THE THEORY AND APPLICATIONS OF UNI-DIMENSIONAL RANDOM RACES WITH PROBABILISTIC HANDICAPS

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THE THEORY AND APPLICATIONS OF UNI-DIMENSIONAL RANDOM RACES WITH PROBABILISITIC HANDICAPS†

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ABSTRACT

We consider the problem of M racers running towards a goal, where, at each instant, racer R_i moves towards the goal with a probability of s_i and stays where he is with a probability of $(1-s_i)$. Additionally, we permit each racer to be granted a certain handicap which allows him to start closer to the goal. This handicap may be stochastically assigned. In the simplest model, the racers run on multiple tracks, and in this scenario, each racer has his own track, thus disallowing interference between the racers. However, in a more general setting, the racers may run on a single track, in which case, interferences between racers are specified in terms of overtaking rules. In this paper, which we believe is a pioneering paper in this area, we first examine random races in a one dimensional space subject to the constraint that the race has multi-tracks and that exactly one racer moves towards the goal at any given instant. A powerful result is derived for the case when the individual racers are given no handicaps at all. Subsequently, a variety of results are proven for the cases when the racers are given handicaps which are either uniformly or geometrically distributed. In each of these cases, the results proved have been obtained for the setting when the length of the track is finite and for the asymptotic condition when the race is arbitrarily long. Analogous results for the single track race are also conjectured and these conjectures are strengthened by numerous simulations.

Keywords : Random Races, Multiple Random Walk, Interacting Markov Chains, Multiple Finite State Markov Chains.

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INTRODUCTION

From the earliest history of Markov Chains, a problem that has been extensively studied involves that of a Random Walk (RW). This problem essentially involves studying the path of a particle in a Multi-Dimensional space, where the particle moves randomly within the space. Humorously enough, the concept of a Random Walk (RW) has been used to model the path of a drunken man. Also, this model has been used extensively to study a host of other problems in Physics, Chemistry, Economics and even the "art" of gambling. The literature on random walks is indeed extensive and we will not even try to survey the literature. But for the sake of completeness, we quote a few helpful references [4,7,11].

In its most elementary setting, the space of the random walk is uni-dimensional. At any time instant, the particle can move forwards, or backwards, or stay where it is, and the length of the step taken may be a random variable with an appropriate distribution. For example, in the uni-dimensional Bernoulli Random Walk (RW) [7,11], the particle can move forwards with probability p or move backwards with probability q.

A RW is said to be recurrent if the probability of a return to the starting point is unity given that an infinite amount of time is available. Alternatively, the RW may have one or more absorbing states, in which case, the walker terminates his motion once he reaches any one of these states. In a physical scenario, a cliff would be an absorbing state for a drunken man, and the condition of bankruptcy will be the absorbing state for the Gambler's Ruin problem. Note that in the above uni-dimensional Bernoulli random walk, the walk is recurrent if and only if $p = q = 1/2$. Additionally, if the walker starts at the origin and the states {+N} and {-N} are absorbing, the mean time to converge to either of these states is MTC, where, (See Ref. [7] pp. 112, Q1)

$$\text{MTC} = \frac{N}{p-q}\left(\frac{1-\left(\frac{q}{p}\right)^N}{1+\left(\frac{q}{p}\right)^N}\right), \qquad \text{if } p \neq q;$$

$$= N^2 \qquad \text{if } p = q = 1/2.$$

Whereas the study of random walks has been quite extensive the question of having particles competing in a "race" has not been studied in the literature. This is the contribution of this paper. In this paper, we shall consider the problem of M particles $\{R_1, R_2, ..., R_M\}$ (which we refer to as racers) each performing a RW. These racers run from a starting point and intend to reach a goal point which is N units away. We shall refer to N as the length of the race (or track), and with no loss of generality, we shall assume that the track starts at the origin. At every time instant, one randomly selected racer is permitted to progress towards the goal by a single step, and at that time the rest of the racers stay where they are. The time invariant probability that racer R_i moves towards the goal is s_i and the probability that he stays where he is $1-s_i$. Note that as a consequence of the constraint that only one racer can move at any one given time instant, the probabilities $\{s_i\}$ are constrained by the equation

$$\sum_{i=1}^{M} s_i = 1. \tag{1}$$

We would like to emphasize that although throughout this paper we have worked with the constraint specified by (1), the more general case in which any number of racers can advance at every time instant remains open and is currently being investigated.

The racer possessing the largest and smallest values of s_i are called the fastest and slowest racers respectively. We shall assume that these racers are unique. Furthermore, we pronounce a particular racer as the winner of the race if he reaches the goal state (or the finishing line) first. This paper, which represents a collection of some of the pioneering results in the area defines the concepts and proves some initial results pertaining to random races.

Random races can be studied from two points of view : namely, from the point of view of handicaps and from the point of view of the number of tracks available for the race. A racer R_i could be given an initial handicap of k_i, where k_i may be a constant, a deterministic variable, or more generally a random variable. In terms of tracks, in a multi-track scenario, each racer has his own track, and so the racers do not interfere with each other. Thus two racers R_i and R_j can both be simultaneously at any position. As opposed to this, in the single-track scenario, only one racer is allowed to be at a particular position at any given time. Consequently, if racer R_i, was just behind racer R_j, and wanted to occupy the position R_j was occupying, then, a passing rule will have to be specified so as to explicitly state how this is achieved. One possible passing (or overtaking) rule is the transposition rule in which R_i would take R_j's place and R_j will move backwards by one unit (See [9], pp. 144-146). Throughout this study, this rule will be the passing rule which we shall employ.

It should be noted that strictly speaking, some of the asymptotic results derived in this paper may be derived by the application of certain central limit theorems. However, the strength of our results does not lie in the fact that they present the asymptotic properties of random races. Indeed, the results that are derived in the paper consider the cases when the tracks are of finite length, and also the special cases when the tracks become arbitrarily long.

I.1 Applications of Random Races

Before we embark on proving the involved mathematical results in the paper it is not inappropriate to state some physical applications of random races. Consider an operating system which is working in a multiple user environment in which M processes have to be executed. Each process has a priority and the operating system assigns a time slot to a particular process based on these priorities. Furthermore, let us assume that the process R_i is assigned the CPU with a probability s_i, where s_i is a monotonic function of the priority associated with the job R_i. For the sake of explanation let us assume that each of the processes requires N time slots of CPU time. The question of job completion is now easily translated into the solution of a random race problem as follows : If s_i is the probability that the process R_i is assigned the CPU, what is the total probability that process R_j would be the first process to be completed. Apart from answering the latter question, the results presented in this paper would also yield various techniques for the systems analyst to forecast various statistics such as the mean time to complete the jobs. Notice that in this scenario, the assumption that the sum of the s_i's is unity is not an unreasonable assumption.

Another example can be found in the area of economic modeling. Suppose that there are M competitive enterprises whose stocks are in the market. Let s_i be the probability that the price of the stock of the enterprise R_i rises by one (discrete monetary) unit. Then the problem of analyzing this economic scenario reduces to the problem of studying the equivalent random race in which the winner of the race is the enterprise whose stock has a value of N. Notice that if N is large enough, this could represent the monopoly of the particular winner in the market place. Of course, in this case the assumption specified by (1) is clearly unrealistic.

A third application of random races is found in the area of robotics. Consider the case when a multi-jointed robot manipulator has to move from a point **A** to a point **B**, where both **A** and **B** are vectors specified in cartesian coordinates [3]. In a joint interpolated motion strategy the joints of the robots are made to move from the initial joint space configuration to the final joint space configuration. In the case when the various joints of the robots have separate actuators the joints can be assigned the tasks of starting and stopping together, and this is usually done using what is called coordinated joint interpolated motion. However, if a single actuator is used to control the motion of more than one joint, the same actuator may drive the joints in succession. Although this motion is easily defined it can be jerky (i.e., not smooth) since the various joints reach their respective final goal positions one at a time. An alternate strategy is to permit each joint to be moved in a number of discrete steps, and to permit a single joint to be actuated at any given time instant. Although the motion strategy can be defined to be deterministic, in certain cases a stochastic scheduling strategy has been shown to be advantageous [8]. Let us assume that a joint J_i moves in the appropriate direction of motion with a probability of s_i. The problem of studying the motion of the robot is now reduced to studying the analogous random race, and questions such as those involving the time taken for the joints to finish their respective courses can be studied using the results presented in this paper.

II RANDOM RACES ON MULTIPLE TRACKS WITHOUT HANDICAPS

In this section we shall study the problem of random racers running on a multiple track. We shall first consider the case in which all the racers start simultaneously at the origin. In the next section we shall then study the cases where the racers are given handicaps which are uniformly or geometrically distributed.

The problem is defined explicitly as follows. Let $\{R_1, R_2, ..., R_M\}$ be a set of M particles whose positions at time 'n' are $\{x_1(n), x_2(n), ..., x_M(n)\}$ respectively. Since the racers are not given any handicaps, the following initial conditions hold for $1 \le i \le M$,

$$x_i(0) = 0. \qquad (2)$$

At any time instant 'n', a particular index j is chosen based on a time invariant probability vector **s**, where $\mathbf{s} = [s_1, s_2, ..., s_M]^T$ and at this time instant $x_j(n)$ is incremented. The vector **s** is called the vector of motion probabilities. The question is now one of computing the probability of racer R_i winning the race if the track is of length N. The following results are true in this scenario.

Theorem I

For any distinct indices u, v ∈ {1, 2, ..., M}, let $\xi_{u,v}$ be the event that either R_u or R_v is advanced. Further, let ${}_uP_v$ be the **conditional** probability that R_u ultimately wins the race over R_v given $\xi_{u,v}$. Then, ${}_uP_v$ increases monotonically with N, the length of the track, and furthermore,

(i) ${}_uP_v$ approaches unity as $N \to \infty$, if $s_u > s_v$;

(ii) ${}_uP_v$ is exactly 0.5 if $s_u = s_v$ even in the limit when $N \to \infty$.

