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A Unified Approach to Input-output Linearization and Concurrent Control of Underactuated Open-chain Multi-body Systems with Holonomic and Nonholonomic Constraints

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Abstract This paper presents a unified geometric framework to input-output linearization of open-chain multi-body systems with symmetry in their reduced phase space. This leads us to an output tracking controller for a class of underactuated open-chain multi-body systems with holonomic and nonholonomic constraints. We consider the systems with multi-degree-of-freedom joints and possibly with non-zero constant total momentum (in the holonomic case). The examples of these systems are free-base space manipulators and mobile manipulators. We first formalize the control problem, and rigorously state an output tracking problem for such systems. Then, we introduce a geometrical definition of the end-effector pose and velocity error. The main contribution of this paper is reported in Section 5, where we solve for the input-output linearization of the highly nonlinear problem of coupled manipulator and base dynamics subject to holonomic and nonholonomic constraints. This enables us to design a coordinate-independent controller, similar to a proportional-derivative with feed-forward, for concurrently controlling a free-base, multi-body system. Finally, by defining a Lyapunov function we prove in Theorem 3 that the closed-loop system is exponentially stable. A detailed case study concludes this paper.

Keywords Open-chain Multi-body System · Dynamical Reduction · Nonholonomic Constraints · Input-output Linearization · Nonlinear Control

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Nomenclature

B_i	Body i of a multi-body system
$r_{cm,i}$	Relative pose of a frame at the centre of mass of B_i with respect to the inertial frame
$r_{i,0}^j$	Initial relative pose of B_i with respect to B_j
L_r	Left composition/translation by r
R_r	Right composition/translation by r
id_n	$n \times n$ identity matrix
0_n	$n \times n$ zero matrix
Ad_r	Adjoint operator corresponding to r
ad_ξ	adjoint operator corresponding to ξ
$[\xi, \eta]$	Lie bracket or matrix commutator
$T_m f$	Tangent map corresponding to the map f at the element m
$T_m^* f$	Cotangent map corresponding to the map f at the element m
$T_m M$	Tangent space of the manifold M at the element m
TM	Tangent bundle of the manifold M
$T_m^* M$	Cotangent space of the manifold M at the element m
$T^* M$	Cotangent bundle of the manifold M
$\exp(\xi)$	Group/matrix exponential of ξ
G_μ	Coadjoint isotropy group for μ in dual of Lie algebra
$\lVert v \rVert_h$	Norm of the vector v with respect to the metric h
$\langle \cdot, \cdot \rangle$	Canonical pairing of the elements of tangent and cotangent space
$\mathcal{L}$	Lagrangian function
ξ_M	Vector field on the manifold M induced by the infinitesimal action of ξ in Lie algebra
$\mathfrak{X}(M)$	Space of all vector fields on the manifold M
$\Omega^2(M)$	Space of all differential 2-forms on the manifold M
$d\Omega$	Exterior derivative of the differential form Ω
df	Exterior derivative of the function f
M/G	Quotient manifold corresponding to a free and proper action of the Lie group G

1 Introduction

Holonomic and nonholonomic open-chain multi-body systems mostly appear in the field of robotics. In the context of geometric mechanics, these systems are considered as Hamiltonian mechanical systems. In this paper, we introduce a unified approach towards the dynamical reduction and output tracking control of such systems in the presence of symmetry. In the following, we first report the existing literature for different topics appearing in this paper. Then, we list the main contributions of the paper, and give the outline of the paper.

An example of a mechanical system with symmetry is a free-base multi-body system, which is mostly studied in the field of robotics and aerospace. Vafa and Dubowsky introduce the notion of Virtual Manipulator [34] (for a free-floating manipulator with zero total momentum), and they show that this approach decouples the system centre of mass translational and rotational motion. Dubowsky and Papadopoulos in [12] use this notion to solve for the inverse dynamics problem that yields to designing linear controllers in joint and task space.

Since the trivial behaviour of a multi-body system due to momentum conservation is eliminated during a reduction process, the behaviour of the system is more explicit in the reduced space. The reduction procedures have been helpful for extracting control laws for space manipulators by restricting the dynamical equations to the submanifold of the phase space where the momentum of the system is constant (and usually equal to zero). Yoshida *et al.* investigate the kinematics of free-floating multi-body systems utilizing the momentum conservation law. They derive a new Jacobian matrix in generalized form and develop a control method based on the resolved motion rate control concept [33,20]. McClamroch *et al.* propose an articulated-body dynamical model for free-floating robots based on Hamilton's equation, and implement it to derive an adaptive motion control law [38]. Based on the concept of Virtual manipulator, Parlaktuna and Ozkan also develop an adaptive controller for free-floating space manipulators [23]. Wang and Xie introduce an adaptive control law for position/force tracking of free-flying manipulators [35,36], and later they use recursive Newton-Euler equations to derive a novel adaptive controller for position tracking of free-floating manipulators in their task space [37]. In this controller, they estimate the inertia tensor of the spacecraft (base body) by a parameter projection algorithm. As an application, Pazelli *et al.* investigate different nonlinear H_∞ control schemes implemented to a free floating space manipulator, subject to parameter uncertainty and external disturbances [24].

In the case of underactuated space manipulators, Mukherjee and Chen in [19] show that even if the unactuated joints do not possess brakes, the manipulator can be brought to a complete rest provided that the system maintains zero momentum. In [32] an alternative path planning methodology is developed for underactuated manipulators using high order polynomials as arguments in cosine functions to specify the desired path directly in joint space. Note that all of the above mentioned control strategies were developed for holonomic multi-body systems with one-degree-of-freedom (d.o.f.) joints and for zero momentum of the system.

Geometric methods have also been used to reduce the dynamical model of free-base multi-body systems and introduce effective control laws. For example, in [30,31] Sreenath symplectically reduces Hamilton's equation by $SO(2)$ for free-base planar multi-body systems with non-zero angular momentum. Chen in his Ph.D. thesis [8] extends Sreenath's approach to spatial multi-body systems with zero angular momentum. Duindam and Stramigioly derive the Boltzmann-Hamel equations for multi-body systems with generalized multi-d.o.f. holonomic and nonholonomic joints by restricting the dynamical equations to the nonholonomic distribution [13]. This is the first attempt to reduce the dynamical equations of an open-chain multi-body systems with generalized holonomic and nonholonomic joints. Furthermore, Shen proposes a novel trajectory planning in shape space for nonlinear control of multi-body systems with symmetry [28, 26, 27]. In his work he performs symplectic reduction for zero momentum and assumes multi-body systems on trivial bundles. Then, in [29] he extends his results to include nonholonomic constraints. Hussein and Bloch study optimal control of nonholonomic mechanical systems, using an affine connection formalism [16]. Sliding mode control of underactuated multi-body systems is also studied in [2]. In the control community, Olfati-Saber in his Ph.D. thesis [21] studies the reduction of underactuated holonomic and nonholonomic Lagrangian mechanical systems with symmetry and its application to nonlinear control of such systems. He uses a feedback linearization method in the reduced phase space to extract control laws for such systems [22]. However, he only considers abelian symmetry groups, and he does not take into account non-zero momentum of the system. As a continuation of Olfati-Saber's work, Grizzle *et al.* in [14] show that a planar mechanism with a cyclic unactuated parameter is always locally feedback linearizable, and they derive a nonlinear control law for such a system. Further, Bloch and Bullo extract coordinate-independent nonlinear control laws for holonomic and nonholonomic mechanical systems with symmetry in [4, 5, 7].

