either of the two configurations depicted in parts (C), (D) of Figure 3. However in both cases it is easy to see that by using the pyramid lemma we can show that the tiling must have at least two 2×2 squares, which is a contradiction. It is also straightforward to see that there is no side of the square with more than two vertical dominoes, since in this case the tiling must have more than one 2×2 square.

It follows easily that two opposite sides of the $n \times n$ square have two dominoes each adjacent to their corners while the other two sides have only dominoes which are not perpendicular to them. Consequently, we can peel off a layer of the $n \times n$ square to produce a new $(n - 2) \times (n - 2)$ square which is tiled by the remaining dominoes of the tiling. Iterating this idea and by peeling off dominoes one layer at a time we reduce to a unique 2×2 square. This completes the proof of the theorem in this case. ■

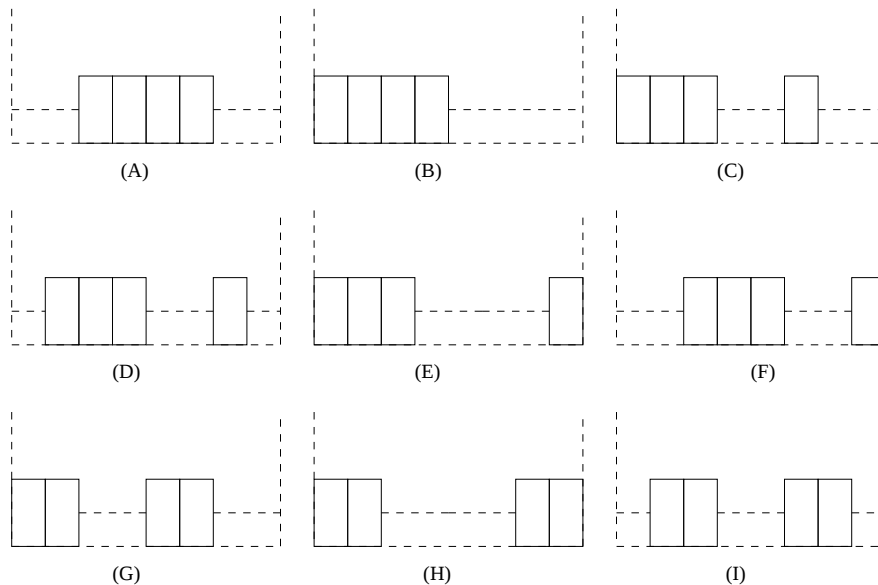


Figure 4: The characterization of domino tilings with exactly two nonoverlapping 2×2 squares.

Tilings with exactly two nonoverlapping 2×2 squares

Let us suppose we are given a tiling T which has exactly two nonoverlapping 2×2 squares. Following the details of the proof it will be shown that T must have a certain canonical representation which is unique.

The proof considers several possible configurations. First of all consider the case where one side (say the horizontal) of the square has four dominoes perpendicular to it. These dominoes may be adjacent in which case they form groups by the number of adjacent dominoes in the group. If they form more than two separate groups then the theorem is immediate from the pyramid lemma. So consider the case they form two groups or less. One case is when one group has all four dominoes (depicted as parts (A) and (B) in Figure 4). Another case is when one group has three dominoes and the other one (depicted as parts (C), (D), (E), (F) in Figure 4). The last case is when there are two groups each consisting of two dominoes (depicted as parts (G), (H), (J) in Figure 4). In all these cases it is straightforward to see using the pyramid lemma that there exist at least three nonoverlapping 2×2 squares in the given tiling.

In particular it follows that no side of the square can have more than two dominoes perpendicular and adjacent to it. This reduces to either of the four configurations depicted in parts (A), (B), (C), (D) of Figure 3. A careful analysis of this shows that the tiling either will have more than two nonoverlapping 2×2 squares or we can peel off a layer of dominoes from the original square; the

resulting square must be a 4×4 square whose domino tiling has exactly two nonoverlapping 2×2 squares. (See Figure 2.) This completes the proof of the theorem in this case. ■

Tilings with exactly three nonoverlapping 2×2 squares

Let us suppose we are given a tiling T which has exactly three nonoverlapping 2×2 squares. Following the details of the proof it will be shown that all the 2×2 of the tiling T must lie within a “central” 8×8 square subtiling of T . The idea is to use the pyramid lemma.

