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Reduction of Hamiltonian Mechanical Systems with Affine Constraints: A Geometric Unification

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ABSTRACT

This paper presents a geometrical approach to the dynamical reduction of a class of constrained mechanical systems. The mechanical systems considered are with affine nonholonomic constraints plus a symmetry group. The dynamical equations are formulated in a Hamiltonian formalism using the Hamilton-d'Alembert equation, and constraint forces determine an affine distribution on the configuration manifold. The proposed reduction approach consists of three main steps: 1) restricting to the constrained submanifold of the phase space, 2) quotienting the constrained submanifold, and 3) identifying the quotient manifold with a cotangent bundle. Finally as a case study, the dynamical reduction of a two-wheeled rover on a rotating disk is detailed. The symmetry group for this example is the relative configuration manifold of the rover with respect to the inertial space.

The proposed approach in this paper unifies the existing reduction procedures for symmetric Hamiltonian systems with conserved momentum, and for Chaplygin systems, which are normally treated separately in the literature. Another characteristic of this approach is that although it tracks the structure of the equations in each reduction step, it does not insist on preserving the properties of the system. For example, the resulting dynamical equations may no longer correspond to a Hamiltonian system. As a result, the invariance condition of the Hamiltonian under a group action that lies at the heart of almost every reduction procedure is relaxed.

Nomenclature

Ad Adjoint operator for a Lie group
ad adjoint operator for a Lie algebra
$[\cdot,\cdot]$ Lie bracket of two Lie algebra elements
Tf Tangent map of the map f
T^*f Cotangent map of the map f
TQ Tangent bundle of the manifold Q
T^*Q Cotangent bundle of the manifold Q
$\mathrm{Lie}(G)$ Lie algebra of the Lie group G
$\mathrm{Lie}^*(G)$ Dual of the Lie algebra of the Lie group G
$\langle\cdot,\cdot\rangle$ Pairing between members of tangent and cotangent spaces
$\mathcal{L}_X$ Lie derivative by the vector field X
ξ_Q Induced vector field by the infinitesimal action of $\xi \in \mathrm{Lie}(G)$
$\iota_X\omega$ Interior product of the differential form ω with the vector field X

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$\Omega^1(Q)$ Space of all 1-forms on Q
$\Omega^2(Q)$ Space of all 2-forms on Q
$\mathfrak{X}(Q)$ Space of all vector fields on Q
$d\omega$ Exterior derivative of the differential form ω
Q/G Quotient space corresponding to an action of G on Q

1 Introduction

Reduction of dynamical systems has been proven helpful in studying their inherent behaviour [1, 2, 3, 4], as well as for designing controllers for such systems [3, 5, 6, 7, 8]. Usually, reduction is performed in the presence of a symmetry group of the system. For unconstrained Hamiltonian systems, the group action preserves both the Hamiltonian and the symplectic 2-form defined on the phase space of the system. For nonholonomic Hamiltonian systems, the nonholonomic distribution is also invariant under the group action.

Although constraints could be generally nonlinear, usually nonholonomic constraints that are considered in the literature form a linear sub-bundle of the tangent bundle of the configuration manifold. An example of an affine constraint is treated in [9], and in [10] the tracking control of nonholonomic mechanical systems with affine constraints is considered. In a more recent article, conservation of energy and momentum corresponding to lifted actions is studied for systems with affine constraints [11]. In [2], affine constraints are mentioned in the geometric treatment of nonholonomic mechanical systems with symmetry. However, in the literature there does not exist a systematic approach for the reduction of dynamical equations of mechanical systems with affine constraints. The purpose of this paper is to provide such an approach, which unifies the symplectic reduction of unconstrained systems with conserved momentum and the reduction of nonholonomically constrained systems.

1.1 Reductions Theories

In the following, the reduction methods investigated in the literature for both unconstrained Hamiltonian systems and nonholonomic systems with symmetry are presented.

1.1.1 Hamiltonian Systems with Symmetry

Emmy Noether in her famous paper [12] showed that any symmetry of the action functional in Hamilton's principle corresponds to a conserved quantity, which is called momentum. In the study of Hamiltonian systems, momentum is unchanged along the flow of a Hamiltonian vector field, if the Hamiltonian and the symplectic 2-form representing the dynamics are invariant under the symmetry [1]. The symplectic reduction theorem [13] provides a construction for expressing the dynamics on a reduced phase space with a symplectic structure. This theorem by Marsden and Weinstein made a huge impact on unifying the reduction methods that were previously developed for Lagrangian and Hamiltonian systems, such as the classical Routh method and the reduction of Lagrangian systems by cyclic parameters [14, 15]. In this theorem the momentum map is assumed Ad^*-equivariant, the 2-form on the phase space is symplectic, and the reduction is at a regular level set of the momentum map. These assumptions were relaxed in later works by, e.g., Planas-Bielsa [16] and Marsden *et al.* [17].

For a mechanical system, the phase space is the cotangent bundle T^*Q of the configuration manifold; it admits a canonical symplectic 2-form. The Hamiltonian of the mechanical system comes from the kinetic energy metric and a potential energy function on Q. Let G be a Lie group acting on the configuration manifold Q. The cotangent lifted action on the phase space is symplectic [1]. In this case, if the Hamiltonian of the system is also invariant under the cotangent lift of the G-action, the group G is called a symmetry group of the mechanical system, and the system is called a Hamiltonian mechanical system with symmetry [1, 15]. Open-chain multibody systems are examples of Hamiltonian mechanical systems with symmetry whose dynamical reduction is detailed in [18].

The phase space T^*Q of a mechanical system also admits a canonical Poisson bracket. Suppose that the symmetry group G acts freely and properly on Q, and hence on T^*Q. The Poisson bracket is invariant under the cotangent lifted action, and it descends to a Poisson bracket on the quotient manifold $(T^*Q)/G$. This process, which has been studied in [1, 3, 19], is called Poisson reduction. The major difference between Poisson reduction and symplectic reduction is the concept of a momentum map, which is not necessary for Poisson reduction, and as a result the reduced Hamilton's equation on the quotient phase space evolves in a bigger space. This approach unifies the Euler-Poincaré and Lagrange-Poincaré equations for mechanical systems with symmetry [1]. Both of the abovementioned reduction theories for mechanical systems with symmetry were developed and extended to Lagrangian systems in the 1990s [20, 21, 22].

1.1.2 Nonholonomic Systems with Symmetry

A historic example of reducing nonholonomic systems is the work of Chaplygin [23]. In this paper he eliminated the Lagrange multipliers in the Lagrange-d'Alembert equation and expressed the dynamical equations in a smaller phase space

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for nonholonomic Lagrangian systems with cyclic parameters. This result was extended to Lagrangian mechanical systems with non-abelian symmetry by Koiller [24]. On the Hamiltonian side, van der Schaft and Maschke [25] eliminated the Lagrange multipliers by expressing Hamilton's equation on the constrained phase space. They worked with the Poisson structure of the cotangent bundles.

A nonholonomic mechanical system with symmetry is usually defined as a mechanical system with symmetry together with a G-invariant distribution $\mathcal{D}$. This distribution is usually taken to be a linear sub-bundle of TQ, where the velocities of the physical trajectories of the system should lie. Generally, this distribution is non-involutive, and it is the result of kinematic nonholonomic constraints such as rolling without slipping. If $\mathcal{D}$ is involutive, the constraints are called holonomic. Hence, the word "nonholonomic" stands for "not necessarily holonomic" and it does not mean "not holonomic". Since nonholonomic systems satisfy the Hamilton-d'Alembert principle [26] instead of the Hamilton principle, the reduction procedures introduced for such systems are normally different from those of Hamiltonian systems with symmetry. A geometric approach for dynamical reduction of nonholonomic mechanical systems with symmetry is reported in [2]; it results in the Lagrange-d'Alembert-Poincaré equation [3,27]. This method is centred at defining a nonholonomic connection as the summation of an Ehresmann connection and the mechanical connection, and introducing a nonholonomic momentum map. The analogue of this approach in Poisson formalism is also explained in [3], which evolved from a paper by van der Schaft and Maschke [25]. The condition that the directions of the group action be complementary to $\mathcal{D}$ determines a class of nonholonomic mechanical systems with symmetry called Chaplygin systems. Reduction of such systems are separately discussed in [2, 3, 24, 28]. An extension of the Chaplygin reduction is also reported in [29, 30], which use an almost symplectic reduction theorem [16] to further reduce the dynamics after the Chaplygin reduction.

On the Hamiltonian side, Bates and Śniatycki realized that the solution of the Hamilton-d'Alembert equation is a section of the distribution $T(\mathbb{F}L(\mathcal{D})) \cap \{v \in T(T^*Q) | T\pi_Q(v) \in \mathcal{D}\} \subseteq T(T^*Q)$. Here, the fibre-wise linear map $\mathbb{F}L\colon TQ \to T^*Q$ is the Legendre transformation, and $\pi_Q\colon T^*Q \to Q$ is the canonical projection map of the cotangent bundle T^*Q. Then under the symmetry hypotheses, after restricting Hamilton's equation to this distribution, they show that the flow of this vector field descends to the quotient manifold $\mathbb{F}L(\mathcal{D})/G$ [31, 32, 33, 34]. Later, based on this method of reduction the Noether theorem is extended to nonholonomic systems and accordingly a two-stage reduction procedure is introduced [35]. This method is further extended to singular reduction of nonholonomic systems, and it is reformulated for almost Poisson manifolds in [36].

Finally, in a recent research by Gay-Balmaz and Yoshimura, the reduction theory of Dirac structures for holonomic and nonholonomic systems on Lie groups with broken symmetry is discussed [37].

1.2 Statement of Contributions

The main contributions of this paper can be listed as follows:

(a) A systematic dynamical reduction procedure for Hamiltonian mechanical systems with *affine* constraints is presented.
(b) The proposed reduction procedure *unifies* two existing reduction theories, i.e.,
  (1) the symplectic reduction of Hamiltonian mechanical systems with symmetry, where the momentum is conserved, and
  (2) the Chaplygin reduction theorem for nonholonomic systems on cotangent bundles.
(c) The invariance assumption of the Hamiltonian under a Lie group action, which lies at the heart of any reduction theory is *relaxed* and substituted by the conditions of Lemma 3.6.

The governing dynamical equations of a constrained Hamiltonian system are identified by the Hamiltonian, a 2-form and a (affine) distribution. Therefore, the existing reduction methods involve reducing the Hamiltonian function, the 2-form and the constrained phase space, separately, and the emphasis is on preserving the structure of the dynamical equations. For example, in the symplectic or Chaplygin reduction theories the reduced system is still Hamiltonian and the reduced dynamics is governed by Hamilton's equation. As opposed to the existing methods, the presented reduction procedure only focuses on the dynamical equations at each reduction step, and the reduced system may not be even Hamiltonian. That is, the resulting structure involves a 2-form that might no longer be symplectic and the right hand side of the dynamical equation is a 1- form that might no longer be exact (coming from a Hamiltonian function).

This proposed reduction theory not only sheds light on the connection between the symplectic and Chaplygin reduction theories, but it also includes more cases that could not be previously treated by those theories, e.g., systems with affine nonholonomic constraints (see the case study) or known variable momentum (orbiting satellites). This theory consists of three main steps:

(a) Calculating the Lagrange multipliers and restriction of the dynamics to the constrained submanifold of T^*Q,
(b) Quotienting the constrained submanifold by a symmetry group, and
(c) Identifying the quotient manifold with a cotangent bundle.

In Section 2, the concept of a constrained Hamiltonian mechanical systems with symmetry that is considered in this paper is defined. It is then shown how this definition can cover a wide range of holonomic and nonholonomic mechanical systems. Section 3 reports the main results of this paper, which includes the detailed explanation of the three reduction steps. A step-by-step application of the proposed reduction method to the dynamical equations of a two-wheeled differential drive rover on a rotating disk is studied in Section 4. Section 5 concludes the paper.

2 Constrained Hamiltonian Mechanical Systems with Symmetry

This section contains a definition of a constrained Hamiltonian mechanical systems with symmetry. The constraints that are considered are in the form of affine distributions, whose special cases are: (a) conservation of momentum for holonomic systems, and (b) linear nonholonomic constraints in the velocity space. Also, some relevant geometric structures for constrained Hamiltonian mechanical systems with symmetry are explained.

Let Q denote the configuration manifold of a mechanical system, and let T^*Q be its cotangent bundle. The cotangent bundle is naturally equipped with a symplectic two form $\omega_{can} \in \Omega^2(T^*Q)$, called the canonical two form. The phase space of the Hamiltonian mechanical system is the symplectic manifold (T^*Q, ω_{can}). Consider the smooth function $H\colon T^*Q \to \mathbb{R}$, called the Hamiltonian, defined by, $\forall (q,p) \in T^*Q$:

$$H(q,p) = \frac{1}{2} K_q(\mathbb{F}L_q^{-1}(p), \mathbb{F}L_q^{-1}(p)) + V(q), \tag{2.1}$$

where $K_q\colon T_qQ \times T_qQ \to \mathbb{R}$ is the kinetic energy Riemannian metric, and where $V\colon Q \to \mathbb{R}$ is a smooth function, called the potential energy function. The Legendre transformation $\mathbb{F}L\colon TQ \to T^*Q$ is the fibre-wise linear isomorphism that is induced by the metric K:

$$\langle \mathbb{F}L_q(v), w \rangle := K_q(v,w) \quad \forall v,w \in T_qQ. \tag{2.2}$$

Physically, the Hamiltonian is the total energy of the mechanical system. The Hamiltonian mechanical system may be represented by the triple (T^*Q, ω_{can}, H). The dynamics of such a system is specified by the vector field X that satisfies Hamilton's equation

$$\iota_X \omega_{can} = dH. \tag{2.3}$$

Generally, constraints can be considered as restrictions on the solution curves of a dynamical system. For a mechanical system, these restrictions may be of the form of a subset of the configuration space Q or the velocity space TQ. In this paper, only affine constraints in TQ are considered. The space of allowed velocities of the system then forms an affine distribution $\mathcal{D}$ whose rank is assumed constant. "Affine" means that there is a vector field Y and a linear distribution Δ on Q such that $\forall q \in Q$,

$$\Delta(q) = \mathcal{D}(q) - Y(q).$$

Special cases of this type of constraints are:

(a) linear nonholonomic distributions, where $0 \in \mathcal{D}(q)$ for all $q \in Q$,
(b) constant momentum, where if $\mathbf{M}$ denotes a momentum map, $\mathbb{F}L(\mathcal{D}) = \mathbf{M}^{-1}(\mu)$, for a value μ of the momentum,
(c) holonomic constraints: a linear involutive distribution, corresponding to a foliation of Q.