In this paper, we use the dynamical reduction of a class of underactuated multi-body systems with holonomic or nonholonomic constraints to derive an output tracking control law. The main contribution of this work is generalizing and unifying the existing approaches to the control of underactuated multi-body systems with symmetry, more specifically, by

- considering holonomic free-base multi-body systems with non-abelian symmetry groups and non-zero conserved momentum,
- considering free-base multi-body systems with nonholonomic constraints,
- unifying the holonomic and nonholonomic cases to introduce a generalized output tracking controller for both types of underactuated multi-body systems.

Section 2 focuses on a quick review of the dynamical reduction of open-chain multi-body systems with holonomic or nonholonomic constraints [11, 9]. Then using this reduction procedure, we formulate an output tracking control problem in Section 3. We define the *output manifold* of a multi-body system and consequently choose a compatible error function and velocity error for the output of the system, in Section 4. Section 5 states one of the main results of this paper, which is the input-output linearization of a class of underactuated multi-body systems with symmetry in their reduced phase space. As the result, we solve the inverse dynamics problem of such systems. Finally, in Section 6 we suggest a nonlinear output tracking control law for free-base open-chain multi-body systems on their output manifold. And, we prove the exponential stability of the closed loop, for any *feasible* desired trajectory of the system. We conclude the paper with a case study in Section 7.

2 Dynamical Reduction of Multi-body Systems

In this section we briefly review our results on the dynamical reduction of open-chain multi-body systems with symmetry stated in [9,11]. The multi-body systems considered in this paper consist of rigid bodies connected with multi-d.o.f. displacement subgroups [10], and possibly with nonholonomic constraints. The reduced dynamical equations of holonomic open-chain multi-body systems with (not necessarily zero) conserved total momentum are first stated. Then, we perform the dynamical reduction for nonholonomic open-chain multi-body systems with symmetry that satisfy the Chaplygin assumption. Although we only focus on these two classes of underactuated open-chain multi-body systems, our results can be generalized to the systems with bigger symmetry groups [9,11].

An *open-chain multi-body system* $MBS(N)$ considered in this paper is a mechanical system consisting of $N+1$ rigid bodies, denoted by B_0 to B_N, and N displacement subgroups [10,15,25], denoted by J_1 to J_N, such that there exists a unique path between any two bodies. In a $MBS(N)$, bodies with only one neighbouring body are called *extremities*. In case the multi-body system is subject to nonholonomic constraints, we call the system a *nonholonomic open-chain multi-body system*.

Let d_i be the number of degrees of freedom of a displacement subgroup in a $MBS(N)$. It is associated with a d_i-dimensional configuration manifold, namely $\mathcal{Q}_i$, which is a Lie subgroup of $SE(3)$. As a result, the configuration manifold of the system is $\mathcal{Q} := \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_N$, which is a Lie subgroup of $\mathcal{P} := SE(3) \times \cdots \times SE(3)$ (N-times). Any state of the system can be realized by an element $q := (q_1, \cdots, q_N) \in \mathcal{Q}$. In this paper, without loss of generality, we assume that B_0 represents an inertial observer, with respect to which we measure the absolute velocities of the bodies in the $MBS(N)$. Let $r_{cm,i} \in SE(3)$ be the initial pose of the centre of mass of B_i with respect to B_0. The pose of the centre of mass of the bodies in $MBS(N)$ are then calculated through the map $F\colon \mathcal{Q} \to \mathcal{P}$:

$$F(q) := (q_1 r_{cm,1}, q_1 q_2 r_{cm,2}, \cdots, q_1 \cdots q_N r_{cm,N}). \tag{2.1}$$

Only the joints in the path connecting B_0 to B_i contribute to the i^{th} component ($i = 1, \cdots, N$) of F.

Any motion of the multi-body system is represented by a curve $t \mapsto q(t) \in \mathcal{Q}$. The absolute velocity of the coordinate frames attached to the centre of mass of bodies is calculated by $\dot{\mathfrak{p}}(t) := \frac{d}{dt}F(q(t)) = T_{q(t)}F(\dot{q}(t))$, where $TF\colon T\mathcal{Q} \to T\mathcal{P}$ is the induced tangent map of F.

The Lagrangian $\mathcal{L}\colon T\mathcal{Q} \to \mathbb{R}$ for the $MBS(N)$ consists of the kinematic energy and a potential function: $\mathcal{L}(v_q) = \frac{1}{2}K_q(v_q, v_q) - V(q)$. The metric K is the metric induced by the left-invariant kinetic energy metric of rigid bodies, and $V\colon \mathcal{Q} \to \mathbb{R}$ is a potential energy function. Let h_i ($i = 1, \cdots, N$) be the left-invariant metric corresponding to B_i. Let $h := h_1 \oplus \cdots \oplus h_N$ be the induced left-invariant metric on $\mathcal{P}$. The kinetic energy metric of an $MBS(N)$ is then calculated by $K := T^*F(h)$, which reads the pull back of the metric h by the map F. That is, $\forall q \in \mathcal{Q}$ and $\forall v_q, w_q \in T_q\mathcal{Q}$ we have

$$\begin{aligned} K_q(v_q, w_q) &= h_{F(q)}\left(T_qF(v_q), T_qF(w_q)\right) \\ &= h_{\mathfrak{e}}\left(T_{F(q)}L_{F(q)^{-1}}(T_qF(v_q)), T_{F(q)}L_{F(q)^{-1}}(T_qF(w_q))\right), \end{aligned} \tag{2.2}$$

where $\mathfrak{e}$ is the identity element of the Lie group $\mathcal{P}$, and for any element $\mathfrak{p} \in \mathcal{P}$, the left translation map is denoted by $L_{\mathfrak{p}}$. The function V can be any function on the configuration manifold $\mathcal{Q}$. The most common potential function is the gravitational potential function induced by a constant field.

Using the Legendre transformation induced by the metric K, we define the Hamiltonian $H\colon T^*\mathcal{Q} \to \mathbb{R}$ for the $MBS(N)$ by

$$H(p_q) := \langle p_q, \mathbb{F}L_q^{-1}(p_q)\rangle - \mathcal{L}(\mathbb{F}L_q^{-1}(p_q)), \tag{2.3}$$

which is the total energy of the system. Here, we remind the reader that $\mathbb{F}L\colon T\mathcal{Q} \to T^*\mathcal{Q}$ is the fibre-wise invertible Legendre transformation defined by $\langle \mathbb{F}L_q(v_q), w_q\rangle := K_q(v_q, w_q)$, $\forall v_q, w_q \in T_q\mathcal{Q}$. For multi-body systems this map is the symmetric tensor corresponding to the metric K. Hence, we write

$$\mathbb{F}L_q = \begin{bmatrix} K_{11}(q) & \cdots & K_{1N}(q) \\ \vdots & \ddots & \vdots \\ K_{N1}(q) & \cdots & K_{NN}(q) \end{bmatrix},$$

where $K_{ij}(q)dq_i \otimes dq_j$ $(i, j = 1, \cdots, N)$ are the block components of the kinetic energy tensor.

Accordingly, any $MBS(N)$ can be considered as a Hamiltonian mechanical system described by the four-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K)$, where $T^*\mathcal{Q}$ is the phase space of the system parametrized by (q, p) and $\Omega_{can} = -dp \wedge dq$ is the canonical 2-form on this space. Here, the metric K and the Hamiltonian H are defined by (2.2) and (2.3), respectively. Hamilton's equation for an $MBS(N)$ has the following form:

$$[\Omega_{can}] \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} := \begin{bmatrix} 0 & -id \\ id & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix},$$

where id is the identity matrix with the appropriate size.