Proof :

Let $p = s_u / (s_u + s_v)$, and $q = s_v / (s_u + s_v)$. Clearly, p is the (conditional) probability of advancing R_u given $\xi_{u,v}$. Similarly, q is the (conditional) probability of advancing R_v given $\xi_{u,v}$, and hence p+q=1

For the rest of this proof we shall assume that all the probabilities and expressions are quantities conditioned on $\xi_{u,v}$.

Let $\langle x_u(n), x_v(n)\rangle$ refer to the positions of R_u and R_v at the n^{th} time instant respectively conditioned on $\xi_{u,v}$. For the sake of simplicity of notation, in the latter pair, we shall not specify the time instant 'n', and hence, unless explicitly stated, the pair $\langle x_u, x_v\rangle$ will refer to the pair $\langle x_u(n), x_v(n)\rangle$.

Clearly, at the n^{th} time instant if $\langle x_u, x_v\rangle$ equals $\langle N, j\rangle$ for $j < N$ then, since R_u has been incremented N times, it will be the winner of the race. Conversely, if $\langle x_u, x_v\rangle = \langle i, N\rangle$ for $i < N$, then R_v is the winner. Thus, the count on the number of increments on x_u and x_v conditioned on $\xi_{u,v}$ obeys the following Markov chain $\mathfrak{M} = (\Phi, F)$, where,

(i) $\Phi = \{0, 1, \ldots, N\} \times \{0, 1, \ldots, N\} - \{\langle N, N\rangle\}$ is the set of states such that $\langle x_u, x_v\rangle \in \Phi$.

(ii) F is the transition function specifying the probability of moving from one state $\langle i, j\rangle$ to another $\langle k, m\rangle$. This transition function F (which is shown schematically in Figure I) can be seen to obey :

$$F_{\langle i,j\rangle,\langle k,m\rangle} = \begin{cases} p & \text{if } k = i+1 \leq N;\ m = j < N \\ q & \text{if } k = i < N;\ m = j+1 \leq N \\ 0 & \text{otherwise} \end{cases} \qquad (3a)$$

Furthermore,

$$F_{\langle i,j\rangle,\langle i,j\rangle} = 1 \qquad \text{if exactly one of i,j is N} \qquad (3b)$$

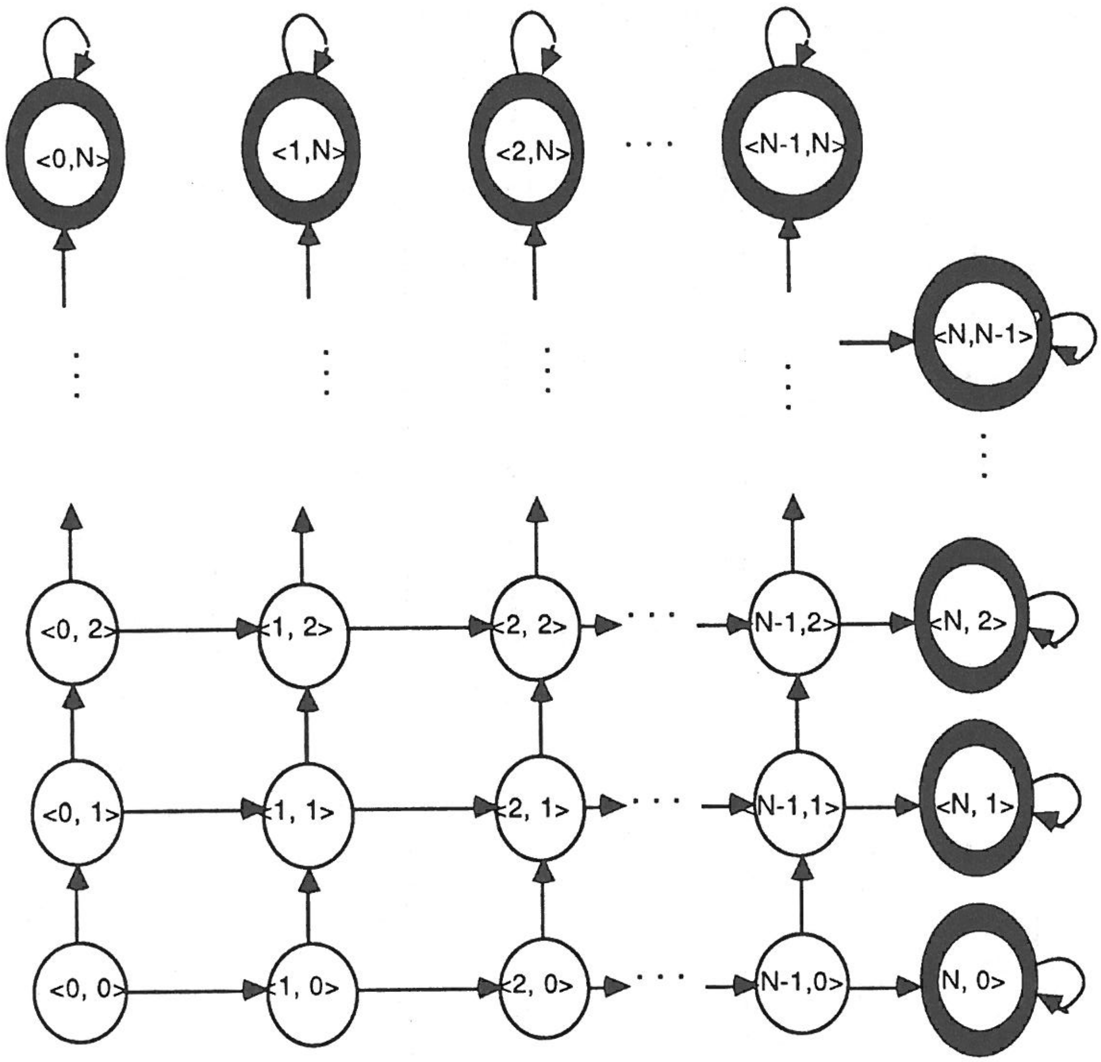

Figure I : The transition map of the Markov Chain defined in Theorem II. All the horizontal transitions are with probability p, and the vertical transitions are with probability q, where p+q=1. The self-loops are with probability one. The bold circles represent the absorbing states.

The Markov chain is clearly absorbing with the set of absorbing barriers being the set {<N, j> | j < N} ∪ {<i, N> | i < N}. The probability of converging to a configuration in which R_U beats R_V is exactly equal to the probability of being absorbed into any of the states in the set {<N, j> | j < N}.

Using the well known theory of absorbing Markov chains [7] one can compute the first passage probabilities for converging in the various absorbing states. In this case the expressions can be derived rather easily, since the transition map of the internal (non-absorbing) states is cycle-free. Thus, consider the conditional probability of converging in <N, j> given $\xi_{i,\,j}$. The Markov chain converges in <N, j> if out of the first (N+j-1) advances, N-1 of them are for R_U and j of them for R_V. The last increment has to be for R_U if it is to converge in <N, j> and thus win the race. Since $\xi_{i,j}$ is the conditional event given, the distribution is binomial, and hence,

$$\Pr[\text{Converging in } <N, j> \mid \xi_{i,j}] = \binom{N+j-1}{j} p^N q^j$$

Hence, the conditional probability of R_U winning over R_V (conditioned on $\xi_{i,j}$) is ${}_uP_V$, where,

$$ {}_uP_v = \sum_{j=0}^{N-1} \binom{N+j-1}{j} p^N q^j = p^N \sum_{j=0}^{N-1} \binom{N-1+j}{j} q^j \qquad (4)$$

Observe that ${}_uP_V$ is the sum of the probabilities of being absorbing in <N, j> where $j<N$.

We shall now consider the limiting value of ${}_uP_V$. Indeed, by virtue of Lemma I, ${}_uP_V$ approaches unity as $N \to \infty$ if $s_U > s_V$, since,

$$\lim_{N \to \infty} {}_uP_v = 1 \qquad (5)$$

The result that ${}_uP_V \to 0.5$ if $s_U = s_V$ is also proven in Lemma I. • • •

Lemma I

$$\text{Let } f_n(p) = p^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} q^k$$

where $p+q = 1$ and $0 \le p \le 1$. Then,

$$\lim_{n \to \infty} f_n(p) = 1 \qquad \text{for } 1/2 < p \le 1$$

$$= 0 \qquad \text{for } 0 \le p < 1/2$$

$$= 1/2 \qquad \text{for } p = 1/2.$$

Proof :

Denote

$$f_n(p) = p^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} q^k$$

where $p+q = 1$ and $0 \le p \le 1$. We intend to find

$$L(p) = \lim_{n \to \infty} f_n(p)$$

From equation (10.4.5) of [6],

$$f_n(p) = I_p(n, n) \quad \text{where,}$$

$$I_p(a, b) = \frac{B_p(a, b)}{B(a, b)}$$

and $B_p(a,b)$ and $B(a,b)$ represent the incomplete and complete Beta function respectively defined as below.

$$B_p(a, b) = \int_0^p t^{a-1} (1-t)^{b-1} \, dt$$

$$B(a, b) = \frac{\Gamma(a) \, . \, \Gamma(b)}{\Gamma(a + b)}$$

From the theory of Beta functions (see equations 26.5.10, 26.5.15, and 26.5.16 of [1]), we easily find that

$$f_{n+1}(p) - f_n(p) = \frac{(2n)!}{2\,(n!)^2} p^n q^n (2p-1)$$

Summing this result,

$$\sum_{n=1}^{J} [\, f_{n+1}(p) - f_n(p) \,] = \frac{1}{2}(2p-1) \sum_{n=1}^{J} \frac{(2n)!}{(n!)^2} p^n q^n \tag{6}$$

for arbitrary $J \geq 1$. Because of cancellations in the sum in the left member,

$$f_{J+1}(p) - f_1(p) = \frac{1}{2}(2p - 1) \sum_{n=1}^{J} \frac{(2n)!}{(n!)^2} p^n q^n$$