A constrained Hamiltonian mechanical system may be represented by a quadruple $(T^*Q, \omega_{can}, H, \mathcal{D})$ as above. If $\alpha_i \in \Omega^1(Q)$ for $i = 1, \ldots, m := \dim(TQ) - \dim(\mathcal{D})$ is a set of differential 1-forms whose point-wise kernel is the distribution Δ, then according to the Hamilton-d'Alembert Principle, the solution curves of the constrained Hamiltonian mechanical system are those curves whose velocity vectors X satisfy the Hamilton-d'Alembert equation:

$$\begin{aligned} \iota_X \omega_{can} &= dH + \sum_{i=1}^{m} \kappa_i T^*\pi_Q(\alpha_i), \\ \langle \alpha_i, T\pi_Q(X) \rangle &= \langle \alpha_i, Y \rangle =: \gamma_i \text{ for } i = 1, \ldots, m. \end{aligned} \tag{2.4}$$

Here, $\pi_Q\colon T^*Q \to Q$ is the natural cotangent bundle projection, and $T^*\pi_Q\colon \Omega^1(Q) \to \Omega^1(T^*Q)$ is the pullback map. The

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constraint equations on the second line of (2.4) are equivalent to the condition $T\pi_Q(X) \subset \mathcal{D}$. Note that near each point (2.4) is the most general dynamical equation for Hamiltonian mechanical systems with linear affine constraints. In the special cases mentioned in the previous paragraph the following conditions respectively hold:

(a) $Y \equiv 0$,
(b) $\kappa_i \equiv 0$ (proved in the reduction process),
(c) Locally $\exists f_i \in C^\infty(Q)$ such that $\alpha_i = df_i$ ($\mathcal{D}$ is a linear involutive distribution).

Let G be a Lie group with the Lie algebra $\mathrm{Lie}(G)$. Consider a free and proper action of G on Q, and denote this action by $\Phi_g \colon Q \to Q$, $\forall g \in G$. This action induces an action of G on T^*Q by the cotangent lift of Φ_g, which is denoted by $T^*\Phi_g \colon T^*Q \to T^*Q$.

Lemma 2.1 *For every $g \in G$, the map $T^*\Phi_g$ is a symplectomorphism, i.e., it preserves ω_{can} [1].* □

Consider the infinitesimal action of $\mathrm{Lie}(G)$ on Q. For any $\xi \in \mathrm{Lie}(G)$, this action gives a vector field $\xi_Q \in \mathfrak{X}(Q)$ such that $\forall q \in Q$,

$$\xi_Q(q) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \left(\Phi_{\exp(\varepsilon\xi)}(q)\right). \tag{2.5}$$

Denote the linear maps corresponding to the infinitesimal action of $\mathrm{Lie}(G)$ by $\phi_q \colon \mathrm{Lie}(G) \to T_qQ$, where $\phi_q(\xi) = \xi_Q(q)$. Likewise, the vector field $\xi_{T^*Q} \in \mathfrak{X}(T^*Q)$ is defined such that $\forall (q,p) \in T_q^*Q$,

$$\xi_{T^*Q}(q,p) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \left(T^*\Phi_{\exp(-\varepsilon\xi)}(q,p)\right). \tag{2.6}$$

Now, consider the fibre-wise linear map $\mathbf{M} \colon T^*Q \to \mathrm{Lie}^*(G)$, defined by $\mathbf{M}(q,p) = \mathbf{M}_q(p)$, where $\mathbf{M}_q = \phi_q^* \colon T_q^*Q \to \mathrm{Lie}^*(G)$. So,

$$\langle \mathbf{M}(q,p), \xi \rangle = \langle p, \xi_Q(q) \rangle. \tag{2.7}$$

Lemma 2.2 *The map $\boldsymbol{M}$ is an Ad^*-equivariant momentum map corresponding to the cotangent lifted action $T^*\Phi_g$, that is, it satisfies*

$$\iota_{\xi_{T^*Q}}\omega_{can} = d\langle \boldsymbol{M}, \xi \rangle, \quad \forall \xi \in Lie(G)$$
$$\boldsymbol{M}(T^*\Phi_g(q,p)) = Ad_g^*\boldsymbol{M}_q(p).$$
□

The following definition refers to the class of systems considered in this paper. In the future, similar terminology may be used with fewer assumptions.

Definition 2.3 *A constrained Hamiltonian mechanical system with symmetry is a constrained Hamiltonian mechanical system $(T^*Q, \omega_{can}, H, \mathcal{D})$ together with an action of a Lie group G and a choice of a connected Lie subgroup E, such that*

(a) For all $q \in Q$

$$T_qQ = \Delta(q) \oplus \phi_q(Lie(G)).$$

(b) The affine sub-bundle $\mathbb{F}L(\mathcal{D})$ is invariant under the cotangent lifted E-action.
(c) The conditions of Lemma 3.6 hold for the cotangent lifted E-action restricted to $\mathbb{F}L(\mathcal{D})$.

Such a system is represented by $(T^*Q, \omega_{can}, H, \mathcal{D}, E \subseteq G)$. □

Let $(T^*Q, \omega_{can}, H, \mathcal{D}, E \subseteq G)$ be a constrained Hamiltonian mechanical system with symmetry. A priori, if no kinematic constraint is defined for the system, i.e., $\mathcal{D} = TQ$, the solution curves of the system satisfy Hamilton's equation, which is a special case of (2.4). In the following, it is shown that even in this unconstrained case the Hamilton-d'Alembert equation may describe the dynamics of the system.

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Proposition 2.4 (Noether's Theorem) *For an unconstrained Hamiltonian mechanical system with symmetry that satisfies $\iota_X \omega_{can} = dH$, the momentum map $\boldsymbol{M}$, defined in (2.7), is constant along the flow of X. That is, $\forall \xi \in Lie(G)$ the condition $\mathcal{L}_X(\langle \boldsymbol{M}, \xi \rangle) = 0$ holds.* □

Noether's Theorem implies that in the presence of a symmetry the solution curves of an unconstrained system may be restricted to a submanifold of T^*Q. This submanifold is the pre-image of a constant element of the dual of the Lie algebra $\mathrm{Lie}^*(G)$ under the momentum map $\mathbf{M}$. Since the G-action is free, for every $\mu \in \mathrm{Lie}^*(G)$ the pre-image $\mathbf{M}^{-1}(\mu)$ is a submanifold of T^*Q. An unconstrained Hamiltonian mechanical system with symmetry generally satisfies the Hamilton-d'Alembert equation (2.4), with

$$\alpha_i(q) = \left\langle \eta_i^*, \mathbb{I}_q^{-1} \circ \mathbf{M}_q \circ \mathbb{F}L_q \right\rangle,$$
$$\gamma_i(q) = \left\langle \eta_i^*, \mathbb{I}_q^{-1}(\mu) \right\rangle,$$

where $\{\eta_i \in \mathrm{Lie}(G) | i = 1, \cdots, m\}$ is a basis for the Lie algebra of G, and $\{\eta_i^* \in \mathrm{Lie}^*(G) | i = 1, \cdots, m\}$ is its dual basis. In this case, the subgroup E is taken to be $G_\mu := \left\{ g \in G | \mathrm{Ad}_g^*(\mu) = \mu \right\}$, the isotropy group corresponding to μ. Here, for any $q \in Q$ the linear map $\mathbb{I}_q\colon \mathrm{Lie}(G) \to \mathrm{Lie}^*(G)$ is defined by

$$\mathbb{I}_q := \phi_q^* \circ \mathbb{F}L_q \circ \phi_q. \tag{2.8}$$

This map is a linear isomorphism for any $q \in Q$, and it is called the locked inertia tensor. For a constrained Hamiltonian mechanical system with symmetry this tensor relates to the kinetic energy metric by $\langle \mathbb{I}_q(\xi), \nu \rangle = K_q(\xi_Q(q), \nu_Q(q)), \forall \xi, \nu \in \mathrm{Lie}(G)$.

In the following, some relevant structures induced by the G-action are introduced. Consider a constrained Hamiltonian mechanical system with symmetry $(T^*Q, \omega_{can}, H, \mathcal{D}, E \subseteq G)$, and $\forall g \in G$ the action map is denoted by $\Phi_g\colon Q \to Q$. The quotient manifold $\overline{Q} := Q/G$ gives rise to the principal bundle $\overline{\pi}\colon Q \to \overline{Q}$ with the base space $\overline{Q}$, and the fibres of the bundle can be identified with the group G. A principal connection on the principle bundle $\overline{\pi}$ is a fibre-wise linear map $\mathcal{A}\colon TQ \to \mathrm{Lie}(G)$, such that $\mathcal{A}_q(\xi_Q(q)) = \xi \;\; \forall \xi \in \mathrm{Lie}(G)$ and $\forall q \in Q$, and it is Ad-equivariant, i.e., $\mathcal{A}(T_q\Phi_g(q,v)) = \mathrm{Ad}_g \mathcal{A}_q(v) \;\; \forall (q,v) \in T_qQ$. Accordingly, for any base element $q \in Q$ the tangent space of Q can be written as the following direct sum

$$T_qQ = \ker(T_q\overline{\pi}) \oplus \ker(\mathcal{A}_q). \tag{2.9}$$

$\overline{\mathcal{V}} := \ker(T\overline{\pi}) = \left\{ \xi_Q(q) = \phi_q(\xi) \,\middle|\, \xi \in \mathrm{Lie}(G),\ q \in Q \right\}$ is called the vertical sub-bundle and $\overline{\mathcal{H}} := \ker(\mathcal{A})$ is called the horizontal sub-bundle of TQ. As the result, any $v \in T_qQ$ can be decomposed into the horizontal and vertical components such that $v = \mathrm{hor}_q(v) + \mathrm{ver}_q(v)$, where $\mathrm{ver}_q(v) := \phi_q \circ \mathcal{A}_q(v)$ and $\mathrm{hor}_q(v) := v - \mathrm{ver}_q(v)$.

For any $q \in Q$ and $\overline{q} := \overline{\pi}(q) \in \overline{Q}$ the restriction of the tangent map $T_q\overline{\pi}\colon T_qQ \to T_{\overline{q}}\overline{Q}$ to the horizontal subspace $\overline{\mathcal{H}}_q$ of T_qQ is a linear isomorphism between and $T_{\overline{q}}\overline{Q}$. Therefore, for any $q \in Q$ the horizontal lift map takes $\overline{v} \in T_{\overline{q}}\overline{Q}$ to

$$\mathrm{hl}_q(\overline{v}) := \left(T_q\overline{\pi}\big|_{\overline{\mathcal{H}}_q} \right)^{-1}(\overline{v}). \tag{2.10}$$

For a constrained Hamiltonian mechanical system with symmetry of the type considered in Definition 2.3, with the constraints given by 1-forms α_is as in (2.4), the principal connection $\mathcal{A}$ is taken to be

$$\mathcal{A} := \sum_{i=1}^{m} \alpha_i \eta_i. \tag{2.11}$$

Thus in this case, $\Delta = \overline{\mathcal{H}}$.

3 Reduction of Constrained Hamiltonian Mechanical Systems with Symmetry

From a geometric point of view, the dynamical equations of a constrained Hamiltonian mechanical system with symmetry are given by the subset of $T(T^*Q)$ that is cut out by the Hamilton-d'Alembert equation (2.4). The reduction process of

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such systems consists of three major steps:

(a) Noting that the Hamilton-d'Alembert equation restricts to the constrained submanifold of the phase space T^*Q,
(b) Quotienting the constrained submanifold by a symmetry group $E \subseteq G$ and transferring the Hamilton-d'Alembert equation to the quotient manifold,
(c) Identifying the quotient manifold with a submanifold of a cotangent bundle.

3.1 Step 1: Restriction to the Constrained Phase Space

In this section, the Hamilton-d'Alembert equation for constrained Hamiltonian mechanical systems with symmetry is formally restricted to the submanifold

$$\mathcal{M} := \mathbb{F}L(\mathcal{D}) \subseteq T^*Q.$$

The constrained phase space $\mathcal{M}$ is an affine sub-bundle since the Legendre transformation is a fibre-wise linear isomorphism for such systems. Throughout this section only Assumption (a) in Definition 2.3 is needed; Assumptions (b) and (c) play a role in the later two steps of the reduction process.