2.1 Holonomic Open-chain Multi-body Systems

Let $\mathcal{G}$ be a Lie group with a free and proper action on the phase space of a Hamiltonian mechanical system. If this action preserves the corresponding Hamiltonian and 2-form of the system, we call the Lie group $\mathcal{G}$ a *symmetry group* and the Hamiltonian system along with this Lie group is a *Hamiltonian system with symmetry*. The $\mathcal{G}$-action induces a momentum map $\mathbf{M}$ from the phase space to $\mathfrak{G}^*$, the dual of the Lie algebra of $\mathcal{G}$, which is constant along the trajectories of the Hamiltonian system with symmetry (Noether's Theorem). As the result, one performs the symplectic reduction, by first restricting to the pre-image of the momentum map at a constant value $\mu \in \mathfrak{G}^*$, and then quotienting by the isotropy group $\mathcal{G}_\mu := \left\{ \mathfrak{g} \in \mathcal{G} | \, \mathrm{Ad}^*_{\mathfrak{g}}(\mu) = \mu \right\}$. This reduction method, which was first introduced by Marsden and Weinstein [18], expresses the equations of motion in a smaller phase space and guarantees the symplecticity of the resulting Hamiltonian system.

In this section, we consider *free-base* holonomic open-chain multi-body systems as Hamiltonian mechanical systems with symmetry and perform the symplectic reduction for such systems. The adjective free-base for a multi-body system refers to the unactuation of the base body, which can be considered to be B_1, without loss of generality. According to the definition of the Kinetic energy metric (2.2), it is easy to show that K is invariant under the action of $\mathcal{G} = \mathcal{Q}_1$ by the left translation on the first component of $\mathcal{Q}$. For any $\mathfrak{g} \in \mathcal{G}$ we denote the $\mathcal{G}$-action by $\Phi_{\mathfrak{g}} \colon \mathcal{Q} \to \mathcal{Q}$ such that $\forall q = (q_1, \cdots, q_N) \in \mathcal{Q}$ we have $\Phi_{\mathfrak{g}}(q) = (\mathfrak{g} q_1, q_2, \cdots, q_N)$ [9]. Note that the canonical 2-form on $T^*\mathcal{Q}$ is also invariant under the lifted $\mathcal{G}$-action. So, if the potential energy is also invariant under the $\mathcal{G}$-action, then $\mathcal{G}$ is a symmetry group of the *MBS*(N). This condition is satisfied, for example, in the case of space manipulators where $V \equiv 0$ and mobile robots where the gravitational field is perpendicular to the plane of the rover motion. We denote an *open-chain multi-body system with symmetry* by the five-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{G})$, as defined above.

In the case of a free-base multi-body system, the momentum map $\mathbf{M} \colon T^*\mathcal{Q} \to \mathfrak{G}^*$ is defined by a fibre-wise linear map

$$\mathbf{M}_q = \begin{bmatrix} T^*_{e_1} R_{q_1} & 0 & \cdots & 0 \end{bmatrix}, \tag{2.4}$$

where $R_{q_1} \colon \mathcal{Q}_1 \to \mathcal{Q}_1$ is the right translation by q_1 on $\mathcal{Q}_1$, and $T^*_{e_1} R_{q_1} \colon T^*_{q_1}\mathcal{Q}_1 \to \mathfrak{Q}^*_1$ is the induced map in the level of the cotangent bundles, where $\mathfrak{Q}^*_1$ is the dual of the Lie algebra of $\mathcal{Q}_1$. This momentum map for a space manipulator is the total Linear and angular momentum of the system with respect to the inertial coordinate frame. If we have additional holonomic constraints at the base body, $\mathbf{M}$ corresponds to the projection of the total momentum of the system onto the first joint axes. For any value $\mu \in \mathfrak{G}^*$ the pre-image of the momentum map $\mathbf{M}^{-1}(\mu)$ is a submanifold of $T^*\mathcal{Q}$ and it is invariant under the action of the isotropy group $\mathcal{G}_\mu$. By reducing the dynamical equations of an *MBS*(N), we mean expressing the dynamical equations in the quotient space $\mathbf{M}^{-1}(\mu)/\mathcal{G}_\mu$ that is symplectomorphic to a vector sub-bundle of the phase space $T^*\widetilde{\mathcal{Q}}$, where $\widetilde{\mathcal{Q}} = \mathcal{Q}/\mathcal{G}_\mu$.

In the following, we only state the final result for the symplectic reduction of a multi-body system as a theorem. In [9] we provide the details of the reduction of holonomic multi-body systems with a conserved momentum.

Theorem 1 (Symplectic Reduction of Multi-body Systems) *Let $\mu \in \mathfrak{G}^*$ be a regular value of the momentum map $\boldsymbol{M}$. A holonomic open-chain multi-body system with symmetry $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{G})$ is reduced to a Hamil-*

tonian system represented by the triple $(\widetilde{\mathcal{S}}, \widetilde{\Omega}, \widetilde{H})$*. We introduce the components of the reduced Hamiltonian system in the following.*

- *The reduced phase space* $\widetilde{\mathcal{S}} = \left\{ (\widetilde{q}, \widetilde{p}) \in T^*\widetilde{\mathcal{Q}} \middle| \widetilde{p}_1 = 0 \right\}$ *is a vector sub-bundle of* $T^*\widetilde{\mathcal{Q}}$*, whose elements are represented by* $(\widetilde{q}_1, \overline{q}, \widetilde{p}_1, \overline{p}) := (\widetilde{q}, \widetilde{p})$*. Therefore, the reduced phase space may be parametrized by* $(\widetilde{q}, \overline{p}) \in \widetilde{\mathcal{S}}$*. Note that* $\overline{q} \in \overline{\mathcal{Q}} := \mathcal{Q}_2 \times \cdots \times \mathcal{Q}_N$ *and* $\overline{p}$ *is its conjugate generalized momentum.*
- *The 2-form* $\widetilde{\Omega} \in \Omega^2(\widetilde{\mathcal{S}})$ *is defined by*

$$\begin{aligned}
\widetilde{\Omega} =& - d\overline{p} \wedge d\overline{q} - \sum_{i'<j'} \Upsilon_{i'j'}(\widetilde{q}) d\widetilde{q}_{i'} \wedge d\widetilde{q}_{j'} \\
:=& - d\overline{p} \wedge d\overline{q} - \sum_{i<j} \sum_{a=1}^{d_1} \mathcal{F}_a \left(\left(\frac{\partial A_j^a}{\partial \overline{q}_i} - \frac{\partial A_i^a}{\partial \overline{q}_j} \right) - \sum_{l<k} \mathcal{E}_{lk}^a (A_i^l A_j^k - A_j^l A_i^k) \right) (d\overline{q}_i \wedge d\overline{q}_j) \\
& - \sum_{i'<j'} \sum_{l<k} \sum_{a=1}^{d_1} \left(\mu_a \mathcal{E}_{lk}^a (\mathcal{H}_{i'}^l \mathcal{H}_{j'}^k - \mathcal{H}_{j'}^l \mathcal{H}_{i'}^k) \right) (d\widetilde{q}_{i'} \wedge d\widetilde{q}_{j'}), \qquad (2.5)
\end{aligned}$$

such that we have the following identities:

$$\begin{aligned}
A :=& \left[\overline{K}_{11}(\overline{q})^{-1} \overline{K}_{12}(\overline{q}) \quad \cdots \quad \overline{K}_{11}(\overline{q})^{-1} \overline{K}_{1N}(\overline{q}) \right], \\
\mathcal{F} :=& \mu^T Ad_{(e_\mu, \widetilde{q}_1)}, \\
\mathcal{H} :=& -A^\mu + \left[T_{(e_\mu, \widetilde{q}_1)} R_{(e_\mu, \widetilde{q}_1)} \quad Ad_{(e_\mu, \widetilde{q}_1)} A \right], \\
A^\mu :=& \left[\widetilde{K}_{11}(\widetilde{q})^{-1} \widetilde{K}'_{11}(\widetilde{q}) \quad \widetilde{K}_{11}(\widetilde{q})^{-1} \widetilde{K}_{12}(\widetilde{q}) \quad \cdots \quad \widetilde{K}_{11}(\widetilde{q})^{-1} \widetilde{K}_{1N}(\widetilde{q}) \right], \\
\sum_{a=1}^{d_1} \mathcal{E}_{lk}^a E_a :=& [E_l, E_k],
\end{aligned}$$