Now $f_1(p) = p$ and hence,

$$L(p) = \lim_{J \to \infty} f_{J+1}(p) = p + \frac{1}{2}(2p - 1) \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} p^n q^n \tag{7}$$

From equation (5.24.31) of [6],

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} p^n q^n = (1 - 4pq)^{-1/2} - 1$$

$$= \begin{cases} \dfrac{1}{1 - 2p} - 1 & \text{for } 0 \leq p < 1/2 \\ \dfrac{1}{2p - 1} - 1 & \text{for } 1/2 < p \leq 1 \end{cases} \tag{8}$$

Therefore, (6) -(8) yield,

$$L(p) = \begin{cases} 0 & \text{for } 0 \le p < 1/2 \\ 1 & \text{for } 1/2 < p \le 1 \end{cases} \qquad (9)$$

We now consider the case p = 1/2. Note that

$$B(n, n) = \int_0^1 t^{n-1} (1 - t)^{n-1}\, dt$$

Dividing the interval of integration into two halves,

$$B(n, n) = \int_0^{1/2} t^{n-1} (1 - t)^{n-1}\, dt + \int_{1/2}^1 t^{n-1} (1 - t)^{n-1}\, dt$$

Replacing t by 1-t in the second integral we find it equals the first integral and hence

$$B(n, n) = 2 \int_0^{1/2} t^{n-1} (1 - t)^{n-1}\, dt$$

$$= 2\, B_{1/2}(n, n).$$

Therefore, for all positive n, $f_n(1/2) = I_{1/2}(n, n) = 1/2$. Thus,

$$L(1/2) = \lim_{n \to \infty} f_n(1/2) = 1/2.$$

Combining the above results we get,

$$L(p) \begin{cases} = 0 & \text{for } 0 \le p < 1/2 \\ = 1/2 & \text{for } p = 1/2 \\ = 1 & \text{for } 1/2 < p \le 1. \end{cases}$$

• • •

The final theorem regarding the asymptotic property of the fastest runner winning the race and of the asymptotic order in which the racers finish the race follows directly.

Theorem II

Let $\mathfrak{R} = \{R_1, R_2, \ldots, R_M\}$ be the list of elements with distinct motion probabilities $\{s_1, \ldots, s_N\}$. Then, if all the racers start at the origin, the probability of the racers finishing the race in the order of their motion probabilities converges to unity as $N \to \infty$.

Proof :

For every distinct pair u, v {1,..., M} x {1,..., M}, we have shown that ${}_uP_v$ tends towards unity if $s_u > s_v$. Let P^* be defined as below :

P^* = Pr [racers finishing in order of decreasing motion probabilities]

Then, since the records are independently drawn according to the distribution $\{s_i\}$, this probability is greater than or equal to the probability that **every single pair** is in the decreasing order of the motion probabilities. Thus,

$$P^* \geq \prod_{u \neq v} \Pr[R_u \text{ beats } R_v \mid s_u > s_v] = \prod_{s_u > s_v} {}_uP_v$$

The result that P^* tends to unity follows since every ${}_uP_v$ does so as $N \to \infty$. • • •

II.1 Remark

Apart from proving the probability that R_i would beat R_j in the race converges to unity as $N \to \infty$, the above results also give us a closed form expression for the probability of any winning sequences. Furthermore this probability has been derived for finite (and infinite) values of N. Let

$\Theta = [\theta_1, \theta_2, \ldots, \theta_M]$

be any permutation of the set {1, 2, ..., M}. Let $\lambda(i,j) = s(\theta_i)/(s(\theta_i)+s(\theta_j))$. Then, using the laws of total probability, the probability that the sequence of the indices of the winners is Θ is Pr(Θ), where, if

$$F(\Theta) = \prod_{i=1\,;\,j>i}^{M-1} {}_{\theta_i}P_{\theta_j}$$

$$= \prod_{i=1\,;\,j>i}^{M-1} \frac{B_{\lambda(i,j)}(N,N)}{B(N,N)}$$

and if Θ is the set of all the permutations of the set {1, 2, ..., M}, then,

$$\Pr(\Theta) = \frac{F(\Theta)}{\sum_{\sigma \in \Theta} F(\sigma)}$$

Indeed, this value can be explicitly computed if the motion probabilities are known since the values of the Incomplete Beta function have been well tabulated. Due to Theorem II, this value tends to unity as $N \to \infty$ if and only if the sequence of the indices is in the decreasing order of their motion probabilities. Otherwise this value tends to zero in the limit. Such an expression would be of great importance in an application of the random race, for example in the analysis of an operating system's scheduling mechanism.

II.2 Simulation Results

The random race that has been studied in this section has been simulated for a number of cases in which two racers competed on a track of length N. The length of the track was varied from 32 to 512 and the value of the motion probability was varied from 0.5 to 0.9. The racers were initially at the origin. To increase the accuracy of the simulations, in each scenario that was studied, a **thousand** races were conducted.

The results that were obtained confirmed the theoretical results claimed in the above sub-section. These results are tabulated in Table I below. The convergence properties of the race are remarkable even when the motion probabilities of the faster racer is as low as 0.7. In this case, the faster racer won the race with a probability of unity even when the length of the track was as small as 32. The results are just as amazing even for the case when the larger motion probability is as low as 0.6. In this case, the probability of the racer winning the race is 0.944 when the race track is of length 32. It rises to 0.998 when N is 128. For all larger values of N the corresponding probability is unity. Such results are typical.

p \ N	32	64	128	256	512
0.5	0.49	0.479	0.509	0.537	0.501
0.55	0.766	0.868	0.928	0.987	0.998
0.6	0.944	0.988	0.998	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1

Table I : Simulation results of the two person multiple track random race with no handicap, and with motion probabilities p and (1-p). The probabilities of the faster racer winning the race is tabulated as a function of the track length, N.

III MULTIPLE TRACK RANDOM RACES WITH RANDOM STARTING POINTS

We shall now consider the general case of the racers running on multiple tracks in which the various racers do not necessarily start at the origin. To do this, as in the previous section, we shall consistently consider the scenario conditioned on the event $\xi_{u,v}$ (i.e., the event that either x_u or x_v is incremented). Observe that in this scenario, if only u or v are considered to be the runners, since only one of them will win the race, arguments similar to those used in Theorem II will have to be employed to consider the properties of the overall race.

III.1 Representation of the Multiple Track Problem

Let $f_{ij} = Pr(R_u \text{ beats } R_v \mid R_u \text{ is at position } i \text{ and } R_v \text{ is at position } j; \xi_{u,v})$. Then, f_{ij} may be recursively defined [7] using the first passage probabilities as,

$$f_{ij} = p.f_{i+1,j} + q.f_{i,j+1} \qquad \text{for all } i<N \text{ and } j<N. \qquad (10)$$

with the boundary condition that

$$f_{N,j} = 1, \qquad \text{for } j<N, \text{ and}$$

$$f_{i,N} = 0 \qquad \text{for } i<N.$$

Using the notation of Theorem I, we can derive the expression for ${}_uP_v(N)$ for any arbitrary starting point for racers R_u and R_v as :

$${}_uP_v(N) = P\,(R_u \text{ winning the race } \xi_{u,v}) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f_{ij}\cdot g_{ij}\,,$$

where $g_{ij} = P(R_u \text{ starts at } i;\ R_v \text{ starts at } j)$.

Using arguments similar to the ones used to derive (4), the recursive function defined for f_{ij} may be written as

$$f_{ij} = \sum_{k=0}^{N-j-1} \binom{N-i+k-1}{k} p^{N-i}.q^{k} \qquad (11)$$

Using this, ${}_uP_v(N)$ can be rewritten to be :

$${}_uP_v(N) = \sum_{i=0}^{N-1} p^{N-i} \sum_{j=0}^{N-1} g_{ij} \sum_{k=0}^{N-j-1} \binom{N-i+k-1}{k}.q^{k} \qquad (12)$$

Furthermore, from the decomposition of f_{ij}, we obtain

$$f_{ij} = p^{N-i} \sum_{k=0}^{N-j-1} \binom{N-j+k-1}{k}\,.\,q^{k}$$

$$= p^{N-i} \sum_{k=0}^{N-j-1} \frac{(N-i+k-1)!}{k!\,(N-i-1)!}\,.\,q^{k}$$

$$= p^{N-i} \sum_{k=0}^{N-j-1} \frac{\Gamma(N-i+k)}{k!\,\Gamma(N-i)} \cdot q^k$$

Using the notation that $(a)_b = \Gamma(a+b)/\Gamma(a)$, we get

$$f_{ij} = p^{N-i} \sum_{k=0}^{N-j-1} \frac{(N-i)_k}{k!} \cdot q^k \tag{13}$$

From the formula (10.4.5) of [6] the above sum has the value given below :

$$f_{ij} = p^{N-i} \frac{N-i}{(N-j-1)!} (N-i+1)_{N-j-1} (1-q)^{-(N-i)} B_{(1-q)}(N-i, N-j) \tag{14}$$

where $B_z(a,b)$ is the incomplete beta function. Since $p = 1-q$, the above formula can be rearranged to get

$$f_{ij} = \frac{N-i}{(N-j-1)!} \cdot \frac{\Gamma(2N-i-j)}{\Gamma(N-i+1)} \cdot B_p(N-i, N-j)$$

$$= \frac{N-i}{(N-j-1)!} \cdot \frac{\Gamma(2N-i-j)}{(N-i)!} \cdot B_p(N-i, N-j)$$

$$= \frac{\Gamma(2N-i-j)}{\Gamma(N-j)\,\Gamma(N-i)} \cdot B_p(N-i, N-j)$$

$$= \frac{B_p(N-i, N-j)}{B(N-i, N-j)} \tag{15}$$

Thus, ${}_uP_v(N)$ can also be expressed as

$${}_uP_v(N) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_{ij} \frac{B_p(N-i, N-j)}{B(N-i, N-j)} \tag{16}$$

An alternate way of expressing ${}_uP_v(N)$ is to rearrange the summations in (12) as follows :

$$\text{Let } y_{ik} = \binom{N-i+k-1}{k} \cdot q^k. \tag{17}$$

$$\text{Then} \quad {}_uP_v(N) = \sum_{i=0}^{N-1} p^{N-i} \sum_{j=0}^{N-1} g_{ij} \sum_{k=0}^{N-j-1} y_{ik} \qquad \text{(from 12)}$$

We rearrange the order of summation of the indices of j and k to give

$${}_uP_v(N) = \sum_{i=0}^{N-1} p^{N-i} \sum_{k=0}^{N-1} y_{ik} \sum_{j=0}^{N-k-1} g_{ij} \tag{18}$$

We shall now prove some limiting results regarding the race when the starting points are uniformly distributed.