To simplify the notation, the 1-form $\beta_i := T^*\pi_Q(\alpha_i)$ is introduced. The Hamilton-d'Alembert equation (2.4) then takes the form

$$\begin{aligned} \iota_X \omega_{can} &= dH + \sum_{i=1}^{m} \kappa_i \beta_i \\ \langle \beta_i, X \rangle &= \gamma_i \ \text{ for } \ i = 1, \cdots, m. \end{aligned} \tag{3.12}$$

Recall that $\{\eta_s \in \mathrm{Lie}(G) | s = 1, \cdots, m\}$ is a basis for $\mathrm{Lie}(G)$. The Lagrange multipliers κ_i are first determined in the following through pairing both sides of the Hamilton-d'Alembert equation by the vector fields $(\eta_s)_{T^*Q}$. For $s = 1, \cdots, m$,

$$\begin{aligned} \langle \iota_X \omega_{can}, (\eta_s)_{T^*Q} \rangle &= \mathcal{L}_{(\eta_s)_{T^*Q}}(H) + \sum_{i=1}^{m} \kappa_i \langle \beta_i, (\eta_s)_{T^*Q} \rangle, \\ -\langle d\langle \mathbf{M}, \eta_s \rangle, X \rangle &= \mathcal{L}_{(\eta_s)_{T^*Q}}(H) + \sum_{i=1}^{m} \kappa_i \langle \alpha_i, (\eta_s)_Q \rangle, \\ \kappa_s &= -\mathcal{L}_X \langle \mathbf{M}, \eta_s \rangle - \mathcal{L}_{(\eta_s)_{T^*Q}}(H). \end{aligned}$$

In the above calculation, the anti-symmetry of the canonical 2-form plus the definition of β_i and the momentum map are used in the second line. The last line is true because of the choice of α_i such that $\langle \alpha_i, (\eta_s)_Q \rangle = \delta_i^s$, where δ_i^s is the Kronecker delta function. To further simplify the notation, let $\mathbf{M}_i := \langle \mathbf{M}, \eta_i \rangle$ and $H_i := \mathcal{L}_{(\eta_i)_{T^*Q}}(H)$. Thus, the Hamilton-d'Alembert equation (3.12) becomes

$$\begin{aligned} \iota_X \omega_{can} &= dH - \sum_{i=1}^{m} \left[(\iota_X d\mathbf{M}_i + H_i) \beta_i \right]. \\ \langle \beta_i, X \rangle &= \gamma_i \ \ (i = 1, \cdots, m) \end{aligned} \tag{3.13}$$

If a tangent vector $X \in T(T^*Q)$ satisfies (3.13), it also satisfies the equations below along with the constraints $\langle \beta_i, X \rangle = \gamma_i$,

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where $\mathcal{A}$ and $\mathcal{B}$ are described further below.

$$\iota_X\left(\omega_{can}+\sum_{i=1}^{m}(d\mathbf{M}_i\wedge\beta_i)\right)=dH-\sum_{i=1}^{m}(\gamma_i d\mathbf{M}_i+H_i\beta_i),$$

$$\iota_X\left(\omega_{can}+\sum_{i=1}^{m}d(\mathbf{M}_i\beta_i)-\mathbf{M}_i d\beta_i)\right)= \\ =dH-\sum_{i=1}^{m}(\gamma_i d\mathbf{M}_i+H_i\beta_i),$$

$$\iota_X(\omega_{can}+d\langle\mathbf{M},\mathcal{A}\rangle-\langle\mathbf{M},d\mathcal{A}\rangle)=dH-\sum_{i=1}^{m}(\gamma_i d\mathbf{M}_i+H_i\beta_i),$$

$$\iota_X(\omega_{can}+d\langle\mathbf{M},\mathcal{A}\rangle-\langle\mathbf{M},\mathcal{B}\rangle-\langle\mathbf{M},[\mathcal{A},\mathcal{A}]\rangle)= \\ =dH-\sum_{i=1}^{m}(\gamma_i d\mathbf{M}_i+H_i\beta_i),$$

$$\iota_X(\omega_{can}+d\langle\mathbf{M},\mathcal{A}\rangle-\langle\mathbf{M},\mathcal{B}\rangle)= \\ =dH-\sum_{i=1}^{m}(\gamma_i d\mathbf{M}_i+H_i\beta_i-\gamma_i\langle\mathbf{M},\mathrm{ad}_{\eta_i}\mathcal{A}\rangle). \tag{3.14}$$

The first equation is valid since X should satisfy the constraints, i.e., $\alpha_i(T\pi_Q(X))=\gamma_i$ for every $i=1,\cdots,m$. The third equation is the consequence of the definition (2.11) of the principal connection $\mathcal{A}$. And the fourth equation uses the Cartan Structure Equation for principal connections [17], i.e., $d\mathcal{A}=\mathcal{B}+[\mathcal{A},\mathcal{A}]$. The 2-form $\mathcal{B}$ is the curvature of $\mathcal{A}$, which is defined by $\mathcal{B}_q(u,v):=(d\mathcal{A})_q(\mathrm{hor}(u),\mathrm{hor}(v))$ for every $u,v\in T_qQ$. Plus, the 2-form $[\mathcal{A},\mathcal{A}]$ acts in the following way $[\mathcal{A},\mathcal{A}]_q(u,v):=[\mathcal{A}_q(u),\mathcal{A}_q(v)]$. The Lie bracket in this formula is the Lie bracket of the Lie algebra Lie(G). The same symbols $\mathcal{A}$ and $\mathcal{B}$ are used to denote these Lie(G)-valued 1-form and 2-form on Q and their pullbacks to T^*Q. In the last equation, the fact that $\mathcal{A}(T\pi_Q(X))=\sum_{i=1}^{m}\gamma_i\eta_i$ is used. Considering the special cases mentioned in the previous section,

(a) for linear nonholonomic constraints $\gamma_i\equiv 0$ in (3.14),
(b) for conservative holonomic systems, the $\mathbf{M}_i$s are constant and the Lie derivatives H_i of H in the group directions vanish, and
(c) for holonomic constraints $d\mathcal{A}=0$ and hence the 2-form on the left hand side of (3.14) is closed.

Defining the 2-form $\omega_{nh}\in\Omega^2(T^*Q)$ and the 1-form $\lambda\in\Omega^1(T^*Q)$ as:

$$\omega_{nh}:=\omega_{can}+d\langle\mathbf{M},\mathcal{A}\rangle-\langle\mathbf{M},\mathcal{B}\rangle, \tag{3.15}$$

$$\lambda:=dH-\sum_{i=1}^{m}\left(\gamma_i d\mathbf{M}_i+H_i\beta_i-\gamma_i\left\langle\mathrm{ad}^*_{\eta_i}\mathbf{M},\mathcal{A}\right\rangle\right), \tag{3.16}$$

equation (3.14) gets the form of Hamilton's equation, i.e.,

$$\iota_X\omega_{nh}=\lambda. \tag{3.17}$$

The difference is that ω_{nh} is not necessarily closed or non-degenerate, and λ is not necessarily exact.

Lemma 3.1 *The interior product of the 2-form $\omega_{nh}\in\Omega^2(T^*Q)$ with any vector in the $2m$-dimensional sub-bundle $\mathcal{Y}:=\mathcal{Y}_1\oplus_{T^*Q}\mathcal{Y}_2\subset TT^*Q$ vanishes, where*

$$\mathcal{Y}_1(q,p):=\left\{\xi_{T^*Q}(q,p)\in T_{(q,p)}(T^*Q)|\xi\in Lie(G)\right\}$$
$$\mathcal{Y}_2(q,p):=\left\{w\in T_{(q,p)}(T^*Q)\,\middle|\,\iota_w\omega_{can}\in span\{\beta_i\}_{i=1,\cdots,m}\right\}.$$

In other words, $\iota(\mathcal{Y})\omega_{nh} = \{0\}$.

Proof. It is first shown that $\forall\xi \in \mathrm{Lie}(G)$, $\iota_{\xi_{T^*Q}}\omega_{nh} = 0$. By the definition of ω_{nh}, this becomes

$$\iota_{\xi_{T^*Q}}\omega_{can} + \iota_{\xi_{T^*Q}} d\langle \mathbf{M}, \mathcal{A}\rangle - \iota_{\xi_{T^*Q}}\langle \mathbf{M}, \mathcal{B}\rangle = 0,$$

By the definition of the momentum map, $\iota_{\xi_{T^*Q}}\omega_{can} = d\langle \mathbf{M}, \xi\rangle$. Further, Cartan's formula gives $\iota_{\xi_{T^*Q}} d\langle \mathbf{M}, \mathcal{A}\rangle = \mathcal{L}_{\xi_{T^*Q}}\langle \mathbf{M}, \mathcal{A}\rangle - d\langle \mathbf{M}, \xi\rangle$. Finally, the definition of the curvature 2-form leads to

$$\iota_{\xi_{T^*Q}}\langle \mathbf{M}, \mathcal{B}\rangle = \langle \mathbf{M}, d\mathcal{A}(\mathrm{hor}(\xi_Q), \mathrm{hor}(\cdot))\rangle = 0,$$

since the horizontal component of a vertical vector field is zero. It remains to show that $\mathcal{L}_{\xi_{T^*Q}}\langle \mathbf{M}, \mathcal{A}\rangle = 0$, or equivalently prove the invariance of $\langle \mathbf{M}, \mathcal{A}\rangle$ under the G-action. For every $g \in G$

$$\begin{aligned} T^*(T^*\Phi_g)\langle \mathbf{M}, \mathcal{A}\rangle &= \left\langle \mathbf{M}\circ T^*\Phi_g, \mathcal{A}\circ T\Phi_{g^{-1}}\right\rangle \\ &= \left\langle \mathrm{Ad}_g^*\mathbf{M}, \mathrm{Ad}_{g^{-1}}\mathcal{A}\right\rangle = \langle \mathbf{M}, \mathcal{A}\rangle, \end{aligned}$$

which is a consequence of the Ad-equivariance of $\mathcal{A}$ and $\mathbf{M}$. This completes the proof that $\iota(\mathcal{Y}_1)\omega_{nh} = \{0\}$.

Secondly, it should be shown that $\iota_W\omega_{nh} = 0$, $\forall W \in \mathfrak{X}(T^*Q)$ where $\iota_W\omega_{can} = \sum_{i=1}^m f_i\beta_i$ for smooth functions f_i. That is, it should be shown that for such W

$$\iota_W\omega_{can} + \iota_W d\langle \mathbf{M}, \mathcal{A}\rangle - \iota_W\langle \mathbf{M}, \mathcal{B}\rangle = 0.$$

Since the interior product of W with any basic differential form with respect to the canonical projection map π_Q vanishes, the relation $\iota_W\langle \mathbf{M}, \mathcal{B}\rangle = 0$ holds. Therefore, it only remains to show that $\iota_W d\langle \mathbf{M}, \mathcal{A}\rangle = -\iota_W\omega_{can}$:

$$\begin{aligned} \iota_W d\langle \mathbf{M}, \mathcal{A}\rangle &= \iota_W \sum_{i=1}^m d(\mathbf{M}_i\beta_i) = \iota_W \sum_{i=1}^m (d\mathbf{M}_i \wedge \beta_i + \mathbf{M}_i(d\beta_i)) \\ &= \sum_{i=1}^m (\iota_W d\mathbf{M}_i)\beta_i = \sum_{i=1}^m \left(-\langle \iota_W\omega_{can}, (\eta_i)_{T^*Q}\rangle\right)\beta_i \\ &= -\sum_{j=1}^m\sum_{i=1}^m \left(f_j\langle \beta_j, (\eta_i)_{T^*Q}\rangle\right)\beta_i \\ &= -\sum_{j=1}^m\sum_{i=1}^m \left(f_j\langle \alpha_j, (\eta_i)_Q\rangle\right)\beta_i \\ &= -\sum_{i=1}^m f_i\beta_i = -\iota_W\omega_{can}. \end{aligned}$$

In the above calculation, the first equality follows from (2.11). The third equality is the result of the relations $\iota_W d\beta_i = 0$ and $\langle \beta_i, W\rangle = 0$. The definition of the momentum map leads to the forth equality, and the fifth and sixth equations follow from the properties of W and β_i. This completes the proof that $\iota(\mathcal{Y}_2)\omega_{nh} = \{0\}$.

Note that the dimensions of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are both equal to m, since the vector fields $\{(\eta_i)_{T^*Q} | i = 1, \cdots, m\}$ (spanning $\mathcal{Y}_1$) and the 1-forms $\{\beta_i | i = 1, \cdots, m\}$ (forming $\mathcal{Y}_2$) are linearly independent at each point of T^*Q.

Lemma 3.2 *The 1-form $\lambda \in \Omega^1(T^*Q)$ vanishes in the directions of the infinitesimal G-action, i.e.,* $\iota(\mathcal{Y}_1)\lambda = \{0\}$.

Proof. Based on the definition of λ, to prove this lemma it is enough to show that the following equations hold $\forall\xi \in \mathrm{Lie}(G)$:

(a) $\mathcal{L}_{\xi_{T^*Q}} H = \sum_{i=1}^m H_i\langle \beta_i, \xi_{T^*Q}\rangle$,
(b) $\mathcal{L}_{\xi_{T^*Q}}\mathbf{M}_i = \left\langle \mathrm{ad}_{\eta_i}^*\mathbf{M}, \mathcal{A}(\xi_Q)\right\rangle$.