for $l, k \in \{1, \cdots, d_1\}$*, and* $i, j \in \{1, \cdots, \dim(\mathcal{Q}) - d_1\}$*, and* $i', j' \in \{1, \cdots, \dim(\mathcal{Q}) - \dim(\mathcal{G}_\mu)\}$*,* $\overline{K}_{1s}(\overline{q}) := K_{1s}((e_1, \overline{q}))$ $(s = 1, \cdots, N)$ *and* $\{E_1, \cdots, E_{d_1}\}$ *being a basis for* $\mathfrak{G}$*, the Lie algebra of* $\mathcal{G}$*. For a local trivialization of* $\mathcal{G}_\mu$ *principal bundle, the block components* $\widetilde{K}_{11}(\widetilde{q})$ *and* $\widetilde{K}'_{11}(\widetilde{q})$ *of the Legendre transformation correspond to the isotropy group* $\mathcal{G}_\mu$ *and* $\mathcal{G}/\mathcal{G}_\mu$ *at the identity element* $e_\mu \in \mathcal{G}_\mu$*, respectively, and the block components* $\widetilde{K}_{1s}(\widetilde{q})$ $(s = 2, \cdots, N)$ *correspond to the successive joints in the MBS*(N) *at the identity element* e_μ*.*
- *The reduced Hamiltonian of the system*

$$\begin{aligned}
\widetilde{H}(\widetilde{q}, \overline{p}) = \frac{1}{2} \Big\langle & (Ad^*_{(e_\mu, \widetilde{q}_1)} \mu, \overline{p} + A^*_{\overline{q}} (Ad^*_{(e_\mu, \widetilde{q}_1)} \mu)), \\
& , \mathbb{F}L^{-1}_{(e_\mu, \widetilde{q})} (Ad^*_{(e_\mu, \widetilde{q}_1)} \mu, \overline{p} + A^*_{\overline{q}} (Ad^*_{(e_\mu, \widetilde{q}_1)} \mu)) \Big\rangle + V(e_\mu, \widetilde{q}). \qquad (2.6)
\end{aligned}$$

Finally in the local coordinates of $\widetilde{\mathcal{S}}$, Hamilton's equation reads

$$\begin{bmatrix} \dot{\widetilde{q}}_1 \\ \dot{\overline{q}} \\ \dot{\overline{p}} \end{bmatrix} = [\widetilde{\Omega}]^{-1}(\widetilde{q}) \begin{bmatrix} \frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} \\ \frac{\partial \widetilde{H}}{\partial \overline{q}} \\ \frac{\partial \widetilde{H}}{\partial \overline{p}} \end{bmatrix}, \tag{2.7}$$

where $[\widetilde{\Omega}]$ is the vector bundle map naturally associated to the 2-form $\widetilde{\Omega}$. This map has the following form:

$$[\widetilde{\Omega}](\widetilde{q}) = \begin{bmatrix} [\widetilde{\Omega}]_{11}(\widetilde{q}) & [\widetilde{\Omega}]_{12}(\widetilde{q}) & 0 \\ -[\widetilde{\Omega}]_{12}^T(\widetilde{q}) & [\widetilde{\Omega}]_{22}(\widetilde{q}) & -id \\ 0 & id & 0 \end{bmatrix},$$

where the zero and identity matrices have the appropriate dimensions and the sub-matrices $[\widetilde{\Omega}]_{11}$, $[\widetilde{\Omega}]_{12}$ and $[\widetilde{\Omega}]_{22}$ are defined based on $\Upsilon_{i'j'}(\widetilde{q})$ $(i', j' = 1, \cdots, \dim(\mathcal{Q}) - \dim(\mathcal{G}_\mu))$.

2.2 Nonholonomic Open-chain Multi-body Systems

Consider a Hamiltonian mechanical system on the cotangent bundle $T^*\mathcal{Q}$ subject to a set of (everywhere) linearly independent nonholonomic constraints $\{\omega_a \in \Omega^1(\mathcal{Q}) \,|\, a = 1, \cdots, f\}$, whose kernel $\mathcal{D}$ forms a distribution on the configuration manifold $\mathcal{Q}$. Let $\mathcal{G}$ be a Lie group with a free and proper action on $\mathcal{Q}$ that leaves the distribution $\mathcal{D}$ and the corresponding Hamiltonian H invariant. The nonholonomic system along with the Lie group $\mathcal{G}$ is called a *nonholonomic Hamiltonian mechanical system with symmetry*. We call such a system a *Chaplygin system*, if $\forall q \in \mathcal{Q}$ it also satisfies the Chaplygin assumption:

$$T_q\mathcal{Q} = \mathcal{D}(q) \oplus T_q\mathcal{O}_q(\mathcal{G}), \tag{2.8}$$

where $\mathcal{O}_q(\mathcal{G})$ is the orbit of the $\mathcal{G}$-action through $q \in \mathcal{Q}$. As a result, one can perform the Chaplygin reduction, by first restricting to $\mathbb{F}L(\mathcal{D})$ the image of the distribution by the Legendre transformation, and then quotienting by the induced $\mathcal{G}$-action on $\mathbb{F}L(\mathcal{D})$. This reduction method, which was first introduced by Koiller [17], expresses the equations of motion in a smaller phase space, but the 2-form representing the equations of motion is not closed any more. Therefore, the resulting mechanical system is not Hamiltonian.

In this section, we study the reduction of the class of free-base nonholonomic open-chain multi-body systems that can be considered as Chaplygin systems. We already know that the kinetic energy of a multi-body system is invariant under the action of $\mathcal{Q}_1$, as defined in the previous subsection. Hence, in the nonholonomic case we assume that there is an f-dimensional Lie subgroup $\mathcal{G} \subset \mathcal{Q}_1$ that leaves $\mathcal{D}$ invariant and satisfies the Chaplygin assumption (2.8). Note that the canonical 2-form on $T^*\mathcal{Q}$ is also invariant under the lifted $\mathcal{G}$-action. As the result, if the potential energy function is also invariant under the $\mathcal{G}$-action, then the open-chain multi-body system $MBS(N)$ is a Chaplygin system. We denote such an open-chain multi-body system by the six-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G})$, as defined above. By reducing the dynamical equations of a nonholonomic $MBS(N)$, we mean expressing the dynamical equations in the quotient space $\mathbb{F}L(\mathcal{D})/\mathcal{G}$ that is symplectomorphic to the phase space $T^*\widehat{\mathcal{Q}}$, where $\widehat{\mathcal{Q}} = \mathcal{Q}/\mathcal{G}$.

In the following, we only state the final result for the Chaplygin reduction of a nonholonomic multi-body system as a theorem. In [11] we provide the details of the reduction of nonholonomic multi-body systems.