Theorem III

Let R_u and R_v be two arbitrary racers with motion probabilities s_u and s_v respectively. Let $p = s_u / (s_u + s_v)$ and $q = s_v / (s_u + s_v)$, and let $p > q$. Then, if the starting positions of R_u and R_v are uniformly distributed in the interval [0, N-1], the probability of R_u winning the race **does not** converge to unity even if the track is arbitrarily long, but has the value ${}_uP_v = (3p - 1) / 2p$.

Proof :

Let the starting positions of R_u and R_v be randomly distributed between the origin and position N-1 (one position away from the goal). The value of g_{ij} is thus,

$$g_{ij} = 1 / (N)^2 \qquad \text{for } i,j = 0,\dots, N\text{-}1.$$

Using (16) we get

$${}_uP_v(N) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \frac{B_p(N-m, N-n)}{B(N-m, N-n)}$$

Performing a substitution of the variables where i = N-m and j = N-n, we get

$${}_uP_v(N) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{B_p(i, j)}{B(i, j)}$$

Let

$$L = \lim_{N \to \infty} \frac{{}_uP_v}{N^2}$$

Our intention is to evaluate L. Since

$$B(i, j) = \frac{(i-1)!\,(j-1)!}{(i+j-1)!} \quad \text{and} \quad B_p(i, j) = \int_0^p t^{i-1} (1-t)^{j-1}\, dt,$$

we have

$$S_N = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} \frac{(n+k+1)!}{k!\, n!} \int_0^p t^k (1-t)^n\, dt$$

where we have replaced i by k+1 and j by n+1. Interchanging the order of the summation and the integration,

$$S_N = \int_0^p \sum_{k=0}^{N-1} (k+1)\, t^k \sum_{n=0}^{N-1} \frac{(k+2)_n}{n!} (1-t)^n \, dt$$

From equation (10.4.5) of [6],

$$\sum_{n=0}^{N-1} \frac{(k+2)_n}{n!} (1-t)^n = \frac{(k+N+1)!}{(N-1)!\,(k+1)!} t^{-k-2} B_t(k+2, N)$$

and, hence,

$$S_N = N\,(N+1) \int_0^p \frac{1}{t^2} \sum_{k=0}^{N-1} \frac{(N+2)_k}{k!} \left(\int_0^t x^{k+1} (1-x)^{N-1} dx\right) dt .$$

From (10.4.5) of [6],

$$\sum_{k=0}^{N-1} \frac{(N-2)_k}{k!} x^k = \frac{(2N+1)!}{(N-1)!\,(N+1)!} (1-x)^{-N-2} B_{1-x}(N+2, N)$$

and, hence,

$$S_N = \frac{(2N+1)!}{(N-1)!^2} \int_0^p \frac{1}{t^2} \left\{\int_0^t \frac{x}{(1-x)^3} B_{1-x}(N+2, N)\, dx\right\} dt .$$

Since $\int_0^p \left(\int_0^t f(x,t)\, dx\right) dt = \int_0^p \left(\int_x^p f(x,t)\, dt\right) dx$,

$$S_N = \frac{(2N+1)!}{(N-1)!^2} \int_0^p \frac{x}{(1-x)^3} \left(\int_x^p \frac{dt}{t^2}\right) b_{1-x}(N+2, N)\, dx$$

$$= \frac{(2N+1)!}{(N-1)!^2} \int_0^p \frac{x}{(1-x)^3} \left(\frac{1}{x} - \frac{1}{p}\right) \left(\int_0^{1-x} y^{N+1} (1-y)^{N-1} dy\right) dx .$$

Since, for $0 \le p \le 1$,

$$\int_0^p \left(\int_0^{1-x} f(x,y)\, dy\right) dx = \int_0^q \left(\int_0^p f(x,y)\, dx\right) dy + \int_q^1 \left(\int_0^{1-y} f(x,y)\, dx\right) dy,$$

we have

$$S_N = \frac{(2N+1)!}{(N-1)!^2} \left\{\int_0^q y^{N+1} (1-y)^{N-1} \left[\int_0^p \frac{x}{(1-x)^3} \left(\frac{1}{x} - \frac{1}{p}\right) dx\right] dy\right.$$

$$+ \int_q^1 y^{N+1} (1-y)^{N-1} [\int_0^{1-y} \frac{x}{(1-x)^3} (\frac{1}{x} - \frac{1}{p}) dx] dy\}$$

$$S_N = \frac{(2N+1)!}{(N-1)!^2} \{ \frac{p}{2q} \int_0^q y^{N+1} (1-y)^{N-1} dy + \int_q^1 y^{N+1} (1-y)^{N-1} (\frac{1}{py} - \frac{q}{1py^2} + \frac{q-2}{2p}) dy\} .$$

Since

$$\int_q^1 y^{a-1} (1-y)^{b-1} dy = B(a, b) - B_q(a, b)$$

$$S_N = \frac{(2N+1)!}{(N-1)!^2} \{ \frac{p}{2q} B_q(N+2, N) + \frac{1}{p} [B(N+1, N) - B_q(N+1, N)]$$

$$- \frac{q}{2p} [B(N, N) - B_q(N, N)] + \frac{q-2}{2p} [B(N+2, N) - B_q(N+2, N)]\} .$$

As in Lemma I, we define $I_p(a, b) = B_p(a, b) / B(a, b)$, then

$$S_N = \frac{p}{2q} N(N+1) I_q (N+2, N) + \frac{1}{p} N (2N+1)[1 - I_q(N+1, N)]$$

$$- \frac{q}{p} N (2N+1) [1 - I_q(N, N)] + \frac{q-2}{2p} N(N+1) [1 - I_q(N+2, N)]$$

From equation 26.5.16 of [1],

$$I_q(a+1, b) = I_q(a, b) - \frac{(a+b-1)!}{a! (b-1)!} q^a p^b$$

Therefore,

$$S_N = \frac{N}{p} [(1-\frac{3}{2}q)N - \frac{q}{2}] + N[N(\frac{1}{2pq} - 2) + \frac{1}{2pq} - 1] I_q(N, N)$$

$$+ \frac{(2N-1)!}{2 (N-1)!^2} (N+1) (2q-1) p^{N-1} q^{N-1}$$

so that

$$\frac{S_N}{N^2} = \frac{1}{p} (1 - \frac{3}{2}q - \frac{q}{2N}) + [\frac{1}{2pq} - 2 + \frac{1}{N}(\frac{1}{2pq} - 1)] I_q(N, N)$$

$$+ \frac{(2N-1)!}{2N! (N-1)!} (1+\frac{1}{N}) (2q - 1) p^{N-1} q^{N-1}$$

For $q = 1/2$, since $I_{1/2}(N, N) = 1/2$,

$$\frac{S_N}{N^2} = \frac{1}{2}$$

Therefore, if p=q=1/2,

$$\lim_{N \to \infty} \frac{S_N}{N^2} = \frac{1}{2}$$

Using Stirling's approximation,

$$\frac{(2N-1)!}{N!\,(N-1)!} \sim \frac{2^{2N-1}}{\sqrt{\pi N}}$$

Therefore,

$$\frac{S_N}{N^2} \sim \frac{1}{p}\left(1 - \frac{3}{2}q\right) + \left(\frac{1}{2pq} - 2\right) I_q(N, N) + \frac{2^{2N-2}}{\sqrt{\pi N}} (2q - 1)\, p^{N-1} q^{N-1}$$

From Lemma I we know that

$$\begin{aligned} \lim_{N \to \infty} I_q(N, N) &= 0 && \text{for } 0 \le q < 1/2, \\ &= 1 && \text{for } 1/2 < q \le 1. \end{aligned}$$

The function $p^{N-1}q^{N-1}$ has its maximum at $p = q = 1/2$. Hence $p^{N-1}q^{N-1} \le 2^{2-2N}$. Hence,

$$\frac{2^{2N-2}}{\sqrt{N}} p^{N-1} q^{N-1} \sim 0\,.$$

Consequently,

$$\begin{aligned} \lim_{N \to \infty} \frac{S_N}{N^2} &= \frac{1}{2} && \text{for } q = \frac{1}{2} \\ &= \frac{1}{p}\left(1 - \frac{3}{2}q\right) && \text{for } 0 \le q < \frac{1}{2} \\ &= \frac{p}{2q} && \text{for } \frac{1}{2} < q \le 1. \end{aligned}$$

Hence the theorem. • • •

III.1.1 Remarks

1. The above result is by no means obvious. Actually, Anderson *et al* [2] who worked on a related problem at an intuitive level erroneously alluded to a claim that would imply that this probability would be unity (see pg. 730-732 of [2]). Although this claim was not entirely right, we would like to emphasize that it does not invalidate Anderson's powerful result.