(a): Let $\xi = \sum_{i=1}^{m} \xi_i \eta_i$ be the coordinate expression of ξ in the chosen basis for Lie(G). The proof of this equation is a consequence of the linearity of the infinitesimal group action and the relation $\langle \beta_i, \xi_{T^*Q} \rangle = \langle \alpha_i, \xi_Q \rangle = \xi_i$:

$$\mathcal{L}_{\xi_{T^*Q}} H = \sum_{i=1}^{m} \xi_i \mathcal{L}_{(\eta_i)_{T^*Q}}(H) = \sum_{i=1}^{m} \xi_i H_i = \sum_{i=1}^{m} H_i \langle \beta_i, \xi_{T^*Q} \rangle .$$

(b): The second equation is the result of the following computation:

$$\begin{aligned} \mathcal{L}_{\xi_{T^*Q}} \mathbf{M}_i = \mathcal{L}_{\xi_{T^*Q}} \langle \mathbf{M}, \eta_i \rangle &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left\langle \mathbf{M} \circ T^* \Phi_{\exp(-\varepsilon\xi)}, \eta_i \right\rangle \\ &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left\langle \mathrm{Ad}^*_{\exp(-\varepsilon\xi)} \mathbf{M}, \eta_i \right\rangle = \left\langle \mathrm{ad}^*_{-\xi} \mathbf{M}, \eta_i \right\rangle \\ &= \langle \mathbf{M}, [\eta_i, \xi] \rangle = \left\langle \mathrm{ad}^*_{\eta_i} \mathbf{M}, \xi \right\rangle = \left\langle \mathrm{ad}^*_{\eta_i} \mathbf{M}, \mathcal{A}(\xi_Q) \right\rangle . \end{aligned} \tag{3.18}$$

Corollary 3.3 *The solution curves of the equation (3.17) are always in the submanifold $\mathcal{M} = \mathbb{F}L(\mathcal{D}) \subseteq T^*Q$.*

Proof. This corollary is a consequence of $\iota(\mathcal{Y}_2)\omega_{nh} = \{0\}$. Let W be a vector field in $\mathcal{Y}_2$, with $\iota_W \omega_{can} = \sum_{i=1}^{m} f_i \beta_i$ for $f_i \in C^\infty(T^*Q)$. Contracting both sides of (3.17) with W leads to the equation $\langle \lambda, W \rangle = 0$. Let V be the Hamiltonian vector field, i.e., $\iota_V \omega_{can} = dH$. Based on the definition of λ and the fact that the contraction of W with any basic form (with respect to the projection map π_Q) vanishes:

$$\begin{aligned} 0 = \langle dH, W \rangle - \sum_{i=1}^{m} \gamma_i \langle d\mathbf{M}_i, W \rangle &= \langle \iota_V \omega_{can}, W \rangle + \sum_{i=1}^{m} \gamma_i f_i \\ &= \langle -\iota_W \omega_{can}, V \rangle + \sum_{i=1}^{m} \gamma_i f_i \\ &= \sum_{i=1}^{m} f_i \left(- \langle \beta_i, V \rangle + \gamma_i \right) = \sum_{i=1}^{m} f_i \left(- \langle \alpha_i, T\pi_Q(V) \rangle + \gamma_i \right). \end{aligned}$$

Since for every element (q,p) of the cotangent bundle $T\pi_Q(V)(q,p) = \mathbb{F}L_q^{-1}(p)$ and since the functions f_is are arbitrary, the above equation hold if and only if

$$\left\langle \alpha_i(q), \mathbb{F}L_q^{-1}(p) \right\rangle = \gamma_i(q),$$

for every $i = 1, \cdots, m$. Thus, $(q,p) \in \mathbb{F}L(\mathcal{D}) = \mathcal{M}$. Therefore, any solution curve of the equation (3.17) must be in $\mathcal{M}$.

Thus, the Hamilton-d'Alembert equation (3.17) does not have a solution outside of the constrained submanifold $\mathcal{M}$. In fact, there exists a unique vector field on $\mathcal{M}$ whose elements satisfy (3.17). Let $\mathfrak{I}_{\mathcal{M}} \colon \mathcal{M} \hookrightarrow T^*Q$ denote the canonical inclusion map. The equation (3.17) can now be pulled back by $\mathfrak{I}_{\mathcal{M}}$ to give the Hamilton-d'Alembert equation on $\mathcal{M}$:

$$\iota_{\widetilde{X}} \widetilde{\omega}_{nh} = \widetilde{\lambda}, \tag{3.19}$$

where $\widetilde{\omega}_{nh} = T^*\mathfrak{I}_{\mathcal{M}}(\omega_{nh})$, $\widetilde{\lambda} = T^*\mathfrak{I}_{\mathcal{M}}(\lambda)$, and $T\mathfrak{I}_{\mathcal{M}}(\widetilde{X}) = X \circ \mathfrak{I}_{\mathcal{M}}$.

Note that Lemmas 3.1 and 3.2 state that the left and right terms in the equation (3.19) trivially vanish in the vertical directions, i.e., the directions of the infinitesimal G-action. Both $\widetilde{\omega}_{nh}$ and $\widetilde{\lambda}$ can be restricted to a horizontal vector sub-bundle. Without loss of generality, assume that the vector field Y is always in the vertical sub-bundle of TQ corresponding to the G-action, i.e., $Y \subset \mathcal{V}$. Therefore, both sides of (3.19) can be restricted to the horizontal sub-bundle $T^\Delta \mathcal{M} := \left\{ \widetilde{w} \in T\mathcal{M} \,\middle|\, T\pi_Q|_{\mathcal{M}}(\widetilde{w}) \in \Delta \right\}$ with the canonical inclusion map $\mathfrak{I}_\Delta \colon T^\Delta \mathcal{M} \hookrightarrow T\mathcal{M}$. The tangent vector X satisfies

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(3.17) if and only if it satisfies

$$\iota_{\widetilde{X}}\, \widetilde{\omega}_{nh}|_{T^\Delta\mathcal{M}} = \widetilde{\lambda}\Big|_{T^\Delta\mathcal{M}}. \tag{3.20}$$

$\widetilde{\omega}_{nh}|_{T^\Delta\mathcal{M}}$ is a non-degenerate 2-form on the sub-bundle $T^\Delta\mathcal{M}$.

3.2 Step 2: Quotienting the Constrained Phase Space

In this section under some conditions, the constrained phase space $\mathcal{M}$ with the Hamilton-d'Alembert equation (3.19) on $\mathcal{M}$ are quotiented by a Lie group action. Let E be a Lie subgroup of G whose infinitesimal action on T^*Q is tangent to $\mathcal{M}$. That is, $\mathcal{M}$ is invariant under the restricted E-action (to $\mathcal{M}$), which is denoted by $\Psi_e\colon \mathcal{M} \to \mathcal{M}$ for every $e \in E$. The dimension of E is denoted by l and without loss of generality it is assumed that $\{\eta_i \in \mathrm{Lie}(G) | i = 1,\cdots,l \le m\}$ is a basis for $\mathrm{Lie}(E) \subseteq \mathrm{Lie}(G)$.

Corollary 3.4 *The 2-form $\widetilde{\omega}_{nh} \in \Omega^2(\mathcal{M})$ and the 1-form $\widetilde{\lambda} \in \Omega^1(\mathcal{M})$ vanish in the directions of the infinitesimal E-action.*

Proof. This is the immediate consequence of Lemmas 3.1 and 3.2.

Lemma 3.5 *The 2-form $\widetilde{\omega}_{nh} \in \Omega^2(\mathcal{M})$ is invariant under the action of E.*

Proof. It suffices to show that $\omega_{nh} \in \Omega^2(T^*Q)$ is invariant under the G-action. Then, this lemma follows immediately, considering the fact that $\mathcal{M}$ is invariant under the E-action.

Invariance of ω_{nh} under the G-action is a consequence of the invariance of each of its terms: (a) The 2-form ω_{can} is invariant under any cotangent lifted group action [1]; (b) Ad-equivariance of $\mathcal{A}$ and $\mathbf{M}$ results in the G-invariance of the 1-form $\langle \mathbf{M}, \mathcal{A}\rangle$ and hence its exterior derivative $d\langle \mathbf{M}, \mathcal{A}\rangle$; (c) Ad-equivariance of $\mathcal{B}$ and $\mathbf{M}$ results in the G-invariance of the 1-form $\langle \mathbf{M}, \mathcal{B}\rangle$.

To simplify the notation, let $\widetilde{H}_i := H_i \circ \mathfrak{I}_\mathcal{M}$, $\widetilde{H}_{ij} := \mathcal{L}_{(\eta_j)_\mathcal{M}}(\widetilde{H}_i)$, $\gamma_{ij} := \mathcal{L}_{(\eta_j)_\mathcal{M}}(\gamma_i)$, $\widetilde{\mathbf{M}}_i := \mathbf{M}_i \circ \mathfrak{I}_\mathcal{M}$ and likewise $\widetilde{\mathbf{M}} := \mathbf{M} \circ \mathfrak{I}_\mathcal{M}$ for $i = 1,\cdots,m$ and $j = 1,\cdots,l$. The notation used for the pullback of β_i, γ_i and $\mathcal{A}$ is the same as the corresponding notation in the original phase space.

Lemma 3.6 *The 1-form $\widetilde{\lambda} \in \Omega^1(\mathcal{M})$ is invariant under the E-action if and only if the following relations hold:*

$$\begin{aligned} d\widetilde{H}_j = \sum_{i=1}^{m} \Big[& \gamma_{ij}\left(d\widetilde{\boldsymbol{M}}_i - \left\langle \widetilde{\boldsymbol{M}}, ad_{\eta_i}\mathcal{A}\right\rangle\right) + \widetilde{H}_{ij}\beta_i + \widetilde{H}_i \left\langle \eta_i^*, ad_{\eta_j}\mathcal{A}\right\rangle \\ & -\gamma_i\left(\left\langle \widetilde{\boldsymbol{M}}, ad_{[\eta_i,\eta_j]}\mathcal{A}\right\rangle - d\left\langle \widetilde{\boldsymbol{M}}, [\eta_i,\eta_j]\right\rangle\right)\Big].\ \ (j = 1,\cdots l) \end{aligned} \tag{3.21}$$

Proof. The proof is a straight forward calculation of the Lie derivative of the 1-form $\widetilde{\lambda}$ in the directions of the infinitesimal E-action. $\forall \zeta \in \mathrm{Lie}(E)$,

$$\iota_{\zeta_\mathcal{M}}\widetilde{\lambda} = \iota_{\zeta_\mathcal{M}}\left[T^*\mathfrak{I}_\mathcal{M}(\lambda)\right] = T^*\mathfrak{I}_\mathcal{M}\left[\iota_{\zeta_{T^*Q}}\lambda\right] = 0,$$

by Lemma 3.2. Hence,

$$\begin{aligned} \mathcal{L}_{\zeta_\mathcal{M}}\widetilde{\lambda} &= \iota_{\zeta_\mathcal{M}}(d\widetilde{\lambda}) + d\left(\iota_{\zeta_\mathcal{M}}\widetilde{\lambda}\right) = \iota_{\zeta_\mathcal{M}}(d\widetilde{\lambda}) \\ &= T^*\mathfrak{I}_\mathcal{M}\left(\iota_{\zeta_{T^*Q}}(d\lambda)\right). \end{aligned}$$

By (3.16),

$$\begin{aligned} d\lambda = -\sum_{i=1}^{m} [& d\gamma_i \wedge (d\mathbf{M}_i - \langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle) + dH_i \wedge \beta_i + H_i d\beta_i \\ & -\gamma_i d\langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle]. \end{aligned}$$

Expressing $\zeta = \sum_{j=1}^{l} \zeta_j \eta_j$ in the chosen basis for $\mathrm{Lie}(E)$, the interior product of $d\lambda$ with ζ_{T^*Q} is calculated as:

$$
\begin{aligned}
\iota_{\zeta_{T^*Q}}(d\lambda) = -\sum_{j=1}^{l} \zeta_j \sum_{i=1}^{m} [\gamma_{ij}\,(d\mathbf{M}_i - \langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle) \\
- \left(\mathcal{L}_{(\eta_j)_{T^*Q}}\mathbf{M}_i - \langle \mathbf{M}, [\eta_i, \eta_j]\rangle\right) d\gamma_i \\
+ H_{ij}\beta_i - \delta_i^j dH_i + H_i \iota_{(\eta_j)_{T^*Q}}(d\beta_i) \\
- \gamma_i \iota_{(\eta_j)_{T^*Q}}\left(d\,\langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle\right)].
\end{aligned}
$$

By (3.18) with ξ replaced by η_j, the coefficient of $d\gamma_i$ is zero for all i. Also,

$$
\begin{aligned}
\iota_{(\eta_j)_{T^*Q}}(d\beta_i) &= \mathcal{L}_{(\eta_j)_{T^*Q}}(\beta_i) - d\left(\iota_{(\eta_j)_{T^*Q}}(\beta_i)\right) = \mathcal{L}_{(\eta_j)_{T^*Q}}(\beta_i) \\
&= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left\langle \eta_i^*, \mathcal{A} \circ T\Phi_{\exp(\varepsilon\eta_j)} \right\rangle = \left\langle \eta_i^*, \mathrm{ad}_{\eta_j}\mathcal{A} \right\rangle,
\end{aligned}
$$

by the Ad-equivariance of $\mathcal{A}$. Further, due to the Ad-equivariance of $\mathcal{A}$ and $\mathbf{M}$:

$$
\begin{aligned}
\iota_{(\eta_j)_{T^*Q}}\left(d\,\langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle\right) &= \mathcal{L}_{(\eta_j)_{T^*Q}}\left(\langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle\right) \\
&\quad - d\,\langle \mathbf{M}, [\eta_i, \eta_j]\rangle \\
&= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left\langle \mathbf{M} \circ T^*\Phi_{\exp(-\varepsilon\eta_j)}, \mathrm{ad}_{\eta_i}\mathcal{A} \circ T\Phi_{\exp(\varepsilon\eta_j)} \right\rangle \\
&\quad - d\,\langle \mathbf{M}, [\eta_i, \eta_j]\rangle \\
&= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} \left\langle \mathrm{Ad}^*_{\exp(-\varepsilon\eta_j)}\mathbf{M}, \mathrm{ad}_{\eta_i}\mathrm{Ad}_{\exp(\varepsilon\eta_j)}\mathcal{A} \right\rangle \\
&\quad - d\,\langle \mathbf{M}, [\eta_i, \eta_j]\rangle = \left\langle \mathbf{M}, \mathrm{ad}_{[\eta_i, \eta_j]}\mathcal{A} \right\rangle - d\,\langle \mathbf{M}, [\eta_i, \eta_j]\rangle.
\end{aligned}
$$