As before, the proof considers several possible configurations. First of all consider the case where one side of the square has four dominoes perpendicular to it. These dominoes may be adjacent in which case they form groups by the number of adjacent dominoes in the group.

If a group consists of four adjacent vertical dominoes (see parts (A), (B) in Figure 4) then it is easy to see that the tiling must have at least four nonoverlapping 2×2 squares unless $n = 4$ in which case there is a unique solution with exactly three nonoverlapping 2×2 squares.

If a group consists of a part with three adjacent vertical dominoes and an isolated one then the situation is depicted in parts (B), (C), (D), (E) of Figure 4. In cases (B), (C), (E) it is easily checked that the tiling must have at least four nonoverlapping 2×2 squares. This leaves only case (D), which is easily seen to have at least four solutions unless the “gap” between the vertical dominoes depicted in part (D) is equal to 2. This gives rise to a set of solutions on a 8×8 subtiling.

If a group consists of two parts each part with two adjacent vertical dominoes then the situation is depicted in parts (G), (H), (I) of Figure 4. In all these cases the tiling is easily seen to have at least four nonoverlapping 2×2 squares. The other group cases are: two parts one of two adjacent vertical dominoes and two isolated or four isolated vertical dominoes and are both easily seen to lead to at least four nonoverlapping 2×2 squares.

Now suppose we have only two dominoes vertical to a side as depicted in part (E) of Figure 3. In this case at least one of the distances of this 2×2 square from the sides of the square must be at least 4 and the other distance at least 2. If one of the distances is exactly 2 we obtain a contradiction to the number of nonoverlapping 2×2 squares. Hence we reduce to the case where both distances are bigger than 2. It follows easily that in this case either we have a subtiling of a 4×4 or a 6×6 square with exactly three nonoverlapping 2×2 squares or else we have at least four nonoverlapping 2×2 squares, which is a contradiction.

Finally this leaves only the case of a group with two vertical isolated dominoes as depicted in part (C) of Figure 3. In this case it is easy to show that if $n \geq 8$ then we can “peel off” an outside layer of dominoes of thickness 1 and reduce to a square of size $n - 2$. This completes the proof of Theorem 6. ■

Some tilings with exactly three nonoverlapping 2×2 squares are depicted in Figure 5.