Theorem 2 (Chaplygin Reduction of Multi-body Systems) *A nonholonomic open-chain multi-body system $MBS(N)$ with symmetry whose dynamics is represented by $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G})$ is reduced to a mechanical system represented by the triple $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H})$. We introduce the components of the reduced mechanical system in the following.*

- *The reduced phase space $T^*\widehat{\mathcal{Q}}$ is the cotangent bundle of the quotient manifold $\widehat{\mathcal{Q}}$, whose elements are represented by $(\widehat{q}_1, \overline{q}, \widehat{p}_1, \overline{p}) := (\widehat{q}, \widehat{p})$.*
- *The almost symplectic 2-form*

$$\begin{aligned}\widehat{\Omega} &= - d\widehat{p} \wedge d\widehat{q} - \sum_{i<j} \widehat{\Upsilon}_{ij}(\widehat{q}, \widehat{p}) d\widehat{q}_i \wedge d\widehat{q}_j \\ &:= - d\widehat{p} \wedge d\widehat{q} - \sum_{i<j} \sum_{a=1}^{f} \widehat{\mathcal{F}}_a \left(\left(\frac{\partial \widehat{A}^a_j}{\partial \widehat{q}_i} - \frac{\partial \widehat{A}^a_i}{\partial \widehat{q}_j} \right) - \sum_{l<k} \widehat{\mathcal{E}}^a_{lk} (\widehat{A}^l_i \widehat{A}^k_j - \widehat{A}^l_j \widehat{A}^k_i) \right) (d\widehat{q}_i \wedge d\widehat{q}_j),\end{aligned} \tag{2.9}$$

such that $\forall \mathfrak{g} \in \mathcal{G}$ we have the following identities:

$$\begin{aligned} Ad_{\mathfrak{g}} \left[T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}} \quad \widehat{A}_{\widehat{q}} \right] &:= \sum_{a=1}^{f} \omega_a \widehat{E}_a, \\ \widehat{\mathcal{F}} &:= \widehat{p}^T \mathbb{F} \widehat{L}_{\widehat{q}}^{-1} \left(-\widehat{A}^* \widehat{K}_{11}(\widehat{q}) + \widehat{K}^*_{12}(\widehat{q}) \right), \\ \sum_{a=1}^{d_1} \widehat{\mathcal{E}}^a_{lk} \widehat{E}_a &:= [\widehat{E}_l, \widehat{E}_k], \end{aligned}$$

for $l, k \in \{1, \cdots, f\}$, and $i, j \in \{1, \cdots, \dim(\mathcal{Q}) - f\}$, and $\left\{ \widehat{E}_1, \cdots, \widehat{E}_f \right\}$ being a basis for $\mathfrak{G}$. For a local trivialization of $\mathcal{G}$ principal bundle, we define the following block components of the Legendre transformation correspond to the Lie group $\mathcal{G}$ and the rest of the joint parameters of the nonholonomic $MBS(N)$. Accordingly, the Legendre transformation $\mathbb{F}\widehat{L}_{\widehat{q}}$ is defined based on the induced metric on the reduced configuration manifold $\widehat{\mathcal{Q}}$.

$$\begin{aligned} \begin{bmatrix} (T^*_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}) \widehat{K}_{11}(\widehat{q}) (T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}) & (T^*_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}) \widehat{K}_{12}(\widehat{q}) \\ \widehat{K}_{21}(\widehat{q}) (T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}) & \widehat{K}_{22}(\widehat{q}) \end{bmatrix} &:= \mathbb{F} L_q, \\ \mathbb{F}\widehat{L}_{\widehat{q}} := \widehat{A}^* \widehat{K}_{11}(\widehat{q}) \widehat{A} - \widehat{K}_{21}(\widehat{q}) \widehat{A} - \widehat{A}^* \widehat{K}_{12}(\widehat{q}) &+ \widehat{K}_{22}(\widehat{q}). \end{aligned}$$

– *The reduced Hamiltonian*

$$\widehat{H}(\widehat{q},\widehat{p}) = \frac{1}{2}\left\langle \widehat{p}, \mathbb{F}\widehat{L}_{\widehat{q}}^{-1}(\widehat{p})\right\rangle + \widehat{V}(\widehat{q}), \tag{2.10}$$

where if $\mathfrak{e} \in \mathcal{G}$ *represents the identity element, in the local trivialization we have* $\widehat{V}(\widehat{q}) := V(\mathfrak{e}, \widehat{q})$.

Finally in the local coordinates of $T^*\widehat{\mathcal{Q}}$*, Hamilton's equation reads*

$$\begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} = [\widehat{\Omega}]^{-1}(\widehat{q},\widehat{p}) \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix}, \tag{2.11}$$

where $[\widehat{\Omega}]$ *is the vector bundle map naturally associated to the 2-form* $\widehat{\Omega}$*. This map has the following form:*

$$[\widehat{\Omega}](\widehat{q},\widehat{p}) = \begin{bmatrix} [\widehat{\Omega}]_1(\widehat{q},\widehat{p}) & -\,id \\ id & 0 \end{bmatrix},$$

where the zero and identity matrices have the appropriate dimensions and the sub-matrix $[\widehat{\Omega}]_1$ *is defined based on* $\widehat{\Upsilon}_{ij}(\widehat{q},\widehat{p})$ $(i, j = 1, \cdots, \dim(\mathcal{Q}) - f)$.

3 Problem Statement

In this section we formally state an output trajectory tracking control problem for free-base, open-chain multi-body systems with multi-d.o.f. holonomic (non-zero momentum) and nonholonomic joints. The output of such systems is usually the pose of the extremities and the base. More generally, one may be interested in controlling only parts of the motion of the extremities and the base. For example, one may want to control the position of the end-effector and the orientation of the base of a free-floating space manipulator. In this case, the *output manifold* of the system is a quotient manifold, which is locally identified by a submanifold of the space of all possible poses of the end-effector and the base. In this paper, by output manifold we mean a submanifold of the smooth manifold that consists of all possible poses of the extremities of a free-base, open-chain multi-body system.

3.1 Mathematical Formalization and Assumptions

Let an open-chain multi-body system with symmetry be denoted by $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{G})$ or $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G})$. In the holonomic case, $\mathcal{G} = \mathcal{Q}_1$ and the momentum is conserved, and $\mathcal{G}$ is an f-dimensional Lie subgroup of $\mathcal{Q}_1$ for a Chaplygin system. Also, we denote the number of independent control directions by n_c; in general, it is equal to $\dim(\mathcal{Q}) - \dim(\mathcal{G})$.

CON1) We assume that $\mathcal{G}$ is a symmetry group of the holonomic or nonholonomic multi-body system, in the sense introduced in Section 2.

We recall that for a mechanical system control directions are modelled by 1-forms on $\mathcal{Q}$.

CON2) We also assume that there is no control input collocated with the $\mathcal{G}$-orbits of the system. That is, for a set of linearly independent differential 1-forms $\{\mathcal{U}_i \in \Omega^1(\mathcal{Q}) \big| i = 1, \cdots, n_c\}$ corresponding to the directions of the (available) control inputs (in the form of control force or torque), $\forall \xi \in \mathfrak{G}$ we have

$$\mathcal{U}_i(\xi_{\mathcal{Q}}) = 0, \qquad i = 1, \cdots, n_c \tag{3.12}$$

where $\xi_{\mathcal{Q}} \in \mathfrak{X}(\mathcal{Q})$ is the vector field induced by the infinitesimal action of $\mathcal{G}$, corresponding to ξ.

This condition guarantees that there is no actuator at the first joint of a holonomic system, and also the actuators of a nonholonomic system do not overlap with the directions of the nonholonomic constraints.

Definition 1 For a free-base holonomic or nonholonomic $MBS(N)$ with symmetry, we call $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{G}, \{\mathcal{U}_i\}_{i=1}^{n_c})$ or $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G}, \{\mathcal{U}_i\}_{i=1}^{n_c})$, respectively, a *controlled multi-body system* with symmetry.