Using the above calculations,

$$
\begin{aligned}
\iota_{\zeta_{T^*Q}}(d\lambda) = -\sum_{j=1}^{l} \zeta_j \Big(-dH_j + \sum_{i=1}^{m} [\gamma_{ij}\,(d\mathbf{M}_i - \langle \mathbf{M}, \mathrm{ad}_{\eta_i}\mathcal{A}\rangle) \\
+ H_{ij}\beta_i + H_i \left\langle \eta_i^*, \mathrm{ad}_{\eta_j}\mathcal{A} \right\rangle \\
- \gamma_i \left(\left\langle \mathbf{M}, \mathrm{ad}_{[\eta_i, \eta_j]}\mathcal{A} \right\rangle - d\,\langle \mathbf{M}, [\eta_i, \eta_j]\rangle\right)]\Big).
\end{aligned}
$$

$\widetilde{\lambda}$ is invariant under the E-action if and only if $\mathcal{L}_{\zeta_{\mathcal{M}}}\widetilde{\lambda} = 0$ for all $\zeta \in \mathrm{Lie}(E)$. By Lemma 3.2 this holds if and only if $\iota_{\zeta_{T^*Q}} d\lambda = 0$, which is equivalent to the condition (3.21).

Corollary 3.7 *In the following special cases the condition (3.21) is satisfied:*

(a) For a Chaplygin system, where the distribution is linear ($\gamma_i = 0$), the Lie group E is all of G, and the Hamiltonian is G-invariant,

(b) For a Hamiltonian mechanical system with symmetry, where the Hamiltonian is G-invariant, the constrained submanifold $\mathcal{M}$ is $\boldsymbol{M}^{-1}(\mu)$ for some constant $\mu \in \mathrm{Lie}^*(G)$, and the Lie subgroup E is the isotropy group G_μ, and

(c) For a Hamiltonian mechanical system with holonomic constraints, where $\alpha_i = df_i$ for $f_i \in C^\infty(Q)$ and $d\widetilde{H}_j = \sum_{i=1}^{m} \gamma_{ij} d\widetilde{\boldsymbol{M}}_i + \widetilde{H}_{ij} T^*\pi_Q(df_i)$.

Based on Corollary 3.4 and under the condition (3.21), both sides of the Hamilton-d'Alembert equation (3.19) can be expressed in the quotient manifold $\mathcal{M}/E$, with the projection map $\pi\colon \mathcal{M} \to \mathcal{M}/E$.

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Theorem 3.8 *Under the assumptions (a), (b) and (c) stated in Definition 2.3, a constrained Hamiltonian mechanical system with symmetry* $(T^*Q, \omega_{can}, H, \mathcal{D}, E \subseteq G)$ *can be reduced to a system whose dynamics is represented by* $(\mathcal{M}/E, \widehat{\omega}_{nh}, \widehat{\lambda})$*, with the Hamilton-d'Alembert equation*

$$\iota_{\breve{X}} \breve{\omega}_{nh} = \breve{\lambda}, \tag{3.22}$$

where $T^*\pi(\breve{\omega}_{nh}) := \widetilde{\omega}_{nh}$*,* $T^*\pi(\breve{\lambda}) := \widetilde{\lambda}$*, and* $\breve{X} := T\pi(\widetilde{X})$*.*

Note that if $l \neq m$, the 2-form $\breve{\omega}_{nh}$ is still degenerate, and in the vertical directions the above equation trivially vanishes. In order to remove the degeneracy, both sides of (3.22) could be restricted to $T\pi(T^{\Delta}\mathcal{M})$, the image of the horizontal distribution under the quotient map.

3.3 Step 3: Identifying the Quotient Manifold $\mathcal{M}/E$

In this section, a map is constructed from the quotient manifold $\mathcal{M}/E$ to a sub-bundle of the cotangent bundle of the quotiented configuration space $\widehat{Q} := Q/E$. The roadmap is to find an E-equivariant diffeomorphism from $\mathcal{M}$ to $\mathbf{M}^{-1}(0)$ and use the theory of cotangent bundle reduction at zero momentum.

Define the fibre-wise linear map $\mathbb{P} \colon T^*Q \to T^*Q$ by

$$\mathbb{P}_q(p) := p - \mathcal{A}_q^* \circ \mathbf{M}_q(p),$$

for all $(q, p) \in T_q^*Q$. The image of this map is the submanifold $\mathbf{M}^{-1}(0) \subset T^*Q$, since

$$\begin{aligned}
\mathbf{M}_q(p - \mathcal{A}_q^* \circ \mathbf{M}_q(p)) &= \mathbf{M}_q(p) - \phi_q^* \circ \mathcal{A}_q^* \circ \mathbf{M}_q(p) \\
&= \mathbf{M}_q(p) - (\mathcal{A} \circ \phi)_q^* \circ \mathbf{M}_q(p) \\
&= \mathbf{M}_q(p) - \mathbf{M}_q(p) = 0.
\end{aligned}$$

This uses the definition of the momentum map and the relation $\mathcal{A}_q \circ \phi_q = \mathbf{Id}_{\mathrm{Lie}(E)}$. The map $\mathbb{P}$ is a projection map whose restriction to the affine sub-bundle $\mathcal{M}$ is a diffeomorphism, denoted $\rho := \mathbb{P} \circ \mathfrak{I}_{\mathcal{M}} \colon \mathcal{M} \to \mathbf{M}^{-1}(0)$. Indeed, this map is a shear transformation along the span of α_is ($i = 1, \cdots, m$), which is everywhere orthogonal to $\mathcal{M}$ with respect to the induced metric on T^*Q by K. In addition, the span of the α_is is always transverse to $\mathbf{M}^{-1}(0)$, since the α_is were chosen such that $\langle \alpha_i, (\eta_i)_Q \rangle = \langle \phi_q^*(\alpha_i), \eta_i \rangle = \langle \mathbf{M}_q(\alpha_i), \eta_i \rangle = 1$ for all $i = 1, \cdots, m$. Considering the fact that $\dim(\mathcal{M}) = \dim(\mathbf{M}^{-1}(0))$ and the preceding argument, it follows that ρ is a diffeomorphism.

It is assumed that $\mathcal{M}$ is E-invariant, also $\mathbf{M}^{-1}(0)$ is invariant under the cotangent lifted G-action (and hence, E-action). Hence, the map ρ is E-equivariant due to the Ad-equivariance of $\mathbf{M}$ and $\mathcal{A}$: $\forall (q, p) \in \mathcal{M}$ and $e \in E$,

$$\begin{aligned}
\rho(\Psi_e(q, p)) &= \Psi_e(q, p) - \mathcal{A}_q^* \circ \mathbf{M}(\Psi_e(q, p)) \\
&= \Psi_e(q, p) - \mathcal{A}_q^* \circ \mathrm{Ad}_e^* \circ \mathbf{M}_q(p) \\
&= \Psi_e(q, p) - \Psi_e \circ \mathcal{A}_q^* \circ \mathbf{M}_q(p) = \Psi_e(\rho(q, p)).
\end{aligned}$$

As a result, this map descends to $\widehat{\rho} \colon \mathcal{M}/E \to \mathbf{M}^{-1}(0)/E$, which is defined by the relation $\pi_0 \circ \rho = \widehat{\rho} \circ \pi$, where π_0 is the canonical projection map for the E-action on $\mathbf{M}^{-1}(0)$. In the following, some notions and maps from the theory of cotangent bundle reduction are briefly introduced. A comprehensive discussion can be found in [17] and the references therein.

In this theory, a map $\tau_0 \colon \mathbf{M}^{-1}(0) \to T^*\widehat{Q}$ is defined by

$$\left\langle \tau_0(q, p), T_q\widehat{\pi}(v) \right\rangle = \langle p, v \rangle, \tag{3.23}$$

for all $(q, p) \in \mathbf{M}^{-1}(0)$ and $v \in T_qQ$, where $\widehat{Q} = Q/E$ and $\widehat{\pi} \colon Q \to \widehat{Q}$ is the projection map. Further, a symplectic embedding $\widehat{\tau}_0 \colon \mathbf{M}^{-1}(0)/E \to T^*\widehat{Q}$ is determined by the relation $\widehat{\tau}_0 \circ \pi_0 = \tau_0$, where $T^*\widehat{\tau}_0\widehat{\omega}_{can} = \omega_0$ with $\widehat{\omega}_{can}$ being the canonical 2-form on $T^*\widehat{Q}$ and ω_0 the reduced symplectic 2-form that satisfies the relation $T^*\mathfrak{I}_0\omega_{can} = T^*\pi_0\omega_0$. Here the map $\mathfrak{I}_0 \colon \mathbf{M}^{-1}(0) \hookrightarrow T^*Q$ is the inclusion map. The image of this embedding is the vector sub-bundle $[T\widehat{\pi}(\overline{\mathcal{V}})]^0 \subseteq T^*\widehat{Q}$. Here, ^{0} indicates the

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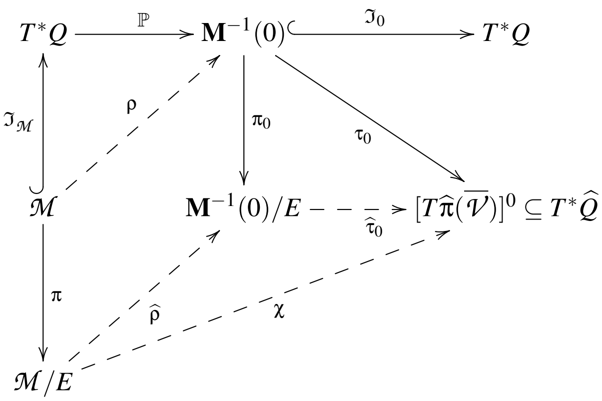
annihilator with respect to the natural pairing of the tangent and cotangent bundle, and $\overline{\mathcal{V}} = \{\xi_Q \in \mathfrak{X}(Q) | \xi \in \mathrm{Lie}(G)\} \subseteq TQ$ is the vertical bundle corresponding to the G-action. Composing with $\widehat{\rho}$ gives an embedding $\chi := \widehat{\tau}_0 \circ \widehat{\rho} \colon \mathcal{M}/E \to T^*\widehat{Q}$, summarized in the following commuting diagram.



(3.24)

Lemma 3.9 *The following relation holds:*

$$T^*(\chi \circ \pi)(\widehat{\omega}_{can}) = T^*\mathfrak{I}_{\mathcal{M}}(\omega_{can} + d\langle \boldsymbol{M}, \mathcal{A}\rangle). \tag{3.25}$$

Proof. The above commuting diagram and the equality $T^*\mathfrak{I}_0(\omega_{can}) = T^*\pi_0(\omega_0)$ give the following relation:

$$\begin{aligned} T^*(\chi \circ \pi)(\widehat{\omega}_{can}) &= T^*(\widehat{\tau}_0 \circ \pi_0 \circ \rho)(\widehat{\omega}_{can}) = T^*(\pi_0 \circ \rho)(\omega_0) \\ &= T^*(\mathfrak{I}_0 \circ \rho)(\omega_{can}) = T^*(\mathfrak{I}_0 \circ \mathbb{P} \circ \mathfrak{I}_{\mathcal{M}})(\omega_{can}) \\ &= T^*\mathfrak{I}_{\mathcal{M}} T^*(\mathfrak{I}_0 \circ \mathbb{P})(\omega_{can}) \\ &= T^*\mathfrak{I}_{\mathcal{M}}(\omega_{can} + d\langle \mathbf{M}, \mathcal{A}\rangle). \end{aligned}$$

The last equality is true because of the definition of the map $\mathbb{P}$, which implies that $T^*(\mathfrak{I}_0 \circ \mathbb{P})\theta_{can} = T^*\mathfrak{I}_{\mathcal{M}}(\theta_{can} - \langle \mathbf{M}, \mathcal{A}\rangle)$, where θ_{can} is the tautological 1-form with $\omega_{can} = d\theta_{can}$.

Define a horizontal lift map $\mathrm{hl}^{\mathcal{M}} \colon [T\widehat{\pi}(\overline{\mathcal{V}})]^0 \to \mathcal{M}$ to the affine sub-bundle $\mathcal{M}$ by

$$\mathrm{hl}^{\mathcal{M}}_q(\widehat{p}) = \rho_q^{-1} \circ \tau_0^{-1}(\widehat{q}, \widehat{p}),$$

where $(\widehat{q}, \widehat{p}) \in [T\widehat{\pi}(\mathcal{V})]^0$ and $\widehat{\pi}(q) = \widehat{q}$. Note that, by (3.23) $\tau_0^{-1}(\widehat{q}, \widehat{p}) = T_q^*\widehat{\pi}(\widehat{p})$.