2. Apart from being far from obvious, this result also shows why certain simple expected value arguments could lead to completely misleading results. To be more specific, let us suppose that the runner R_u and R_v were the only runners in the race, and that $p = s_u$ and $q = s_v$. Furthermore, assume that R_u and R_v are initially at positions i and j respectively. Thus, the conditional expected times for R_u and R_v to finish the race are (N-i)/p and (N-j)/q respectively given their initial starting positions. Averaging over all possible values of i and j and taking the limit as $N \to \infty$ leads to the conclusion that the expected time for R_u to reach the goal is **always** (i.e., with probability 1) strictly less than the expected time for R_v to reach the goal. Using this one can deduce (erroneously) that the **probability** of R_u reaching the goal before R_v is asymptotically unity. Indeed, this is not the case, as shown by Theorem III.

3. For any finite N we can now write an explicit expression that $\Theta=[\theta_1, \theta_2, \ldots, \theta_M]$ is the sequence of indices of the winners. Let $\lambda(i,j) = s(\theta_i)/(s(\theta_i)+s(\theta_j))$. Then the probability that the sequence of the indices of the winners is Θ is Pr(Θ), where, if $F(\Theta)$ is

$$F(\Theta) = \sum_{u=1;\, v>u}^{M-1} \prod_{i,j=1}^{N} \frac{B_{\lambda(u,v)}(i,j)}{B(i,j)}$$

and if Θ is the set of all the permutations of the set {1, 2, ..., M}, then,

$$\Pr(\Theta) = \frac{F(\Theta)}{\sum_{\sigma \in \Theta} F(\sigma)}$$

Knowing the motion probabilities this value can be explicitly computed using the well tabulated Incomplete Beta function. Note that due to Theorem III the value can be exactly computed for the case when $N \to \infty$.

III.1.2 Simulation Results

To compute the effectiveness of the results computed in the previous sub-section, the random race that has been studied above has been simulated for a number of cases in which two racers competed on a track of length N which increased from 32 to 512. As in the case of when the racers started at the origin, the value of the motion probability was varied from 0.55 to 0.9. The racers were initially positioned at locations $x_1(0)$ and $x_2(0)$, both of which were uniform in the interval [0,N-1]. Also, to increase the accuracy of the simulations, in each case that was studied **two thousand** races were conducted.

The results that were obtained are tabulated in Table II. The changing values of N seem to have little effect on the winning probability. For example, when the motion probability of the faster racer is 0.8 and the value of N is 32, the faster racer won the race with a probability of 0.856. This probability fluctuates to the value of 0.883 for the case when N=512. Note that the final **theoretical** asymptotic value is to 0.875. Similar results can be observed from Table II.

p \ N	32	64	128	256	512
0.55	0.5825	0.5745	0.592	0.5845	0.5835
0.6	0.6735	0.6495	0.6585	0.6655	0.6705
0.7	0.7805	0.787	0.7845	0.788	0.783
0.8	0.856	0.877	0.8745	0.864	0.883
0.9	0.9415	0.94	0.947	0.943	0.946

Table II : Simulation results of the two person multiple track random race with random initial starting positions, and with motion probabilities p and (1-p). The probabilities of the faster racer winning the race is tabulated as a function of the track length, N.

III.2 Multiple Track Races With Handicaps

Till now we have considered the case when the random racers were moving along the track and at every time instant one racer was permitted to progress based on the vector of motion probabilities **s.** We were able to show that if the racers were all given no handicap the fastest racer (the one with the largest value of s_i) would win the race with a probability which increased as the length of the track increased, and this probability approached unity as the track was infinitely long. As opposed to this, we showed too that if the racers were given initial positions that were uniformly distributed the probability of the fastest racer winning the race would **never** approach unity even if the track was infinitely long.

We shall now consider the case when the fastest racer is always forced to start at the origin. With no loss of generality let R_1 be this racer whose motion probability s_1 obeys $s_1 > s_i$ for all $i > 1$. The slower racers (i.e., R_i for $i \geq 2$) are given handicaps and are thus placed at positions $x_i(0)$ obeying :

$$x_i(0) > 0 \qquad\qquad i \geq 2.$$

Obviously the handicaps can be constants, deterministic variables or random variables. In the case when the handicaps are constants or deterministic variables the question of the fastest racer winning the race over R_i is trivially solved since we need to merely compute the probability of R_1 winning the race when he has to make N increments and R_i has to make only $N\text{-}x_i(0)$ increments. Indeed, expressions analogous to (12) can be used to actually derive the closed form equations for the associated probabilities.

The problem is however far more complex when the handicaps are random variables. A simple corollary of Theorem III is that if the fastest racer is initially positioned at the origin but the weaker racers are given handicaps which are **uniformly** distributed in the interval [0,N], then the probability that a weaker racer winning the race will definitely be positive for finite **and infinite** tracks. We would like to note that although this result is a direct bi-product of Theorem III it is far from obvious and is a significant result in its own right.

However, the situation changes when the handicaps are distributed with a non-uniform distribution. It is clear that if the racer R_i is given an initial handicap of $x_i(0)$, and if $x_i(0)$ is large, then the **conditional** probability of R_i wining the race over R_1 subject to the constraint that the initial handicap is $x_i(0)$ is going to be correspondingly large. However, if the probability of this condition occurring is small, the **total** probability of R_1 winning over R_i may still be high. Indeed, we shall now derive a powerful result that if the handicaps for the slower racers are geometrically distributed the fastest racer will still win the race with a probability that tends to unity as the length of the track tends to unity.

In an operating systems application the implication of the result is as follows. Let us suppose M users submit jobs, where the user R_i has a job requiring y_i units of time. Let us assume that the y_i's are geometrically distributed. Then, if the users are assigned priorities such that the probability of R_i obtaining a time slot is a monotonic function of the priority, the probability that the user with the highest priority can have his job finished first tends towards unity even if his job requires the most processing time.

Although the result which we shall prove is true for the case when the fastest racer is compared to all the other racers, in the following theorem, we shall prove a more general version of the theorem in which we compare any two racers R_u and R_v and prove that if the handicap for the weaker of the two is geometrically distributed the stronger racer will still win the race (w.p. 1) if the track is arbitrarily long.

Theorem IV

Let R_u and R_v be two arbitrary racers and let $p = s_u/(s_u+s_v)$ and $q = s_v/(s_u+s_v)$ where $p > q$. Furthermore, let $x_u(0) = 0$. Then, given $\xi_{u,\,v}$, the event that either R_u or R_v is advanced, the probability of R_u winning the race tends to unity as the length of the track increases even if R_v, the weaker racer, has an initial advantage that is geometrically distributed with **any** parameter $\beta>0$.

Proof :

We are given that $x_u(0)=0$, and that $x_v(0)$ is an integer j, where j is geometrically distributed with a parameter β. Thus, for a track of length N,

$$\Pr[x_v(0) = j] = \beta^j(1-\beta).k_N$$

where,

$$k_N = \frac{1}{\sum_{j=0}^{N-1} \beta^j (1-\beta)} \tag{19}$$

Using (18), we know that given an arbitrary initial distribution, g_{ij}, the total probability of R_u winning the race is ${}_uP_v$ This probability can be expressed under parameters N and β with fixed motion probabilities of u and v as $h_N(\beta)$, where

$$h_N(\beta) = {}_uP_v = \sum_{i=0}^{N-1} p^{N-i} \sum_{k=0}^{N-1} y_{ik} \sum_{j=0}^{N-k-1} g_{ij}$$

In this case, due to the form of $x_V(0)$, defined by (19), the terms in g_{ij} contribute non-zero components only when i=0. Hence,

$$h_N(\beta) = p^N \sum_{k=0}^{N-1} y_{0k} \sum_{j=0}^{N-k-1} g_{0j}$$

$$= p^N \sum_{k=0}^{N-1} y_{0k} \sum_{j=0}^{N-k-1} \beta^j (1-\beta).k_N$$

Extracting the terms not found in each summation we get,

$$h_N(\beta) = p^N.k_N \sum_{k=0}^{N-1} y_{0k} (1-\beta) \sum_{j=0}^{N-k-1} \beta^j$$

Summing the final geometric series yields,

$$h_N(\beta) = p^N.k_N \sum_{k=0}^{N-1} y_{0k} (1-\beta) . \frac{(1-\beta^{N-k})}{1-\beta}$$

Cancelling (1-β) and separating the terms, we get

$$h_N(\beta) = k_N.[\, p^N \sum_{k=0}^{N-1} y_{0k} - p^N \sum_{k=0}^{N-1} y_{0k}\, \beta^{N-k}]$$

Substituting for y_{0k} from (17) and simplifying by extracting β from the second summation, we get

$$h_N(\beta) = k_N.[\, p^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} q^k - (p\beta)^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{q}{\beta}\right)^k] \qquad (20)$$

We can now take the limit as N approaches infinity. Let this limit be $h^*(\beta)$. Then,

$$h^*(\beta) = \lim_{N\to\infty} h_N(\beta) = \lim_{N\to\infty} k_N . \lim_{N\to\infty} [\, p^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} q^k - (p\beta)^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{q}{\beta}\right)^k].$$

But $\lim_{N\to\infty} k_N = 1$, since g_{0j} is a valid probability distribution and the geometric distribution converges as N→∞. Furthermore, using Lemma I,

$$\lim_{N\to\infty} [\, p^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} q^k] = 1, \text{ since } p > 0.5.$$

Thus, since p > 0.5, (20) reduces to :

$$h^{*}(\beta) = 1 - \lim_{N\to\infty} \left[(p\beta)^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{q}{\beta}\right)^k \right]$$

Consider the second term on the R.H.S. of the above equation, $z(\beta)$, where,

$$z(\beta) = \lim_{N\to\infty} (p\beta)^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{q}{\beta}\right)^k \qquad (21)$$

We shall show that $z(\beta)$ converges to zero for all β and $p>0.5$. Rearranging (21) :

$$z(\beta) = \lim_{N\to\infty} \left[\frac{(p\beta)^N}{\left(1-\frac{q}{\beta}\right)^N} \cdot \left(1-\frac{q}{\beta}\right)^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{q}{\beta}\right)^k \right] \qquad (22)$$