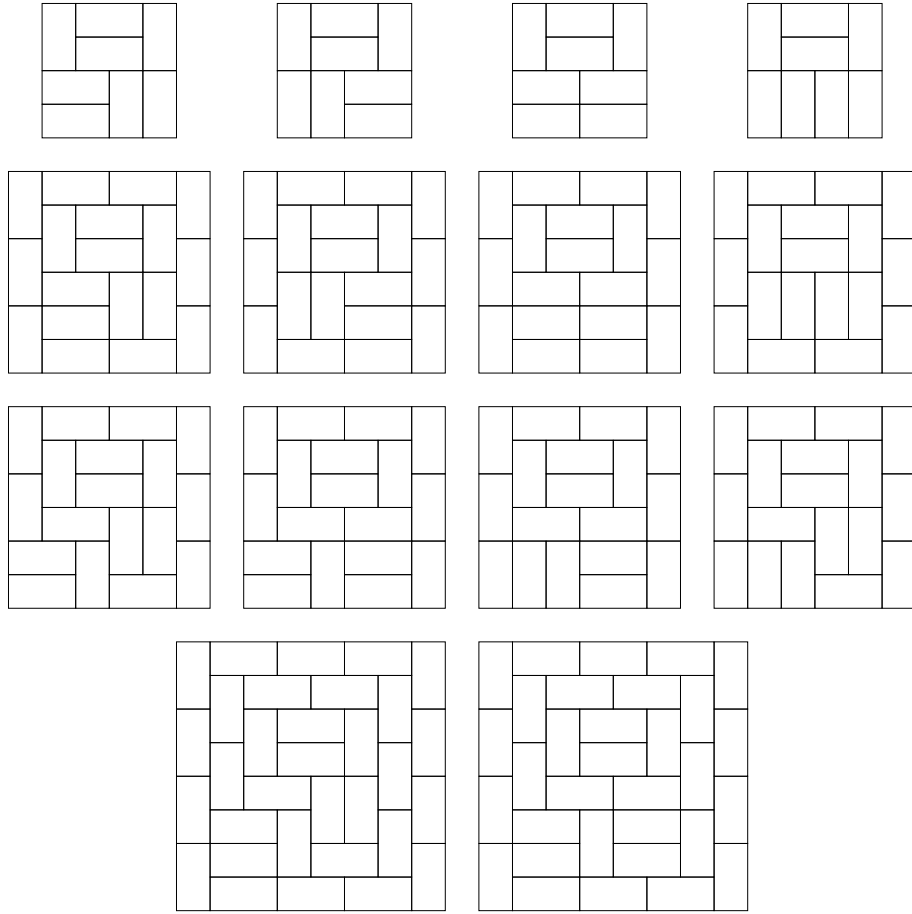


Figure 5: Tilings of squares with exactly three nonoverlapping 2×2 squares. The top four tile 4×4 squares, the next two rows tile 6×6 squares, and the bottom 8×8 squares. More tilings can be obtained from these by 90 degree rotation.

Tilings with exactly four nonoverlapping 2×2 squares

The situation is different for the case of tilings of an $n \times n$ square with exactly four nonoverlapping 2×2 squares, because these need not be located within a bounded subtiling of the original tiling. To see this, for a given n which is divisible by 4 we can construct the following configuration. Take the unique tiling of an $(n/2) \times (n/2)$ square and join four copies of this to form a tiling of $n \times n$ square with exactly four nonoverlapping 2×2 squares.

This argument raises the conjecture that in all such tilings either all four nonoverlapping 2×2 squares lie within a bounded subtiling located at the center or else there is a partition of the $n \times n$ square into four $(n/2) \times (n/2)$ squares each of which has exactly one 2×2 square located at its center. However this is still unproven.

4 Tilings of Rectangles

The following result shows that the number of nonoverlapping 2×2 squares imposes a bound on the size of the rectangle.

THEOREM 10 *Domino tilings of rectangles of dimension $m \times n$, where $1 < m \leq n$, must always have at least $\lfloor n/(m+1) \rfloor$ nonoverlapping 2×2 squares.*

PROOF. We use the pyramid lemma. If there is vertical domino at level $y = 0$ then we search for a 2×2 square starting either with an L - or with an R -domino configuration. Since $m \leq n$ such a 2×2 square must exist within $y \leq m$. If there is no vertical domino at $y = 0$ then we look for a vertical domino at $y = 1$. If such a vertical domino exists then again arguing as before we find a 2×2 square within $y \leq m + 1$. However, if there is no vertical domino either at level 0 or at level 1 then there is a 2×2 square within $y \leq 1$. In either case we can always find a 2×2 square within $y \leq m + 1$. Now we continue this process searching for a 2×2 square within $m + 2 \leq y \leq 2(m + 1)$, and so on. Iterating we obtain the desired result. ■

As a corollary we also obtain that if the rectangle has a domino tiling with at least k nonoverlapping 2×2 squares then $n \leq (k + 1)(m + 1)$.

The lower bound $\lfloor n/(m+1) \rfloor$ is attained as shown by the following example. Tile an $(3k + 1) \times 2$ rectangle iterating k copies of a vertical domino followed by two horizontal adjacent dominoes (i.e. of the form $\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}$) and ending in a vertical domino. The resulting tiling has exactly k nonoverlapping 2×2 squares.

5 Open Problem

An interesting question left open for further investigation concerns the characterization of domino tilings which have a given “polyomino” pattern.

References

- [1] S. W. Golomb, “Checkerboards and Polyominoes”, American Mathematical Monthly LXI, December 1954, 10, pp. 672 - 682.
- [2] S. W. Golomb, “Polyominoes”, Princeton Science Library, Princeton University Press, 1994. (Original edition published by Charles Scribner’s Sons, 1965.
- [3] S. W. Golomb, “Polyominoes which tile rectangles”, Journal of Combinatorial Theory, Series A 51, no 1 (1989), pp. 117 - 124.
- [4] R. L. Graham, D. E. Knuth, and O. Patashnik, “Concrete Mathematics”, Addison-Wesley, 2nd edition, 1994.
- [5] B. Grünbaum and G. C. Shephard, “Tilings and Patterns”, W. H. Freeman and Company, New York, 1987.