Lemma 3.10 *The map* $\mathrm{hl}^{\mathcal{M}}$ *is E-equivariant.*

Proof. It suffices to show that $\forall e \in E$ and $\forall q \in Q$,

$$\mathrm{hl}^{\mathcal{M}}_{\Phi_e(q)} = T^*\Phi_{e^{-1}} \circ \mathrm{hl}^{\mathcal{M}}_q.$$

The definition of $\mathrm{hl}^{\mathcal{M}}$ and the E-equivariance of ρ give

$$\begin{aligned} \mathrm{hl}^{\mathcal{M}}_{\Phi_e(q)} &= \rho^{-1}_{\Phi_e(q)} \circ T^*_{\Phi_e(q)}\widehat{\pi} = \rho^{-1}_{\Phi_e(q)} \circ T^*\Phi_{e^{-1}} \circ T_q^*\widehat{\pi} \\ &= (T^*\Phi_e \circ \rho_{\Phi_e(q)})^{-1} \circ T_q^*\widehat{\pi} = (\rho_q \circ T^*\Phi_e)^{-1} \circ T_q^*\widehat{\pi} \\ &= T^*\Phi_{e^{-1}} \circ \rho_q^{-1} \circ T_q^*\widehat{\pi} = T^*\Phi_{e^{-1}} \circ \mathrm{hl}^{\mathcal{M}}_q. \end{aligned}$$

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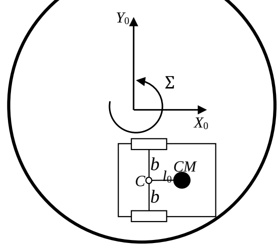

Fig. 1. A rover on a rotating disk with angular velocity Σ

Based on this lemma and the Ad-equivariance of the maps $\mathcal{A}$ and $\mathbf{M}$, a 2-form $\widehat{\Xi}$ on $[T\widehat{\pi}(\overline{\mathcal{V}})]^0$ is then defined by

$$\widehat{\Xi} = \left\langle \mathbf{M} \circ \mathfrak{J}_{\mathcal{M}} \circ \mathrm{hl}^{\mathcal{M}}, \mathcal{B}(\mathrm{hl}(\cdot), \mathrm{hl}(\cdot)) \right\rangle, \tag{3.26}$$

where $\mathrm{hl}_q = (T_q\widehat{\pi}|_{\widehat{\mathcal{H}}_q})^{-1}$ is the horizontal lift map of the quotient map $\widehat{\pi}$ corresponding to the E-action, and $\widehat{\mathcal{H}}$ is its horizontal bundle. Also, there exists a unique 1-form $\widehat{\lambda} \in \Omega^1([T\widehat{\pi}(\overline{\mathcal{V}})]^0)$ that is the pullback of $\widetilde{\lambda}$ by $\mathrm{hl}^{\mathcal{M}}$:

$$\widehat{\lambda} = T^*\mathrm{hl}^{\mathcal{M}}(\widetilde{\lambda}).$$

Theorem 3.11 *The dynamics of the reduced constrained Hamiltonian mechanical system with symmetry* $(\mathcal{M}/E, \breve{\omega}_{nh}, \breve{\lambda})$ *can be presented in the sub-bundle* $[T\widehat{\pi}(\overline{\mathcal{V}})]^0 \subseteq T^*\widehat{Q}$ *through the Hamilton-d'Alembert equation*

$$\iota_{\widehat{X}}\widehat{\omega}_{nh} = \widehat{\lambda}, \tag{3.27}$$

where $\widehat{\omega}_{nh} := \widehat{\omega}_{can} - \widehat{\Xi}$, $\widehat{\lambda} := T^*\mathrm{hl}^{\mathcal{M}}(\widetilde{\lambda})$, *and* $\widehat{X} := T\chi(\breve{X})$.

4 Case Study

In this section the dynamics of a symmetric Hamiltonian mechanical system with affine nonholonomic constraints is studied. The geometric approach introduced in this paper is used to systematically reduce the dynamical equations of a two-wheeled, differential drive rover on a rotating disk (with infinite radius). The top view of the system is shown in Fig. 1. To maintain the stability, a roller caster is needed in front of the system mass center; its dynamics is disregarded in this case study. The configuration manifold Q of this system is of dimension five and it is diffeomorphic to the Lie group $\mathrm{SE}(2) \times \mathrm{SO}(2) \times \mathrm{SO}(2)$. In a parametrization, any element of Q can be represented by $q = (x, y, \theta, \vartheta_1, \vartheta_2)$, where

(a) (x, y) specifies the position of the point C with respect to the inertial coordinate frame,
(b) θ is the heading angle of the rover with respect to the X_0-axis, and
(c) ϑ_1 and ϑ_2 are the rotation angles of the wheels.

The coordinate frame $X_0Y_0Z_0$ is an inertial frame at the center of the rotating disk. The Z_0-axis is perpendicular to the plane of motion. The constant angular velocity of the disk about Z_0 is denoted by Σ.

The momenta conjugate to the states of the system are then denoted by the vector $p = [p_x, p_y, p_\theta, p_{\vartheta_1}, p_{\vartheta_2}]^T$; hence, (q, p) is in the cotangent bundle T^*Q. Here, $[p_x, p_y]^T$ is the total linear momentum of the system with respect to $X_0Y_0Z_0$ and p_θ is the total angular momentum of the system about the point C. In this section the proposed reduction theory is formulated step-by-step. Throughout this section, the relations are presented as matrix equations, wherever this is convenient. However, for calculating exterior and Lie derivatives of forms, the language of differential forms is used. Then, the final results are presented in matrix form.

Based on the kinetic energy of the system, the Legendre transformation $\mathbb{F}L$: $TQ \to T^*Q$ is calculated in the local coordinates by the fibre-wise linear map:

$$\mathbb{F}L_q = \begin{bmatrix} M & 0 & -Ml_0\sin(\theta) & 0 & 0 \\ 0 & M & Ml_0\cos(\theta) & 0 & 0 \\ -Ml_0\sin(\theta) & Ml_0\cos(\theta) & J_r & 0 & 0 \\ 0 & 0 & 0 & J_w & 0 \\ 0 & 0 & 0 & 0 & J_w \end{bmatrix},$$

where M is the total mass of the rover, l_0 is the distance of the centre of mass from the point C, J_r is the moment of inertia of the system about C, and J_w is the wheels moment of inertia about their axle. The Hamiltonian is the total energy of the system:

$$\begin{aligned} H(q,p) &= \frac{1}{2}p^T\mathbb{F}L_q^{-1}p \\ &= \frac{p_{\vartheta_1}^2+p_{\vartheta_2}^2}{2J_w} + \frac{1}{2(J_r-Ml_0^2)}((J_r-Ml_0^2\cos^2(\theta))\frac{p_x^2}{M} \\ &+ (J_r-Ml_0^2\sin^2(\theta))\frac{p_y^2}{M} - l_0^2p_xp_y\sin(2\theta) + p_\theta^2 \\ &+ 2l_0p_\theta(p_x\sin(\theta)-p_y\cos(\theta))) \end{aligned}$$

The rover is experiencing nonholonomic constraints corresponding to the no-slip conditions at the wheels. The three linearly independent 1-forms describing these constraints are

$$\begin{aligned} &-\sin(\theta)dx+\cos(\theta)dy, \\ &\cos(\theta)dx+\sin(\theta)dy-bd\theta-r_wd\vartheta_1, \\ &\cos(\theta)dx+\sin(\theta)dy+bd\theta-r_wd\vartheta_2. \end{aligned}$$

Here, b denotes the distance from the point C to each wheel and r_w is the radius of the wheels. In this case study, the vector field Y that defines the affine nonholonomic distribution is determined as

$$Y(q) = \Sigma\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+\frac{\partial}{\partial\theta}\right).$$

This vector field specifies the linear and angular velocity of a coordinate frame attached to the point C induced by the rotating disk. The linear distribution Δ can then be written as the point-wise span of the following smooth vector fields:

$$\Delta(q) = \text{span}\left\{\frac{\partial}{\partial\vartheta_1}+\frac{r_w}{2}\left(\cos(\theta)\frac{\partial}{\partial x}+\sin(\theta)\frac{\partial}{\partial y}-\frac{1}{b}\frac{\partial}{\partial\theta}\right),\right.$$

$$\left.\frac{\partial}{\partial\vartheta_2}+\frac{r_w}{2}\left(\cos(\theta)\frac{\partial}{\partial x}+\sin(\theta)\frac{\partial}{\partial y}+\frac{1}{b}\frac{\partial}{\partial\theta}\right)\right\}.$$

When the angular velocity of the disk is zero, these vector fields summarize the relationship between the speed of the wheels and the speed of the rover in the plane of motion. The affine distribution is then $\mathcal{D} = \Delta + Y$. Let the Lie group G be SE(2), in this case study, with an action on Q defined by the left translation on the relative configuration manifold of the rover with respect to the inertial coordinate frame $X_0Y_0Z_0$. That is, $\forall g = (x_0, y_0, \theta_0) \in \mathrm{SE}(2)$

$$\Phi_g(q) = (x\cos(\theta_0) - y\sin(\theta_0) + x_0, x\sin(\theta_0) + y\cos(\theta_0) + y_0,\\ \theta + \theta_0, \vartheta_1, \vartheta_2).$$

In the matrix representation, the following elements of $\mathrm{se}(2)$ are considered as the basis for the Lie algebra:

$$\eta_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \eta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \eta_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which correspond to the linear motion in the X and Y directions and angular motion about the Z axis, respectively. Then in this basis, the infinitesimal action of $\mathrm{se}(2)$ can be represented by the fibre-wise linear map:

$$\phi_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -y & x & 1 & 0 & 0 \end{bmatrix}^T,$$

and the momentum map is determined through $\mathbf{M}_q = \phi_q^T$, such that

$$\mathbf{M}_q(p) = \begin{bmatrix} p_x & p_y & -yp_x + xp_y + p_\theta \end{bmatrix}^T.$$

The last component of the momentum map is the total angular momentum of the rover about the Z_0 axis.

Based on the definition of the infinitesimal action, the induced vector fields corresponding to the basis elements of $\mathrm{se}(2)$ are determined as

$$(\eta_1)_Q = \frac{\partial}{\partial x}, (\eta_2)_Q = \frac{\partial}{\partial y}, (\eta_3)_Q = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}.$$

The quotient manifold $\overline{Q} = Q/\mathrm{SE}(2)$ gives rise to the principal bundle $\overline{\pi}\colon Q \to \overline{Q}$ with the Ad-equivariant principal connection $\mathcal{A}\colon TQ \to \mathrm{se}(2)$ whose matrix representation in the chosen basis for $\mathrm{se}(2)$ is

$$\mathcal{A}_q = \sum_{i=1}^{3} \alpha_i \eta_i \\ := \begin{bmatrix} 1 & 0 & y & -\frac{r_w}{2}\left(\cos(\theta) - \frac{y}{b}\right) & -\frac{r_w}{2}\left(\cos(\theta) + \frac{y}{b}\right) \\ 0 & 1 & -x & -\frac{r_w}{2}\left(\sin(\theta) + \frac{x}{b}\right) & -\frac{r_w}{2}\left(\sin(\theta) - \frac{x}{b}\right) \\ 0 & 0 & 1 & \frac{r_w}{2b} & -\frac{r_w}{2b} \end{bmatrix},$$

where the relation $\langle \alpha_i, (\eta_j)_Q \rangle = \delta_i^j$ holds for every $i, j = 1, 2, 3$. In this case study, the functions γ_i are then calculated through

$$\gamma_1(q) = \langle \alpha_1, Y \rangle = 0, \gamma_2(q) = \langle \alpha_2, Y \rangle = 0, \gamma_3(q) = \langle \alpha_3, Y \rangle = \Sigma.$$

The quotient manifold $\overline{Q}$ is the shape space of the system whose elements correspond to the internal degrees of freedom of the rover, i.e., the wheel angles. Further, the principal connection is used to define the vertical projection map by $\mathrm{ver}_q = \phi_q \mathcal{A}_q$.

Therefore, the horizontal projection corresponding to this connection is

$$\mathrm{hor}_q = \mathbf{1}_{5\times5} - \phi_q \mathcal{A}_q = \begin{bmatrix} 0\ 0\ 0 & \frac{r_w \cos(\theta)}{2} & \frac{r_w \cos(\theta)}{2} \\ 0\ 0\ 0 & \frac{r_w \sin(\theta)}{2} & \frac{r_w \sin(\theta)}{2} \\ 0\ 0\ 0 & -\frac{r_w}{2b} & \frac{r_w}{2b} \\ 0\ 0\ 0 & 1 & 0 \\ 0\ 0\ 0 & 0 & 1 \end{bmatrix}.$$

At each q, this transformation maps a velocity of the system to its corresponding velocity in the distribution Δ.