Consider the constraint that

$$q/\beta < 1. \qquad (23)$$

By Lemma I, subject to (23),

$$\lim_{N\to\infty} \left(1-\frac{q}{\beta}\right)^N \sum_{k=0}^{N-1} \binom{N+k-1}{k} \left(\frac{q}{\beta}\right)^k \quad \text{is a value in the set } \{0, 1/2. 1\}. \qquad (24)$$

Indeed, combining (22) and (24), we observe that subject to (23)

$$z(\beta) = \lim_{N\to\infty} \left(\frac{p\beta}{1-\frac{q}{\beta}}\right)^N . C \qquad \text{, where } C \in \{0, 1/2, 1\}.$$

To render $z(\beta)$ to have the value zero, it suffices to show that subject to (23),

$$\lim_{N\to\infty} \left(\frac{p\beta}{1-\frac{q}{\beta}}\right)^N = 0$$

Thus it is sufficient to prove that if $p > 0.5$, and $\beta > q$,

$$\left| \frac{p\beta}{1-\frac{q}{\beta}} \right| < 1. \qquad (25)$$

From (23), we know that the L.H.S. of (25) has a positive denominator. Thus, (23) and (25) yields,

$$\frac{p\beta}{1-\frac{q}{\beta}} < 1.$$

which implies that $\beta > (1/p) - 1.$ (26)

The theorem follows from Lemma II using (23) and (26). •••

Lemma II

Let $h_N(\beta_1)$ and $h_N(\beta_2)$ be the corresponding values of R_u beating R_v subject to the conditions of Theorem IV, where in $h_N(\beta_1)$ R_v has the advantage based on the geometric distribution with parameter β_1, and in $h_N(\beta_2)$, R_v has the advantage based on the geometric distribution with parameter β_2. Let

$$\Delta^* = \lim_{N\to\infty} \left[h_N(\beta_2) - h_N(\beta_1)\right] .$$

Then $\Delta^* \leq 0$ if $\beta_2 > \beta_1$.

Proof :

We intend to prove that $\Delta^* \to 0$, where if $m_j = g_{0j}(\beta_2) - g_{0j}(\beta_1)$,

$$\Delta^* = \lim_{N\to\infty} \Big[\sum_{j=0}^{N-1} f_{0j} \, [g_{0j}(\beta_2) - g_{0j}(\beta_1)]\Big]$$

$$= \lim_{N\to\infty} \Big[\sum_{j=0}^{N-1} f_{0j} \, m_j\Big]$$

To do this, we shall first show that m_j is a monotonically increasing function with respect to j. This is clearly true since,

$$m_j = g_{0j}(\beta_2) - g_{0j}(\beta_1) = (1 - \beta_2)\, \beta_2{}^j - (1 - \beta_1)\, \beta_1{}^j$$

$$= \beta_1^j \left[(1-\beta_2) \left(\frac{\beta_2}{\beta_1}\right)^j - (1-\beta_1)\right]$$

Note that $m_0 = (1 - \beta_2) - (1 - \beta_1) = \beta_1 - \beta_2 < 0$ and thereafter m_k is strictly greater than m_{k+1} implying that there exist an integer r where $m_r < 0$ and $m_{r+1} \geq 0$. Moreover, for all N,

$$\sum_{j=1}^{r} m_j = - \sum_{j=r+1}^{\infty} m_i \tag{27}$$

The next step in the proof is to show that for all N,

$$\sum_{j=0}^{N} m_j = 0. \tag{28}$$

(28) is easily proved since,

$$\sum_{j=0}^{N} m_j = \sum_{j=0}^{N} \left[\, g_{0j}(\beta_2) - g_{0j}(\beta_1)\right]$$

$$= \sum_{j=0}^{N} g_{0j}(\beta_2) - \sum_{j=0}^{N} g_{0j}(\beta_1)$$

which is zero since both $g_{0j}(\beta_2)$ and $g_{0j}(\beta_1)$ are valid probability distributions.

To evaluate Δ^*, we first make use of the fact that f_{0j} is a monotonically decreasing function with respect to j. So,

$$\Delta^* = \sum_{j=0}^{r} (f_{0j} \cdot m_j) + \sum_{j=r+1}^{\infty} (f_{0j} \cdot m_j)$$

where r is as defined in (28). Thus,

$$\Delta^* \leq f_{0r} \sum_{j=0}^{r} m_j + f_{0(r+1)} \sum_{j=r+1}^{\infty} m_j$$

since we are minimizing the negative m_j's and maximizing the positive m_j's.

$$\Delta^* \leq (f_{0(r+1)} - f_{0r}) \sum_{j=r+1}^{\infty} m_j$$

The result follows since $\sum_{j=r+1}^{\infty} m_i > 0$ and $f_{0,(r+1)} - f_{0r} < 0$. •••

III.2.1 Simulation Results

Random races with geometrically varying handicaps, which were studied theoretically in the previous subsection, were simulated for a variety of scenarios. A few **typical** results are given below in Table III for the case when the length of the track is 512. The motion probabilities were varied from 0.5 to 0.9. The faster racer was initially stationed at the origin and the weaker racer was given an initial handicap that was distributed with a truncated geometric distribution with parameter β. The parameter β was increased from 0.3 to 0.9. Unlike the other simulations, to increase the accuracy of this simulation, for each scenario, 50 races were conducted for each of the 512 possible handicap positions. Each of the results is then multiplied by the probability of occurrence of such a handicap and then summing over all the products.

The results obtained, which are tabulated in Table III, remarkably agreed with the theoretical results already derived. The convergence properties of the race are significant even when the motion probabilities of the faster racer is as low as 0.55. In this case, the faster racer won the race with a probability of 0.99789 (5 s.f.) even when the value of β is as large as 0.9. The probability is as large as unity when β is 0.3. As can be expected, the convergence to unity is faster for smaller values of β.

p \ β	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	0.41572	0.41898	0.42162	0.42290	0.42050	0.40863	0.36583
0.55	1	1	1	1	0.99999	0.99990	0.99789
0.6	1	1	1	1	1	1	1
0.7	1	1	1	1	1	1	1
0.8	1	1	1	1	1	1	1
0.9	1	1	1	1	1	1	1

Table III : Simulation results of the two person multiple track random race with a handicap geometrically distributed with parameter β, and with motion probabilities p and (1-p). The probabilities of the faster racer winning the race is tabulated as a function of the geometric parameter β.

IV SINGLE TRACK RANDOM RACES

We shall now shift our emphasis from considering racers moving on a multiple track setting to the racers moving along a single track setting. The formulation of the problem is indeed quite similar to the one described earlier except for the case when more than one racer intends to occupy the same geographical position. We explicitly define the problem as follows.

Let $\{R_1, R_2, ..., R_M\}$ be a set of M particles (racers) whose positions at time 'n' are $\{x_1(n), x_2(n), ..., x_M(n)\}$ respectively. Since the track does not permit more than one racer to occupy a given position, we have, for all $1 \le i, j \le M$,

$$x_i(n) \neq x_j(n) \qquad \text{whenever } i \neq j. \qquad (29)$$

The initial positions of the racers are given as $x_i(0)$, where,

$$x_i(0) \neq x_j(0) \qquad \text{For } 1 \le i, j \le M, \text{ and } i \neq j. \qquad (30)$$

If the racers are not given handicaps they are initially positioned as close to the origin as possible. Hence, subject to (30), $x_i(0)$ obeys :

$$x_i(0) \in \{1,2,.., M\} \qquad (31)$$

Thus, if the racers are not given handicaps, the initial position of $x_i(0)$ may be randomly assigned based on an initial probability vector on the M! permutations. We assume that the M! initial permutations are all equally likely.

As in the multiple track case, at any time instant 'n', a particular index j is chosen based on a time invariant probability vector **s**, where $\mathbf{s} = [s_1, s_2, ..., s_M]^T$ and at this time instant $x_j(n)$ is incremented. Since only one racer is allowed to be at a particular position at any given time, if racer R_i, was just behind racer R_j, and it's position was incremented so as to make it try to occupy the position that R_j was occupying, the motion is permitted only on the basis of the passing (overtaking) rule described as the transposition rule in which R_i would take R_j's place and R_j will move backwards by one unit. More explicitly, for all $1 \le i, j \le M$, if

$$x_i(n) = x_j(n) - 1$$

and R_i is to be incremented by one unit, then,

$$x_i(n+1) = x_i(n) + 1, \qquad \text{and,}$$

$$x_j(n+1) = x_j(n) - 1. \qquad (32)$$

To compute the probabilities of the individual racers winning the race we shall try to obtain a recursive definition of the latter. As before, for any distinct indices $u, v \in \{1, 2, ..., M\}$, let $\xi_{u,v}$ be the event that either R_u or R_v is advanced, and let $p = s_u / (s_u + s_v)$, and $q = s_v / (s_u + s_v)$. Since p is the (conditional) probability of advancing R_u given $\xi_{u,v}$ and q is the (conditional) probability of advancing R_v given $\xi_{u,v}$, $p+q=1$. For the rest of the discussion we shall assume that all the probabilities and expressions are quantities conditioned on $\xi_{u,\,v}$.

Let $<x_u(n),\ x_v(n)>$ refer to the positions of R_u and R_v at the n^{th} time instant respectively conditioned on $\xi_{u,\,v}$. The count on the number of increments on x_u and x_v conditioned on $\xi_{u,\,v}$ obeys the following Markov chain $\mathfrak{M} = (\Phi, G)$:

(i) $\Phi = \{0, 1, \ldots, N\} \times \{0, 1, \ldots, N\} - \{<N, N>\}$ is the set of states such that $<x_u, x_v> \in \Phi$.