Let $\{u_i \in C^2(T^*\mathcal{Q} \times \mathbb{R}) \big| i = 1, \cdots, n_c\}$ be a set of twice differentiable functions on the extended phase space (by the time direction). We define the *control input* for a controlled multi-body system with symmetry by

$$\sum_{i=1}^{n_c} u_i \mathcal{U}_i. \tag{3.13}$$

We write the *control Hamilton's equation* for a controlled multi-body system with symmetry as

$$\begin{aligned} [\Omega_{can}] \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} - \begin{bmatrix} \sum_{a=1}^{f} \kappa_a \omega_a \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n_c} u_i \mathcal{U}_i \\ 0 \end{bmatrix}, \\ \omega_a(\dot{q}) = 0 & \qquad a = 1, \cdots, f \end{aligned} \tag{3.14}$$

where ω_a's are the constraint 1-forms defining the nonholonomic distribution $\mathcal{D}$ and κ_a's are the Lagrange multipliers.

Remark 1 Note that this equation is Hamilton's equation for both holonomic and nonholonomic multi-body systems, where in the holonomic case $\omega_a \equiv 0$.

We are interested in controlling the motion of the extremities of a multi-body system with symmetry, in certain directions. Let n_e be the number of extremities of a controlled multi-body system with symmetry, and let $FK_i \colon \mathcal{Q} \to SE(3)$ (for $i = 1, \cdots, n_e$) be the forward kinematics maps for the extremities, defined by

$$FK_i(q) = q_1 \cdots q_{i_0} r^0_{i_0,0}, \qquad i = 1, \cdots, n_e \tag{3.15}$$

where B_{i_0} is the i^{th} extremity (base body is considered as the first extremity, i.e., $1_0 = 1$), $r^0_{i_0,0} \in P_0 \cong SE(3)$ is the initial pose of the i^{th} extremity with respect to the inertial coordinate frame. Note that only the elements of the relative configuration manifolds of the joints that appear in the path connecting B_0 to B_{i_0} are involved in the above equation. For the i^{th} extremity, we identify the corresponding output manifold by the embedded submanifold $\mathcal{R}_i \subseteq SE(3)$, which corresponds to the directions of motion of B_{i_0} that we are interested in. This submanifold comes with the canonical inclusion map $\iota_{\mathcal{R}_i} \colon \mathcal{R}_i \hookrightarrow SE(3)$, and a projection map $\mathfrak{r}_i \colon SE(3) \to \mathcal{R}_i$, such that we have the identity $\mathfrak{r}_i \circ \iota_{\mathcal{R}_i} = id_{\mathcal{R}_i}$, where $id_{\mathcal{R}_i}$ indicates the identity map on $\mathcal{R}_i$. By $FK \colon \mathcal{Q} \to \mathcal{R} := \mathcal{R}_1 \times \cdots \times \mathcal{R}_{n_e}$ we denote the collection of FK_i composed with the projection maps. We also denote the induced projection map that projects $\mathcal{P}_e := SE(3) \times \cdots \times SE(3)$ $(n_e - times)$ to $\mathcal{R}$ by $\mathfrak{r} \colon \mathcal{P}_e \to \mathcal{R}$. The manifold $\mathcal{R}$ is called the *output manifold*, and $n_o := \dim(\mathcal{R})$ indicates the dimension of the output.

Remark 2 Note that in general the projection map $\mathfrak{r}$ is not defined globally. Whenever the projection map does not make sense globally, we define it in a tubular neighbourhood in $\mathcal{P}_e$ around the submanifold $\mathcal{R}$.

Consider a curve $\gamma \colon \mathbb{R} \to \mathcal{R}$ in the output manifold, corresponding to the desired motion of the extremities.

CON3) It is always assumed that the curve $t \mapsto \gamma(t)$ is a feasible trajectory for the system. That is, it respects the nonholonomic constraints and the momentum conservation, and also it is in the image of the forward kinematics map FK with a full rank Jacobian.

CON4) We also assume that the number of control inputs n_c is greater than or equal to the dimension of the output manifold, i.e., $D = \dim(\mathcal{Q}) \geq n_c \geq n_o$.

Remark 3 Condition CON4 together with the fact that the control directions are linearly independent guarantee local controllability of a controlled holonomic or nonholonomic open-chain multi-body system with symmetry at the configurations away from the singularities of the Jacobian [26]. In the following we assume that the system is always away from singular configurations.

Problem 1 (Control Problem) Let $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G}, \{\mathcal{U}_i\}_{i=1}^{n_c})$ be a controlled multi-body system with symmetry, and let $\gamma \colon \mathbb{R} \to \mathcal{R}$ be a desired motion of its extremities. Find a set of twice differentiable functions $\left\{ u_i \in C^2(T^*\mathcal{Q} \times \mathbb{R}) \,\middle|\, i = 1, \cdots, n_c \right\}$, such that the output $FK(q(t))$ tracks the curve γ with an exponentially decreasing error. We can formulate the controlled system as

$$\text{Controlled System: } [\Omega_{can}] \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} - \begin{bmatrix} \sum_{a=1}^{f} \kappa_a \omega_a \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n_c} u_i \mathcal{U}_i \\ 0 \end{bmatrix},$$

$$\text{Nonholonomic Constraints: } \omega_a(\dot{q}) = 0, \qquad a = 1, \cdots, f \qquad (3.16)$$

$$\text{Output: } FK(q(t)) = (\mathfrak{r}_1(q_1(t)), \mathfrak{r}_2(q_1(t) \cdots q_{2_0}(t)), \cdots, \mathfrak{r}_{n_e}(q_1(t) \cdots q_{n_{e0}}(t))).$$

We defined the problem for a nonholonomic open-chain multi-body system with symmetry. In the holonomic case, we have no constraint 1-form ω_a in (3.16), the distribution $\mathcal{D} = T\mathcal{Q}$. This problem is not precise, since we have neither defined the error, nor the exponential stability on manifolds. After reformulating the problem in the following section, we rigorously define the error in Section 4 and the exponential stability in Definition 5.

3.2 Reduced Hamilton's Equation and Reconstruction

In this section, we use the reduction theories stated in Section 2 and their corresponding *reconstruction* equations to reformulate Problem 1, in the reduced phase space. One of the premises of this section is to introduce a notation to treat holonomic and nonholonomic cases at the same time. We denote a reduced holonomic (nonholonomic) open-chain multi-body system by $(\check{\mathcal{S}}, \check{\Omega}, \check{H})$, where $\check{\mathcal{S}}$ is the reduced phase space, $\check{\Omega} \in \Omega^2(\check{\mathcal{S}})$ is the (almost) symplectic 2-form on the reduced phase space, $\check{H}\colon \check{\mathcal{S}} \to \mathbb{R}$ is the reduced Hamiltonian. An element of $\check{\mathcal{S}}$ is denoted by $(\check{q}, \check{p})$, which has a configuration part $\check{q}$ and a momentum part $\check{p}$ that may not have the same dimensions. In the holonomic case, the dimension of $\check{q}$ is larger than the dimension of $\check{p}$, and for nonholonomic Chaplygin systems both dimensions are equal. Then the reduced Hamilton's equation for the system $(\check{\mathcal{S}}, \check{\Omega}, \check{H})$ reads

$$\begin{bmatrix} \dot{\check{q}} \\ \dot{\check{p}} \end{bmatrix} = [\check{\Omega}]^{-1}(\check{q}, \check{p}) \begin{bmatrix} \frac{\partial \check{H}}{\partial \check{q}} \\ \frac{\partial \check{H}}{\partial \check{p}} \end{bmatrix}, \tag{3.17}$$

where $[\check{\Omega}]$ is the vector bundle map naturally associated to the 2-form $\check{\Omega}$. For the holonomic case, this equation is equivalent to (2.7), and in the nonholonomic case it is (2.11). In order to control the extremities of a controlled multi-body with symmetry in the inertial coordinate frame, not only we need the reduced Hamilton's equation but also the equations corresponding to the reduced parameters of the system. The process of recovering these equations is called *reconstruction*. The reconstruction equations are a set of first order differential equations for the symmetry group parameters (the whole or part of the first joint parameters) that involve the relative positions and velocities of other joints.