Let dot denote the time derivative. Then the vector field $X = [\dot{q}, \dot{p}]^T$ satisfies the Hamilton-d'Alembert equation (2.4):

$$\begin{bmatrix} \mathbf{0}_{5\times5} & -\mathbf{1}_{5\times5} \\ \mathbf{1}_{5\times5} & \mathbf{0}_{5\times5} \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \partial H/\partial q \\ \partial H/\partial p \end{bmatrix} + \kappa_1 \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ y \\ -\frac{r_w}{2}(\cos(\theta) - \frac{y}{b}) \\ -\frac{r_w}{2}(\cos(\theta) + \frac{y}{b}) \end{bmatrix} \\ \mathbf{0}_{5\times1} \end{bmatrix}$$

$$+\kappa_2 \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -x \\ -\frac{r_w}{2}(\sin(\theta) + \frac{x}{b}) \\ -\frac{r_w}{2}(\sin(\theta) - \frac{x}{b}) \end{bmatrix} \\ \mathbf{0}_{5\times1} \end{bmatrix} + \kappa_3 \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{r_w}{2b} \\ -\frac{r_w}{2b} \end{bmatrix} \\ \mathbf{0}_{5\times1} \end{bmatrix}, \tag{4.28}$$

with the constraint equation $\mathcal{A}_q \dot{q} = [0\ \ 0\ \ \Sigma]^T$. As expected, the right hand side of this constraint equation is non-zero.

As the result of eliminating the Lagrange multipliers and restricting the dynamical equations to the constrained phase space of the system, the two form ω_{nh} and the 1- form λ are determined in the following.

The components of the 2-form $\omega_{nh} = \omega_{can} + d\langle \mathbf{M}, \mathcal{A}\rangle - \langle \mathbf{M}, \mathcal{B}\rangle$ are then calculated, and later they are presented in matrix form for the chosen local coordinates.

$$\begin{aligned} \omega_{can} = & -dp_x \wedge dx - dp_y \wedge dy - dp_\theta \wedge d\theta \\ & -dp_{\vartheta_1} \wedge d\vartheta_1 - dp_{\vartheta_2} \wedge d\vartheta_2, \\ d\langle \mathbf{M}, \mathcal{A}\rangle = & \, dp_x \wedge dx + dp_y \wedge dy + dp_\theta \wedge d\theta \\ & -\frac{r_w}{2}\left[d\left(p_x \cos(\theta) + p_y \sin(\theta) - \frac{p_\theta}{b}\right) \wedge d\vartheta_1 \right. \\ & \left. + d\left(p_x \cos(\theta) + p_y \sin(\theta) + \frac{p_\theta}{b}\right) \wedge d\vartheta_2 \right]. \end{aligned}$$

In order to determine $\langle \mathbf{M}, \mathcal{B}\rangle = \langle \mathbf{M}, d\mathcal{A}(\mathrm{hor}(\cdot), \mathrm{hor}(\cdot))\rangle$, first the 2-form $\langle \mathbf{M}, d\mathcal{A}\rangle$ is calculated, then $\langle \mathbf{M}, \mathcal{B}\rangle$ is formed in the matrix form.

$$\begin{aligned} \langle \mathbf{M}, d\mathcal{A}\rangle = & \, p_x \left[dy \wedge d\theta - \frac{r_w}{2} d\left(\cos(\theta) - \frac{y}{b}\right) \wedge d\vartheta_1 \right. \\ & \left. -\frac{r_w}{2} d\left(\cos(\theta) + \frac{y}{b}\right) \wedge d\vartheta_2 \right] + p_y \left[-dx \wedge d\theta \right. \\ & \left. -\frac{r_w}{2} d\left(\sin(\theta) + \frac{x}{b}\right) \wedge d\vartheta_1 - \frac{r_w}{2} d\left(\sin(\theta) - \frac{x}{b}\right) \wedge d\vartheta_2 \right] \end{aligned}$$

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Hence, as a 2-form on Q

$$\langle \mathbf{M}, \mathcal{B} \rangle = \mathrm{hor}_q^T \begin{bmatrix} 0 & 0 & p_y & \frac{p_y r_w}{2b} & -\frac{p_y r_w}{2b} \\ 0 & 0 & -p_x & -\frac{p_x r_w}{2b} & \frac{p_x r_w}{2b} \\ -p_y & p_x & 0 & -\frac{p_0 r_w}{2} & -\frac{p_0 r_w}{2} \\ -\frac{p_y r_w}{2b} & \frac{p_x r_w}{2b} & \frac{p_0 r_w}{2} & 0 & 0 \\ \frac{p_y r_w}{2b} & -\frac{p_x r_w}{2b} & \frac{p_0 r_w}{2} & 0 & 0 \end{bmatrix} \mathrm{hor}_q$$
$$= \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 2} \\ \mathbf{0}_{2\times 3} & \begin{bmatrix} 0 & \frac{r_w^2 p_0}{2b} \\ -\frac{r_w^2 p_0}{2b} & 0 \end{bmatrix} \end{bmatrix},$$

where $p_0 = p_x \sin(\theta) - p_y \cos(\theta)$. As the result, in matrix form the 2-form ω_{nh} is given by

$$\omega_{nh} = \begin{bmatrix} \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 3} \\ \mathbf{0}_{3\times 2} & U \end{bmatrix} & \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 2} \\ K & -\mathbf{1}_{2\times 2} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0}_{3\times 3} & -K^T \\ \mathbf{0}_{2\times 3} & \mathbf{1}_{2\times 2} \end{bmatrix} & \mathbf{0}_{5\times 5} \end{bmatrix},$$

where

$$U = \frac{r_w p_0}{2} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -\frac{r_w}{b} \\ 1 & \frac{r_w}{b} & 0 \end{bmatrix}, \quad K = -\frac{r_w}{2} \begin{bmatrix} \cos(\theta) & \sin(\theta) & -\frac{1}{b} \\ \cos(\theta) & \sin(\theta) & \frac{1}{b} \end{bmatrix}.$$

As expected, the point-wise kernel of the nonholonomic 2-form ω_{nh} is:

$$\mathrm{span}\left\{ \omega_{can}^{-1}(\beta_i), (\eta_i)_{T^*Q} \middle| i = 1, 2, 3 \right\},$$

where

$$(\eta_1)_{T^*Q} = \frac{\partial}{\partial x}, (\eta_2)_{T^*Q} = \frac{\partial}{\partial y},$$
$$(\eta_3)_{T^*Q} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial \theta} - p_y \frac{\partial}{\partial p_x} + p_x \frac{\partial}{\partial p_y}$$

are the induced vector fields on the phase space of the system (T^*Q), due to the G-action.

The 1-form λ can be calculated through (3.16). Note that this equation can be simplified knowing the fact that the functions $\gamma_1 = \gamma_2 = H_1 = H_2 = H_3 \equiv 0$. The non-zero terms of this equation in matrix form are calculated as:

$$dH = \begin{bmatrix} \mathbf{0}_{2\times 1} \\ \frac{l_0^2 \sin(2\theta)(p_x^2 - p_y^2) - 2l_0 p_x p_y \cos(2\theta) + 2l_0 p_\theta (p_x \cos(\theta) + p_y \sin(\theta))}{2(J_r - Ml_0^2)} \\ \mathbf{0}_{2\times 1} \\ \frac{2(J_r - Ml_0^2 \cos^2(\theta))p_x - Ml_0^2 p_y \sin(2\theta) + 2Ml_0 p_\theta \sin(\theta)}{2M(J_r - Ml_0^2)} \\ \frac{2(J_r - Ml_0^2 \sin^2(\theta))p_y - Ml_0^2 p_x \sin(2\theta) - 2Ml_0 p_\theta \cos(\theta)}{2M(J_r - Ml_0^2)} \\ \frac{p_\theta + l_0(p_x \sin(\theta) - p_y \cos(\theta))}{J_r - Ml_0^2} \\ \frac{p_{\vartheta_1}}{J_w} \\ \frac{p_{\vartheta_2}}{J_w} \end{bmatrix},$$

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$$\gamma_3(\langle \mathrm{ad}^*_{\eta_3}\mathbf{M}, \mathcal{A}\rangle - d\mathbf{M}_3) = \Sigma \begin{bmatrix} \mathbf{0}_{2\times1} \\ xp_x + yp_y \\ \frac{r_w}{2}\left(p_0 + \frac{xp_x+yp_y}{b}\right) \\ \frac{r_w}{2}\left(p_0 - \frac{xp_x+yp_y}{b}\right) \\ y \\ -x \\ -1 \\ \mathbf{0}_{2\times1} \end{bmatrix};$$

hence, $\lambda = dH + \gamma_3(\langle \mathrm{ad}^*_{\eta_3}\mathbf{M}, \mathcal{A}\rangle - d\mathbf{M}_3)$, and (4.28) could be rewritten in the following form:

$$\omega_{nh}\begin{bmatrix}\dot{q} \\ \dot{p}\end{bmatrix} = \lambda. \tag{4.29}$$

In this case study, λ contains not only the derivative of the Hamiltonian but also the rotational velocity of the disk. This 1-form, unlike in Hamilton's equation, fails to be exact, and it vanishes in the directions of the infinitesimal action of se(2) on T^*Q.

Let the vector $[p_{\vartheta_1}\ p_{\vartheta_2}]^T$ be denoted by $\widetilde{p}$. The constrained submanifold $\mathcal{M} = \mathbb{F}L(\mathcal{D})$ can then be parametrized using $(q, \widetilde{p})$ with the inclusion map:

$$\begin{aligned} \mathfrak{I}_{\mathcal{M}}(q,\widetilde{p}) &= A_q\widetilde{p} + B_q := \\ &\begin{bmatrix} \frac{Mr_w}{2J_w}\left(\cos(\theta) + \frac{l_0}{b}\sin(\theta)\right) & \frac{Mr_w}{2J_w}\left(\cos(\theta) - \frac{l_0}{b}\sin(\theta)\right) \\ \frac{Mr_w}{2J_w}\left(\sin(\theta) - \frac{l_0}{b}\cos(\theta)\right) & \frac{Mr_w}{2J_w}\left(\sin(\theta) + \frac{l_0}{b}\cos(\theta)\right) \\ -\frac{J_r r_w}{2bJ_w} & \frac{J_r r_w}{2bJ_w} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\widetilde{p} \\ &\qquad + \Sigma \begin{bmatrix} M(-y - l_0\sin(\theta)) \\ M(x + l_0\cos(\theta)) \\ Ml_0(y\sin(\theta) + x\cos(\theta)) + J_r \\ \mathbf{0}_{2\times1} \end{bmatrix}. \end{aligned}$$

Note that $\mathcal{M}$ is an affine vector sub-bundle of T^*Q, and the inclusion map $\mathfrak{I}_{\mathcal{M}}$ is the identity map on Q. The associated tangent map $T\mathfrak{I}_{\mathcal{M}}\colon T\mathcal{M} \to T(T^*Q)$ is then given by:

$$T\mathfrak{I}_{\mathcal{M}} = \begin{bmatrix} \mathbf{1}_{5\times5} & \mathbf{0}_{5\times5} \\ \frac{\partial A_q}{\partial q}\widetilde{p} + \frac{\partial B_q}{\partial q} & A_q \end{bmatrix},$$

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where

$$\frac{\partial A_q}{\partial q}\widetilde{p} = \frac{Mr_w}{2J_w}\begin{bmatrix} \mathbf{0}_{2\times 5} \\ \begin{bmatrix} -\sin(\theta)(p_{\vartheta_1}+p_{\vartheta_2})+\frac{l_0}{b}\cos(\theta)(p_{\vartheta_1}-p_{\vartheta_2}) \\ \cos(\theta)(p_{\vartheta_1}+p_{\vartheta_2})+\frac{l_0}{b}\sin(\theta)(p_{\vartheta_1}-p_{\vartheta_2}) \\ \mathbf{0}_{3\times 1} \end{bmatrix}^T \\ \mathbf{0}_{2\times 5} \end{bmatrix}^T,$$

$$\frac{\partial B_q}{\partial q} = \Sigma M \begin{bmatrix} \begin{bmatrix} 0 & -1 & -l_0\cos(\theta) \\ 1 & 0 & -l_0\sin(\theta) \\ l_0\cos(\theta) & l_0\sin(\theta) & l_0(y\cos(\theta)-x\sin(\theta)) \end{bmatrix} & \mathbf{0}_{3\times 2} \\ \mathbf{0}_{2\times 3} & \mathbf{0}_{2\times 2} \end{bmatrix}.$$

This tangent map can be used to express a tangent vector to $\mathcal{M}$ (at a point) in the ambient manifold, i.e. T^*Q. Hence, the pullback of the nonholonomic 2-form ω_{nh} and the 1-form λ are computed through restricting to the submanifold $\mathcal{M}$ and then multiplying by $T\mathfrak{I}_{\mathcal{M}}$ and its transpose:

$$\widetilde{\omega}_{nh} = (T\mathfrak{I}_{\mathcal{M}})^T(\omega_{nh}\circ\mathfrak{I}_{\mathcal{M}})(T\mathfrak{I}_{\mathcal{M}}), \quad \widetilde{\lambda} = (T\mathfrak{I}_{\mathcal{M}})^T(\lambda\circ\mathfrak{I}_{\mathcal{M}}). \tag{4.30}$$

The Hamilton-d'Alembert equation on $\mathcal{M}$ then reads:

$$\widetilde{\omega}_{nh}\begin{bmatrix}\dot{q}\\ \dot{\tilde{p}}\end{bmatrix} = \widetilde{\lambda}. \tag{4.31}$$