(ii) G is the transition function specifying the probability of moving from one state <i, j> to another <k, m>. This transition function G is shown schematically in Figure II and obeys :

$$
\begin{aligned}
F_{<i,\,j>,\,<k,\,m>} &= p && \text{if } k = i+1 \leq N;\ m = j < N\\
&= q && \text{if } k = i < N;\ m = j+1 \leq N\\
&= p && \text{if } i = j+1 \textbf{ and } k = m-1\\
&= q && \text{if } i = j-1 \textbf{ and } k = m+1\\
&= 0 && \text{otherwise}
\end{aligned}
\qquad (34)
$$

Furthermore,

$$F_{<i,\,j>,\,<i,\,j>} = 1 \qquad \text{if exactly one of } i,j \text{ is } N \qquad (35)$$

Notice that the **only** difference between the formulation of the multiple track and the single track setting is the two transitions in (34) which correspond to the transition between the states <i,i-1> and <i+1,i> for all $i \in \{1, \ldots, N-1\}$. Note that this renders the transition graph to be cyclic (See Figure II).

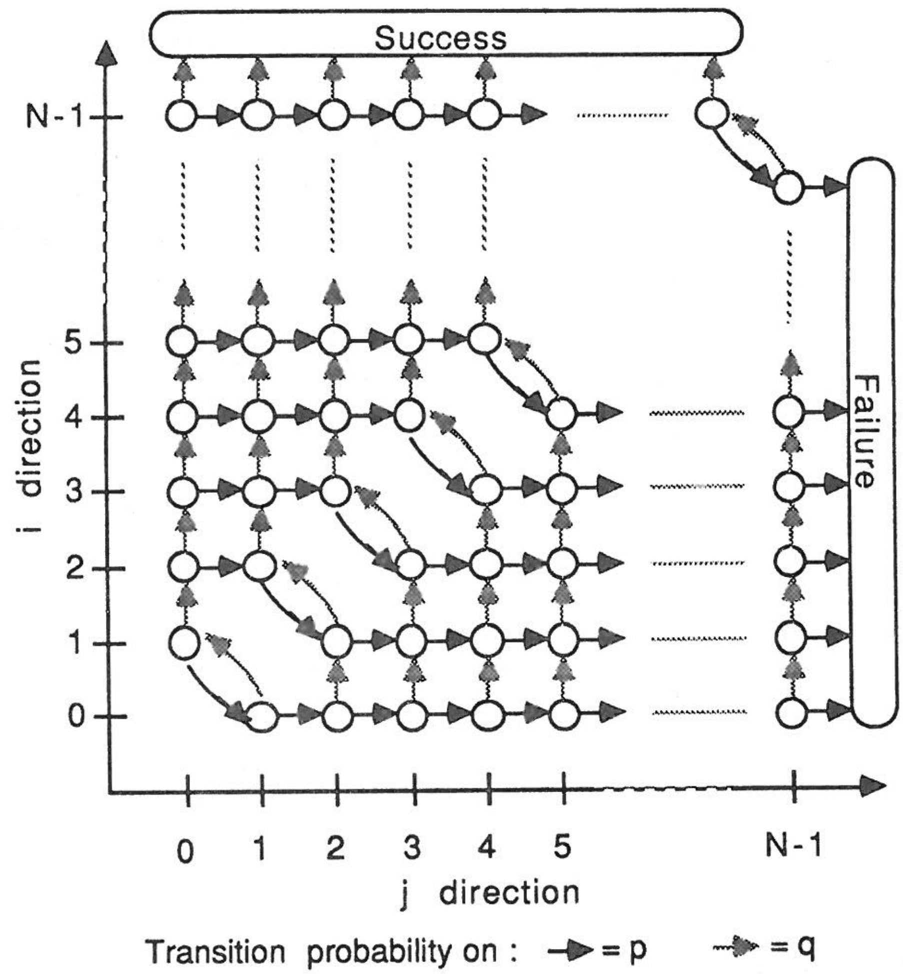

Figure 2 : Representation of single track random race with the transposition used as a passing rule. Each node represents a state $f_{i,j}$, the transition denoted by the bold arrow has probability p and the transition denoted by the shaded arrow has probability q.

As before, the Markov chain is clearly absorbing with the set of absorbing barriers being the set $\{<N, j> \mid j < N\} \cup \{<i, N> \mid i < N\}$. The probability of converging to a configuration in which R_u beats R_v is exactly equal to the probability of being absorbed into any of the states in the set $\{<N, j> \mid j < N\}$. To compute the winning probabilities, we shall now consider the first passage probabilities [7] of the chain being absorbed in a particular state given that the starting state is $<i,j>$. Let

$$f_{ij} = P[\, R_u \text{ reach the end before } R_v \mid x_u(0) = i;\ x_v(0) = j].$$

Then [7], $f_{i,j}$ obeys the following recursive equations :

$$\begin{aligned} f_{ij} &= p.f_{i+1,j-1} + q.f_{i,j+1} && \text{if } i = j-1 \text{ (lower diagonal points)} \\ &= p.f_{i+1,j} + q.f_{i-1,j+1} && \text{if } i-1 = j \text{ (upper diagonal points)} \\ &= p.f_{i+1,j} + q.f_{i,j+1} && \text{otherwise} \end{aligned} \qquad (36)$$

The boundary conditions for (36) are given by :

$f_{i,N} = 0$ for all $i < N$

$f_{N,j} = 1$ for all $j < N$ (37)

Using arguments similar to those introduced in Section III, we have,

$$Pr(R_u \text{ reaches the goal before } R_v) = \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ j \neq i}}^{N-1} f_{ij} \cdot g_{ij}$$

where $g_{ij} = P(R_u$ start at position i ; R_v start at position j).

Clearly, when $i = j \pm 1$, f_{ij} and f_{ji} are interdependent functions. We rewrite the third and fourth lines of (36) as below :

$$f_{j-1,j} = p.f_{j,j-1} + q.f_{j-1,j+1} \quad (38)$$

$$f_{j,j-1} = p.f_{j+1,j-1} + q.f_{j-1,j} \quad (39)$$

Substituting (39) into (38), we have

$$f_{j-1,j} = p(p.f_{j+1,j-1} + q.f_{j-1,j}) + q.f_{j-1,j+1}$$

$$\Rightarrow \quad (1-pq).f_{j-1,j} = p^2.f_{j+1,j-1} + q.f_{j-1,j+1}$$

$$\Rightarrow \quad f_{j-1,j} = (p^2.f_{j+1,j-1} + q.f_{j-1,j+1}) / (1-pq) \quad (40)$$

IV.1 Single Track Random Races Without Handicaps

Given that the starting positions for R_u and R_v are in the set {0,1}we now consider the the probability that R_u will reach the end before R_v. The initial conditions for g_{ij} specify the following constraints :

$g_{01} = g_{10} = 1/2$

$g_{ij} = 0$ otherwise. (41)

Using the notation of Sections II and III, h_N the conditional probability of R_u winning the race conditioned on $\xi_{u,v}$ are

$$h_N = (1/2).f_{01} + (1/2).f_{10} .$$

Unfortunately the closed form expression for h_N cannot easily be derived because, unlike the analogous results in the previous sections, the transitions graph is not acyclic. However, we have computed the value of h_N for small values of N using the symbolic manipulation package *MACSYMA*. The exact closed form expressions for h_N are tabulated in Table IV below for the cases when $N \leq 7$.

N	h_N
2	$\frac{p^2 + p}{2(1 - pq)}$
3	$\frac{p^3q^2 + (-p^4 + p^3 + p^2)\, q + p^3 + p^2}{2(p^2q^2 - 2pq + 1)}$
4	$\frac{p^5q^4 + (-p^6 + p^5 - 3p^4)\, q^3 + (-p^6 + 3p^5 - 3p^4 - 2p^3)\, q^2 + (2p^5 - p^4 - 2p^3)\, q - p^4 - p^3}{2(p^3q^3 - 3p^2q^2 + 3pq - 1)}$
5	$\left[2p^7q^6 + (-2p^8 + 2p^7 - 7p^6)\, q^5 + (-2p^8+8p^7-7p^6+8p^5)q^4 + (-p^8+7p^7-11p^6+8p^5+5p^4)q^3 + (3p^7-7p^6+2p^5+5p^4)q^2 + (-3p^6 + 5p^5 + 3p^4)q + p^5 + p^4 \right]$ $/\, 2\,(p^4q^4 - 4p^3q^3 + 6p^2q^2 - 4pq + 1)$
6	$\left[5p^9q^8 + (-5p^{10}+5p^9-22p^8)\, q^7 + (-5p^{10}+25p^9-22p^8+35p^7)q^6 + (-3p^{10}+23p^9-48p^8+35p^7-20p^6)\, q^5 + (-p^{10} + 13p^9-39p^8+40p^7-20p^6-14p^5)\, q^4 + (4p^9-20p^8+25p^7-3p^6-14p^5)\, q^3 + (-6p^8+12p^7-9p^5)\, q^2 + (4p^7-p^6-4p^5)q-p^6-p^5 \right]$ $/\, 2\,(p^5q^5 - 5p^4q^4 + 10p^3q^3 - 10p^2q^2 + 5pq - 1)$
7	$\left[14p^{11}q^{10}+(-14p^{12}+14p^{11}-75p^{10})q^9+(-14p^{12}+84p^{11}-75p^{10}+157p^9)q^8 +(-9p^{12}+79p^{11}-205p^{10}+157p^9-151p^8)q^7+(-4p^{12}+49p^{11}-178p^{10}+251p^9-151p^8+45p^7)q^6 +(-p^{12}+21p^{11}-105p^{10}+194p^9-141p^8+45p^7+42p^6)q^5+(5p^{11}-43p^{10}+106p^9-86p^8-3p^7+42p^6)q^4 +(-10p^{10}+42p^9-43p^8-9p^7+28p^6)q^3+(10p^9-18p^8-3p^7+14p^6)q^2+(-5p^8+p^7-5p^6)q+p^7+p^6 \right]$ $/\, 2(p^6q^6-6p^5q^5+15p^4q^4-20p^3q^3+15p^2q^2-6pq + 1)$

Table IV : Close form expressions of the two person single track random race with no handicap, and with motion probabilities p and q. The close form expression of the faster racer winning the race is tabulated as a function of the track length, N.