For a holonomic open-chain multi-body system with symmetry, where $\mathcal{G} = \mathcal{Q}_1$, the reconstruction yields the velocity of B_1 (base) with respect to the inertial coordinate frame and expressed in the coordinate frame attached to B_1 (body velocity):

$$T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}(\dot{\mathfrak{g}}) = \overline{K}_{11}^{-1}(\overline{q}) \mathrm{Ad}^*_{\overline{q}_1}(\mu) - A_{\overline{q}} \dot{\overline{q}}, \tag{3.18}$$

where $\mathfrak{g}$ is an element of $\mathcal{G}$, $\mu \in \mathfrak{G}^*$ is the constant momentum of the system, and $A_{\overline{q}}\colon T_{\overline{q}}\overline{\mathcal{Q}} \to \mathfrak{G}$ and $\overline{K}_{11}(\overline{q})$ are defined in Theorem 1.

For a nonholonomic system with symmetry, where $\mathcal{G} \subseteq \mathcal{Q}_1$ as defined in Section 2, the reconstruction leads to the body velocities of the base.

$$T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}(\dot{\mathfrak{g}}) = -\widehat{A}_{\widehat{q}} \dot{\widehat{q}}, \tag{3.19}$$

where in the local trivialization, we have $q_1 = (\mathfrak{g}, \widetilde{q}_1)$, and $\widehat{A}_{\widehat{q}} \colon T_{\widehat{q}} \widehat{\mathcal{Q}} \to \mathfrak{G}$ is defined in Theorem 2.

For a multi-body system with symmetry, we can uniquely identify n_c control directions $\{\check{\mathcal{U}}\}_{i=1}^{n_c}$ in the reduced phase space $\check{\mathcal{S}}$. Further, we call the four tuple $(\check{\mathcal{S}}, \check{\Omega}, \check{H}, \{\check{\mathcal{U}}_i\}_{i=1}^{n_c})$ the *reduced controlled multi-body system.*

Problem 2 Let $(\check{\mathcal{S}}, \check{\Omega}, \check{H}, \{\check{\mathcal{U}}_i\}_{i=1}^{n_c})$ be the reduced controlled multi-body system, and let $\gamma \colon \mathbb{R} \to \mathcal{R}$ be a desired motion of the extremities of the original controlled multi-body system with symmetry. Find a set of twice differentiable functions $\left\{ \check{u}_i \in C^2(\mathcal{G} \times \check{\mathcal{S}} \times \mathbb{R}) \,\middle|\, i = 1, \cdots, n_c \right\}$, such that the output $FK(q(t))$ tracks the curve γ with an exponentially decreasing error. We can reformulate the controlled multi-body system as

$$\text{Controlled System: } [\check{\Omega}] \begin{bmatrix} \dot{\check{q}} \\ \dot{\check{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \check{H}}{\partial \check{q}} \\ \frac{\partial \check{H}}{\partial \check{p}} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n_c} \check{u}_i \check{\mathcal{U}}_i \\ 0 \end{bmatrix},$$

Reconstruction Equation: (3.18) or (3.19)

$$\text{Output: } FK(q(t)) = (\mathfrak{r}_1(q_1(t)), \mathfrak{r}_2(q_1(t) \cdots q_{2_0}(t)), \cdots, \mathfrak{r}_{n_e}(q_1(t) \cdots q_{n_{e0}}(t))).$$

We write the above equations for a controlled holonomic multi-body system:

$$\text{Controlled System: } [\check{\Omega}](\widetilde{q}) \begin{bmatrix} \dot{\widetilde{q}}_1 \\ \dot{\overline{q}} \\ \dot{\overline{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} \\ \frac{\partial \overline{H}}{\partial \overline{q}} \\ \frac{\partial \overline{H}}{\partial \overline{p}} \end{bmatrix} + \begin{bmatrix} 0 \\ \overline{u} \\ 0 \end{bmatrix},$$

$$\text{Reconstruction Equation: } T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}(\dot{\mathfrak{g}}) = \overline{K}_{11}^{-1}(\overline{q}) \mathrm{Ad}^*_{\widetilde{q}_1}(\mu) - A_{\overline{q}} \dot{\overline{q}}, \tag{3.20}$$

$$\text{Output: } FK(q(t)) = (\mathfrak{r}_1(q_1(t)), \mathfrak{r}_2(q_1(t) \cdots q_{2_0}(t)), \cdots, \mathfrak{r}_{n_e}(q_1(t) \cdots q_{n_{e0}}(t))),$$

where $\overline{u}$ is the control input in the given coordinate chart.

And, for a controlled nonholonomic multi-body system we have

$$\text{Controlled System: } [\widehat{\Omega}](\widehat{q}, \widehat{p}) \begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix} + \begin{bmatrix} \widehat{u} \\ 0 \end{bmatrix},$$

$$\text{Reconstruction Equations: } T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}}(\dot{\mathfrak{g}}) = -\widehat{A}_{\widehat{q}} \dot{\widehat{q}}, \tag{3.21}$$

$$\text{Output: } FK(q(t)) = (\mathfrak{r}_1(q_1(t)), \mathfrak{r}_2(q_1(t) \cdots q_{2_0}(t)), \cdots, \mathfrak{r}_{n_e}(q_1(t) \cdots q_{n_{e0}}(t))),$$

where $\widehat{u}$ are the control input in the given coordinate chart, and $\mathfrak{g} \in \mathcal{G} \subseteq \mathcal{Q}_1$ is an element of the subgroup of $\mathcal{Q}_1$. The equations (3.20) and (3.21) formally define the control problem in the reduced phase space.

4 End-effector Pose and Velocity Error

4.1 Error Function

In this section we introduce a quadratic error function, based on an induced metric on the output manifold $\mathcal{R}$ from a left-invariant metric on $\mathcal{P}_e$. This error function represents the distance between the actual output and the desired output of the system in the ambient manifold $\mathcal{P}_e$. Different methods of defining error functions and their corresponding gradients are discussed in Bullo's thesis [5]. The definition of the error function adopted in this paper is due to its geometrical interpretation. But, the following development can readily be applied to other definitions of the error function.

Definition 2 A smooth two variable function $E_r\colon \mathcal{R}\times\mathcal{R}\to\mathbb{R}^{\geq 0}$ is a *symmetric error function* on $\mathcal{R}$, if $E_r(r_1,r_2)\geq 0$ for every $r_1,r_2\in\mathcal{R}$ where the equality holds if and only if $r_1=r_2$, and $E_r(r_1,r_2)=E(r_2,r_1)$.