This equation includes 7 ordinary differential equations. Note that the 2-form $\widetilde{\omega}_{nh}$ is still degenerate. Restricting this equation to the horizontal vector sub-bundle $T^{\Delta}\mathcal{M}$ would eliminate the degeneracy in the equation and it results in 4 ordinary differential equations on the 7-dimensional phase space $\mathcal{M}$. Let $\dot{\tilde{q}}$ denote the vector $[\dot{\vartheta}_1\ \dot{\vartheta}_2]^T$, which corresponds to the velocities in the direction of ϑ_1 and ϑ_2. In the local coordinates, the vector sub-bundle $T^{\Delta}\mathcal{M}$ can be parametrized by $(q,\dot{\tilde{q}})$, such that the inclusion map $\mathfrak{I}_{\Delta}$ has the following form:

$$\mathfrak{I}_{\Delta}(q,\dot{\tilde{q}}) = C_q\dot{\tilde{q}} := \begin{bmatrix} \frac{r_w\cos(\theta)}{2} & \frac{r_w\cos(\theta)}{2} \\ \frac{r_w\sin(\theta)}{2} & \frac{r_w\sin(\theta)}{2} \\ -\frac{r_w}{2b} & \frac{r_w}{2b} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\dot{\tilde{q}}.$$

This map is the identity map on the phase space $\mathcal{M}$ and the portion of its tangent bundle corresponding to the momenta p_{ϑ_1} and p_{ϑ_2}. Therefore, the restriction to the horizontal vector sub-bundle $T^{\Delta}\mathcal{M}$ is given by:

$$\begin{bmatrix} C_q^T & \mathbf{0}_{5\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2}\end{bmatrix}\widetilde{\omega}_{nh}\begin{bmatrix} C_q & \mathbf{0}_{5\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2}\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}}\\ \dot{\tilde{p}}\end{bmatrix} = \begin{bmatrix} C_q^T & \mathbf{0}_{5\times 2} \\ \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2}\end{bmatrix}\widetilde{\lambda}.$$

In this case study, consider the subgroup $E = \mathrm{SO}(2)$ of $G = \mathrm{SE}(2)$. The distribution $\mathcal{D}$ and the Legendre transformation $\mathbb{F}L$ are both invariant under the action of E. Therefore, $\mathcal{M}$ is invariant under the E-action. The condition (3.21) holds with both sides equal to zero. In the local coordinates, the projection map $\widehat{\pi}\colon Q \to \widehat{Q} = Q/\mathrm{SO}(2)$ becomes

$$\widehat{q} = \widehat{\pi}(q) = (\sigma := \sqrt{x^2+y^2}, \psi := \tan^{-1}(\frac{y}{x}) - \theta, \vartheta_1, \vartheta_2),$$

where $(\sigma, \psi+\theta)$ can be interpreted as the polar coordinates for the position of the point C on the rover. That is, σ is the distance of C from the origin of the inertial frame and $\psi+\theta$ is the angle of the position vector from the X_0-axis. The induced tangent map $T\widehat{\pi}$ is given in the local coordinates:

$$T_q\widehat{\pi} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 & 0 & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the corresponding cotangent lifted map is the transpose of this matrix.

In order to identify the Hamilton-d'Alembert equation on the cotangent bundle of the quotient manifold $\widehat{Q}$, the fibre-wise linear map $\mathbb{P}$ should be formed first. This map is from the original phase space T^*Q to $\mathbf{M}^{-1}(0)$, which can be expressed in the local coordinates as:

$$\mathbf{M}^{-1}(0) = \{\, (q,p) \in T^*Q |\, p_x = p_y = p_\theta = 0 \}.$$

In matrix form, the map $\mathbb{P}_q$ reads,

$$\mathbb{P}_q = \mathbf{1}_{5\times5} - \mathcal{A}_q^*\mathbf{M}_q = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times2} \\ \frac{r_w}{2}\begin{bmatrix} \cos(\theta) & \sin(\theta) & -\frac{1}{b} \\ \cos(\theta) & \sin(\theta) & \frac{1}{b} \end{bmatrix} & \mathbf{1}_{2\times2} \end{bmatrix}.$$

This map is a projection to the submanifold $\mathbf{M}^{-1}(0)$. Restricting the domain and range of $\mathbb{P}_q$ to the submanifolds $\mathcal{M}$ and $\mathbf{M}^{-1}(0)$, respectively, gives the map ρ. Since the nonholonomic distribution is affine, this map is no longer linear, and its inverse is computed by:

$$\rho_q^{-1}(\overline{p}) = (\overline{\mathbb{P}}_q A_q)^{-1}(\overline{p} - \overline{\mathbb{P}}_q B_q) =: \overline{S}^{-1}(\overline{p} - \overline{T}),$$

where in local coordinates $(q, \overline{p}) \in \mathbf{M}^{-1}(0)$, and

$$\overline{\mathbb{P}}_q = \begin{bmatrix} \frac{r_w}{2}\begin{bmatrix} \cos(\theta) & \sin(\theta) & -\frac{1}{b} \\ \cos(\theta) & \sin(\theta) & \frac{1}{b} \end{bmatrix} & \mathbf{1}_{2\times2} \end{bmatrix},$$
$$\overline{S} = \frac{r_w^2}{4J_w}\begin{bmatrix} M + \frac{J_r}{b^2} + \frac{4J_w}{r_w^2} & M - \frac{J_r}{b^2} \\ M - \frac{J_r}{b^2} & M + \frac{J_r}{b^2} + \frac{4J_w}{r_w^2} \end{bmatrix},$$
$$\overline{T} = -\frac{r_w M\Sigma}{2}\begin{bmatrix} \sigma\sin(\psi) + \frac{l_0}{b}\sigma\cos(\psi) + \frac{J_r}{bM} \\ \sigma\sin(\psi) - \frac{l_0}{b}\sigma\cos(\psi) - \frac{J_r}{bM} \end{bmatrix}.$$

Let the vector $\widehat{p} = [\widehat{p}_\sigma\ \widehat{p}_\psi\ \widehat{p}_{\vartheta_1}\ \widehat{p}_{\vartheta_2}]^T$ denote the momenta conjugate to $\widehat{q} = (\sigma, \psi, \vartheta_1, \vartheta_2)$, such that an element of the cotangent bundle of the reduced configuration manifold $\widehat{Q}$ may be represented by $(\widehat{q}, \widehat{p})$. Hence, the vector sub-bundle $[T\widehat{\pi}(\overline{\mathcal{V}})]^0$ can be parametrized through:

$$[T\widehat{\pi}(\overline{\mathcal{V}})]^0 = \left\{ (\widehat{q}, \widehat{p}) \in T^*\widehat{Q} \middle| \widehat{p}_\sigma = \widehat{p}_\psi = 0 \right\}.$$

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Note that the maps τ_0 and $T^*\widehat{\pi}$ are fibre-wise identity maps, when they are restricted to this vector sub-bundle. Therefore, the horizontal lift map $\mathrm{hl}_q^{\mathcal{M}}$ is given by:

$$\mathrm{hl}_q^{\mathcal{M}}(\widehat{p}_1) = \overline{S}^{-1}(\widehat{p}_1 - \overline{T}),$$

where the vector $\widehat{p}_1 = [\widehat{p}_{\vartheta_1} \;\; \widehat{p}_{\vartheta_2}]^T$ is used to parametrize the reduced phase space $[T\widehat{\pi}(\overline{\mathcal{V}})]^0$. This map relates the momenta in the reduced phase space of the system to the feasible momenta of the system in the original phase space. Now, through introducing the horizontal lift map hl_q for the E-action that relates the corresponding velocities, the 2-form $\widehat{\Xi} \in \Omega^2([T\widehat{\pi}(\overline{\mathcal{V}})]^0)$ in the matrix form becomes

$$\widehat{\Xi} = \frac{r_w^2 \widehat{p}_0}{2b} \begin{bmatrix} \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{bmatrix} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \mathbf{0}_{2\times 2} \end{bmatrix},$$

where

$$\widehat{p}_0 = \frac{M l_0 r_w}{2 b J_w}(p_{\vartheta_1} - p_{\vartheta_2}) - \Sigma M(l_0 + \sigma \cos(\psi)),$$

and where

$$\begin{bmatrix} p_{\vartheta_1} \\ p_{\vartheta_2} \end{bmatrix} = \mathrm{hl}_q^{\mathcal{M}}(\widehat{p}_1).$$

This 2-form corresponds to the constraint forces at the wheels due to the no-slip condition on a rotating disk.

Hence, the 2-form ω_{nh} descends to the following 2-form on the reduced phase space $[T\widehat{\pi}(\overline{\mathcal{V}})]^0$:

$$\widehat{\omega}_{nh} = \widehat{\omega}_{can} - \widehat{\Xi} = \begin{bmatrix} \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \begin{bmatrix} 0 & -\frac{r_w^2 \widehat{p}_0}{2b} \\ \frac{r_w^2 \widehat{p}_0}{2b} & 0 \end{bmatrix} \end{bmatrix} & \begin{bmatrix} \mathbf{0}_{2\times 2} \\ -\mathbf{1}_{2\times 2} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{1}_{2\times 2} \end{bmatrix} & \mathbf{0}_{2\times 2} \end{bmatrix}.$$

Finally in order to calculate $\widehat{\lambda} \in \Omega^1([T\widehat{\pi}(\overline{\mathcal{V}})]^0)$, the following relations are substituted in the formula (4.30) for $\widetilde{\lambda}$:

$$x = \sigma \cos(\psi + \theta), \;\; y = \sigma \sin(\psi + \theta), \;\; \widetilde{p} = \mathrm{hl}_q^{\mathcal{M}}(\widehat{p}_1).$$

Then the matrix $\widehat{W}$ is formed, such that $\widehat{\lambda} = \widehat{W}^T \widetilde{\lambda}$:

$$\widehat{W} = \begin{bmatrix} \begin{bmatrix} \cos(\psi+\theta) & -\sigma \sin(\psi+\theta) & 0 & 0 \\ \sin(\psi+\theta) & \sigma \cos(\psi+\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \mathbf{0}_{5\times 2} \\ -\overline{S}^{-1} \frac{\partial \overline{T}}{\partial \widehat{q}} & \overline{S}^{-1} \end{bmatrix},$$

where

$$\frac{\partial \overline{T}}{\partial \widehat{q}} = \begin{bmatrix} \sin(\psi) + \frac{l_0}{b}\cos(\psi) & \sigma\cos(\psi) - \frac{l_0}{b}\sigma\sin(\psi) & 0 & 0 \\ \sin(\psi) - \frac{l_0}{b}\cos(\psi) & \sigma\cos(\psi) + \frac{l_0}{b}\sigma\sin(\psi) & 0 & 0 \end{bmatrix}.$$

Although the matrix $\widehat{W}$ and the 1-form $\widetilde{\lambda}$ depend on variable θ, the 1-form $\widehat{\lambda}$ is independent of this variable, by Lemma 3.6. Therefore, the Hamilton-d'Alembert equation on $[T\widehat{\pi}(\overline{\mathcal{V}})]^0$ reads:

$$\widehat{\omega}_{nh} \begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}}_1 \end{bmatrix} = \widehat{\lambda},$$

whose first two equations for $\dot{\sigma}$ and $\dot{\psi}$ are trivially satisfied. Let $\dot{\widehat{q}}_1 = [\dot{\vartheta}_1 \;\; \dot{\vartheta}_2]^T$. Therefore the full dynamics of the system can be captured by the following 4 differential equations on the reduced phase space $[T\widehat{\pi}(\overline{\mathcal{V}})]^0$:

$$\begin{bmatrix} \mathbf{0}_{4\times 2} & \mathbf{1}_{4\times 4} \end{bmatrix} \widehat{\omega}_{nh} \begin{bmatrix} \mathbf{0}_{2\times 4} \\ \mathbf{1}_{4\times 4} \end{bmatrix} \begin{bmatrix} \dot{\widehat{q}}_1 \\ \dot{\widehat{p}}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{4\times 2} & \mathbf{1}_{4\times 4} \end{bmatrix} \widehat{\lambda}. \tag{4.32}$$

The reconstruction equations on $\widehat{Q}$ are then used to determine the rest of the states of the system:

$$\begin{bmatrix} \dot{\sigma} \\ \dot{\psi} \end{bmatrix} = \frac{r_w}{2} \begin{bmatrix} \cos(\psi) & \cos(\psi) \\ -\frac{1}{\sigma}\sin(\psi) + \frac{1}{b} & -\frac{1}{\sigma}\sin(\psi) - \frac{1}{b} \end{bmatrix} \begin{bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix}. \tag{4.33}$$

5 Conclusions

A geometric approach to the dynamical reduction of a class of constrained Hamiltonian mechanical systems with symmetry was introduced. This approach considers constraints in the form of affine nonholonomic distributions and unifies the symplectic and Chaplygin reduction theorems for holonomic and nonholonomic systems, respectively. The proposed reduction approach consists of three main steps:

(a) The Hamilton-d'Alembert equation is restricted to the constrained phase space $\mathcal{M} = \mathbb{F}L(\mathcal{D})$, through calculating the Lagrange multipliers and introducing the 2-form ω_{nh}. Then, both sides of the restricted equations are pulled back by the inclusion map $\mathfrak{I}_{\mathcal{M}} : \mathcal{M} \to T^*Q$.
(b) Under the conditions of Lemma 3.6, which replace the invariance of the Hamiltonian, the dynamical equations on the constrained phase space $\mathcal{M}$ are E-invariant. Hence, the dynamical equations of the mechanical system can be expressed in the quotient manifold $\mathcal{M}/E$.
(c) The quotient manifold is identified with the affine sub-bundle $[T\widehat{\pi}(\mathcal{V})]^0 \subseteq T^*(Q/E)$.

As a case study, the abovementioned reduction method was performed for a two-wheeled rover on a rotating disk. The dynamical equations of this system was reduced from 10 first-order differential equations with 3 constraints to 4 dynamical equations plus 2 reconstruction equations. These equations were expressed on a sub-bundle of the cotangent bundle that could be parametrized by the momenta conjugate to the wheel angles.

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