To observe the behaviour of the race for large values of N the random race that has been simulated for a number of cases in which R_u and R_v competed in the race on a track of length N, and the length of the track was varied from 32 to 512. The value of the motion probability was varied from 0.5 to 0.9, and the racers were initially started off using the initial distribution defined in (41). The performance was studied for a **thousand** races.

The results that were obtained are remarkably similar to the results obtained for the multiple track scenario. Of course, when N is small, the difference between the two types of races is more prominent. However, as N increases, the probability of the racers being in adjacent positions decreases rapidly and the results are quite similar. The results are tabulated in Table V. The convergence properties of the race are evident even when p is but 0.7. Just as in the multi-track scenario, the faster racer won the race with a probability of unity even when N was as small as 32. When p=0.6, the probability of R_u winning the race increases from 0.96 for N=32 to 0.987 for N=64 and is unity thereafter. Observe that the corresponding values for the multi-track case are 0.944 for N=32 and 0.988 for N=64 and unity thereafter. Based on these results we conjecture the following claim.

Conjecture I

For any distinct indices u, v $\in$ {1, 2, ..., M}, let $\xi_{u,v}$ be the event that either R_u or R_v is advanced in a single track random race, where the initial handicap is given by (41). Further, let ${}_uP_v$ be the **conditional** probability that R_u ultimately wins the race over R_v given $\xi_{u,v}$. Then, ${}_uP_v$ increases monotoncally with N, the length of the track, and furthermore,

(i) ${}_uP_v$ approaches unity as $N \to \infty$, if $s_u > s_v$;

(ii) ${}_uP_v$ is exactly 0.5 if $s_u = s_v$ even in the limit when $N \to \infty$.

N / p	32	64	128	256	512
0.5	0.509	0.509	0.526	0.511	0.497
0.55	0.81	0.892	0.942	0.988	1
0.6	0.962	0.987	1	1	1
0.7	1	1	1	1	1
0.8	1	1	1	1	1
0.9	1	1	1	1	1

Table V : Simulation results of the two person single track random race with no handicap, and with motion probabilities p and (1-p). The probabilities of the faster racer winning the race is tabulated as a function of the track length, N.

IV.1 Single Track Random Races With Random Starts

Given an random initial positioning anywhere along a track of size N we shall now consider the probability that R_u will reach the end before R_v given the initial conditions of $x_u(0)$ and $x_v(0)$. If both R_u and R_v can start anywhere on the track, we observe that there are N^2-N possible starting configurations. If we impose the constraints of Theorem III that all these configurations are equally likely, we have:

$$g_{ij} = 1 / (N^2-N) \text{ for all } i, j \text{ with } i \neq j. \qquad (42)$$

Thus,

$$h_N = \frac{1}{N^2-N} \sum_{i=0}^{N-1} \sum_{\substack{j=0 \\ j \neq i}}^{N-1} f_{ij} .$$

Using the symbolic manipulation package *MACSYMA* we have computed the value of h_N for small values of N. The closed form expressions for h_N are tabulated in Table VI for N ≤6.

N	h_N
2	$\dfrac{p^2 + p}{2(1 - pq)}$
3	$\dfrac{(2p^3 - p^2)\, q^2 + (-2p^4 - 2p^2 + p)q + 2p^3 + 2p^2 + 2p}{6(p^2q^2 - 2pq + 1)}$
4	$\Big[(2p^5+p^4-p^3)q^4 + (-2p^6+2p^5-4p^4-2p^3+2p^2)q^3 + (-3p^6+5p^5-6p^4-4p^3+3p^2-p)q^2$ $+ (6p^5+3p^3+5p^2-2p)q - 3p^4-3p^3-3p^2-3p \Big]$ $/\ 12(p^3q^3-3p^2q^2+3pq-1)$
5	$\Big[(4p^7+2p^6-p^4)q^6 + (-4p^8+4p^7-12p^6-6p^5+3p^3)q^5$ $+ (-6p^8+16p^7-13p^6+12p^5+2p^4+2p^3-3p^2)q^4$ $+ (-4p^8+20p^7-24p^6+11p^5-2p^4+5p^3-4p^2+p)q^3$ $+(12p^7-18p^6+12p^5+9p^4+12p^3-6p^2+2p)q^2+(-12p^6-4p^4-7p^3-10p^2+3p)q$ $+4p^5+4p^4+4p^3+4p^2+4p\Big]\ /\ 20(p^4q^4 - 4p^3q^3+6p^2q^2-4pq+1)$
6	$\Big[(10p^9+5p^8+p^7-p^6-p^5)q^8+ (-10p^{10}+10p^9-40p^8-20p^7-4p^6+4p^5+4p^4)q^7$ $+(-15p^{10}+51p^9-42p^8+56p^7+28p^6+4p^5-4p^4-6p^3)q^6$ $+(-12p^{10}+68p^9-102p^8+64p^7-26p^6-13p^5+p^4-2p^3+4p^2)q^5$ $+(-5p^{10}+51p^9-116p^8+94p^7-39p^6-19p^5+8p^4-6p^3+5p^2-p)q^4$ $+(20p^9-76p^8+83p^7-21p^6+3p^5+19p^4-15p^3+7p^2-7p)q^3$ $+(-30p^8+42p^7-20p^6-16p^5-20p^4-28p^3+11p^2-3p)q^2$ $+(20p^7+5p^5+9p^4+13p^3+17p^2-4p)q -5p^6-5p^5-5p^4-5p^3-5p^2-5p\Big]$ $/\ 30(p^5q^5-5p^4q^4+10p^3q^3-10p^2q^2+5pq - 1)$

Table VI : Close form expressions of the two person single track random race with random start, and with motion probabilities p and q. The close form expression of the faster racer winning the race is tabulated as a function of the track length, N. A similar but more involved expression has been derived for N=7 but omitted for the sake of brevity.

The behaviour of the race for large values of N has been simulated. As in the above case N was increased from 32 to 512, and p was varied from 0.5 to 0.9. The racers were initially positioned using (42), and a **thousand** races were studied. In this case the results are **not** too similar to the case of the multiple track scenario. This is because in computing the **total probability** there are many more cases where the overtaking rule becomes effective, and the chain operates along the cyclic principal "diagonal" of the transition graph. The results of the simulation are tabulated in Table VII. For example, when p=0.7, the probability of R_U winning the race varies from 0.772 for N=256 to 0.802 for N=32. Based on these results and the corresponding results derived in Theorem III we conjecture that the value of h_N will **not** converge to unity but by will be **bounded** by (3p-1)/2p. We are not certain whether this bound is tight even when N approaches infinity.

p \ N	32	64	128	256	512
0.5	0.514	0.497	0.467	0.509	0.488
0.55	0.575	0.575	0.602	0.594	0.564
0.6	0.698	0.671	0.699	0.656	0.688
0.7	0.802	0.78	0.775	0.772	0.781
0.8	0.872	0.865	0.878	0.869	0.874
0.9	0.956	0.935	0.937	0.94	0.948

Table VI : Simulation results of the two person single track random race with random starting point, and with motion probabilities p and (1-p). The probabilities of the faster racer winning the race is tabulated as a function of the track length, N.

As opposed to the case when the initial positions are uniform in the state space, we have also simulated the scenario when R_u starts at the origin and R_v starts at a position $x_v(0)$ which has a truncated geometric distribution with a parameter β. In this case, however, the results are quite similar to the multiple track case, because, although the situations where the overtaking rule has to be used is "large" the probability of the situations occurring is correspondingly "small" because of the initial geometric distribution of $x_v(0)$. The simulation results are presented in Table VIII and the parameters for the simulations are exactly as described in III.2.1. Based on these results, we conjecture the following claim:

Conjecture II

Let R_u and R_v be two arbitrary racers and let $p = s_u/(s_u+s_v)$ and $q = s_v/(s_u+s_v)$ where $p > q$. Furthermore, let $x_u(0) = 0$. Then, given $\xi_{u,v}$, the event that either R_u or R_v is advanced, the probability of R_u winning the race tends to unity as the length of the track increases even if R_v, the weaker racer, has an initial advantage that is geometrically distributed with **any** parameter $\beta>0$.

p \ β	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	0.45248	0.44926	0.44532	0.43975	0.43031	0.41168	0.36327
0.55	1	1	1	1	0.99999	0.99990	0.99784
0.6	1	1	1	1	1	1	1
0.7	1	1	1	1	1	1	1
0.8	1	1	1	1	1	1	1
0.9	1	1	1	1	1	1	1

Table VII : Simulation results of the two person single track random race with a handicap geometrically distributed with parameter β, and with motion probabilities p and (1-p). The probabilities of the faster racer winning the race is tabulated as a function of the geometric parameter β.

V CONCLUSIONS

In this paper we considered the problem of M random racers running towards a goal. At each instant, racer R_i moves towards the goal with a probability of s_i and stays where he is with a probability of $(1-s_i)$. We also permit each racer to be granted a certain handicap which allows him to start closer to the goal. This handicap may be a constant, a deterministic variable or may be stochastically assigned.

We have considered two distinct random race models called the multiple track and the single track model respectively. In the simplest model, the multiple track model, each racer runs on has his own track and interference between the racers is prohibited. In the more general setting, the racers run on a single track, and interferences between racers are specified in terms of an overtaking rule, which in our paper is the transposition rule.

In this paper we first studied random races in a one dimensional space subject to the constraint that the race has multi-tracks and that exactly one racer moves towards the goal at any given instant. Various results are proven for the cases when the racers are not given handicaps and for the case when the racers are given handicaps which are either uniformly or geometrically distributed. In each of these cases, the results proved have been obtained for the setting when the length of the track is finite and for the asymptotic condition when the race is arbitrarily long. Analogous results for the single track race are also conjectured and these conjectures are strengthened by numerous simulations. Various applications of these races are also alluded to.

We are currently investigating the use of these principles in stochastic robot control and in the scheduling of tasks in a multi-tasking environment.

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