Let $\iota_{\mathcal{R}}\colon \mathcal{R}\hookrightarrow\mathcal{P}_e$ be the inclusion map, and let $\mathcal{K}_i$ (for $i=1,\cdots,n_e$) be an arbitrary left invariant Riemannian metric on $SE(3)$ corresponding to the i^{th} extremity. These metrics induce a left invariant metric $\mathcal{K}:=\mathcal{K}_1\oplus\cdots\oplus\mathcal{K}_{n_e}$ on $\mathcal{P}_e$. Consider two elements $r_1,r_2\in\mathcal{R}$ and the one-parameter subgroup $\sigma\colon\mathbb{R}\to\mathcal{P}_e$ in $\mathcal{P}_e$ that connects $\iota_{\mathcal{R}}(r_1)$ to $\iota_{\mathcal{R}}(r_2)$. We define the distance between $r_1\in\mathcal{R}$ and $r_2\in\mathcal{R}$ by the length of the portion of σ that connects $\iota_{\mathcal{R}}(r_1)\in\mathcal{P}_e$ to $\iota_{\mathcal{R}}(r_2)\in\mathcal{P}_e$ in the ambient manifold. That is,

$$
\begin{aligned}
dis(r_1,r_2) &= \int_0^1\sqrt{\mathcal{K}_{\sigma(\mathfrak{s})}\left(\frac{d\sigma(\mathfrak{s})}{d\mathfrak{s}},\frac{d\sigma(\mathfrak{s})}{d\mathfrak{s}}\right)}d\mathfrak{s} \qquad \sigma(0)=\iota_{\mathcal{R}}(r_1),\sigma(1)=\iota_{\mathcal{R}}(r_2)\\
&= \int_0^1\sqrt{\mathcal{K}_{e_{\mathcal{P}}}\left(\sigma^{-1}(\mathfrak{s})\frac{d\sigma(\mathfrak{s})}{d\mathfrak{s}},\sigma^{-1}(\mathfrak{s})\frac{d\sigma(\mathfrak{s})}{d\mathfrak{s}}\right)}d\mathfrak{s},
\end{aligned}
$$

where $\mathfrak{s}\in\mathbb{R}$ is the curve parameter for σ, and $e_{\mathcal{P}}$ is the identity element of $\mathcal{P}_e$. Since one-parameter subgroups are the integral curves of left invariant vector fields, we can write the curve as

$$
\sigma(\mathfrak{s})=\iota_{\mathcal{R}}(r_1)\exp\left(\mathfrak{s}\exp^{-1}(r_e)\right),
$$

where $r_e:=\iota_{\mathcal{R}}(r_1)^{-1}\iota_{\mathcal{R}}(r_2)$ that is called *output pose error*. Consequently,

$$
\sigma^{-1}(\mathfrak{s})\frac{d\sigma(\mathfrak{s})}{d\mathfrak{s}}=\exp^{-1}(r_e),
$$

which is a constant vector in $\mathfrak{P}_e$, the Lie algebra of $\mathcal{P}_e$. As the result,

$$
dis(r_1,r_2)=\sqrt{\mathcal{K}_{e_{\mathcal{P}}}\left(\exp^{-1}(r_e),\exp^{-1}(r_e)\right)}=\parallel\exp^{-1}(r_e)\parallel_{\mathcal{K}_{e_{\mathcal{P}}}},
$$

where $\|\cdot\|_{\mathcal{K}_{e_{\mathcal{P}}}}$ is the induced norm on $\mathfrak{P}_e$ by the left invariant metric $\mathcal{K}$. This length is also equal to the length of the one-parameter subgroup that connects $e_{\mathcal{P}}\in\mathcal{P}_e$ to r_e. It is easy to show that the error function defined by

$$
E_r(r_1,r_2)=\frac{1}{2}dis(r_1,r_2)^2=\frac{1}{2}\parallel\exp^{-1}(r_e)\parallel^2_{\mathcal{K}_{e_{\mathcal{P}}}} \tag{4.22}
$$

is a quadratic, smooth, symmetric error function on $\mathcal{R}$.

Remark 4 Although it would have been natural to define the length of the geodesic corresponding to the induced metric on $\mathcal{R}$ by $\mathcal{K}$ as the error function, the error function defined by 4.22 is more efficient computationally.

We denote the exterior derivative of the error function with respect to the first and second input by d_1 and d_2, respectively. Also, let us use $\mathcal{K}_P$ for the self-adjoint positive definite map between $\mathfrak{P}_e$ and $\mathfrak{P}_e^*$ corresponding to the induced metric on the Lie algebra. Then, for all $v_{r_1} \in T_{r_1}\mathcal{R}$ and $w_{r_2} \in T_{r_2}\mathcal{R}$ we have the following equations:

$$\begin{aligned}\langle d_1 E_r(r_1, r_2), v_{r_1}\rangle &= \frac{1}{2}\left\langle d_1\mathcal{K}_{e_\mathcal{P}}\left(\exp^{-1}(r_e), \exp^{-1}(r_e)\right), v_{r_1}\right\rangle \\ &= -\mathcal{K}_{e_\mathcal{P}}\left(\exp^{-1}(r_e), \Theta(r_e)^{-1}\mathrm{Ad}_{\iota_\mathcal{R}(r_2)^{-1}}(T_{\iota_\mathcal{R}(r_1)}R_{\iota_\mathcal{R}(r_1)^{-1}})(T_{r_1}\iota_\mathcal{R})(v_{r_1})\right) \\ &= \left\langle -\mathcal{K}_P\exp^{-1}(r_e), \Theta(r_e)^{-1}\mathrm{Ad}_{\iota_\mathcal{R}(r_2)^{-1}}(T_{\iota_\mathcal{R}(r_1)}R_{\iota_\mathcal{R}(r_1)^{-1}})(T_{r_1}\iota_\mathcal{R})(v_{r_1})\right\rangle \\ &= \left\langle -(T_{r_1}^*\iota_\mathcal{R})(T_{\iota_\mathcal{R}(r_1)}^*R_{\iota_\mathcal{R}(r_1)^{-1}})\mathrm{Ad}_{\iota_\mathcal{R}(r_2)^{-1}}^*(\Theta(r_e)^{-1})^*\mathcal{K}_P\exp^{-1}(r_e), v_{r_1}\right\rangle,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\langle d_2 E_r(r_1, r_2), w_{r_2}\rangle &= \frac{1}{2}\left\langle d_2\mathcal{K}_{e_\mathcal{P}}\left(\exp^{-1}(r_e), \exp^{-1}(r_e)\right), w_{r_2}\right\rangle \\ &= \mathcal{K}_{e_\mathcal{P}}\left(\exp^{-1}(r_e), \Theta(r_e)^{-1}(T_{\iota_\mathcal{R}(r_2)}L_{\iota_\mathcal{R}(r_2)^{-1}})(T_{r_2}\iota_\mathcal{R})(w_{r_2})\right) \\ &= \left\langle \mathcal{K}_P\exp^{-1}(r_e), \Theta(r_e)^{-1}(T_{\iota_\mathcal{R}(r_2)}L_{\iota_\mathcal{R}(r_2)^{-1}})(T_{r_2}\iota_\mathcal{R})(w_{r_2})\right\rangle \\ &= \left\langle (T_{r_2}^*\iota_\mathcal{R})(T_{\iota_\mathcal{R}(r_2)}^*L_{\iota_\mathcal{R}(r_2)^{-1}})(\Theta(r_e)^{-1})^*\mathcal{K}_P\exp^{-1}(r_e), w_{r_2}\right\rangle,\end{aligned} \tag{4.24}$$

where $d_1E_r(r_1, r_2) \in T_{r_1}^*\mathcal{R}$ and $d_2E_r(r_1, r_2) \in T_{r_2}^*\mathcal{R}$. The map $\Theta(r_e)\colon \mathfrak{P} \to \mathfrak{P}$

$$\Theta(r_e) := \int_0^1 \mathrm{Ad}_{\exp(\mathfrak{s}\exp^{-1}(r_e))}d\mathfrak{s},$$

which is the linear map that appears in the tangent map of the exponential map, corresponding to non-commutativity of Lie algebra elements [10]. The tangent map to $\exp\colon \mathfrak{P} \to \mathcal{P}$ at the element $\xi \in \mathfrak{P}$ is defined as

$$T_\xi \exp = T_e L_{\exp(\xi)}\Theta(\exp(\xi)).$$

The Lie algebra of $\mathcal{P}$ is denoted by $\mathfrak{P}$. The map Θ is invertible in an open neighbourhood of the identity element. From now on, we always assume that $r_e = \iota_\mathcal{R}(r_1)^{-1}\iota_\mathcal{R}(r_2)$ belongs to a symmetric neighbourhood of identity such that $\Theta(r_e)$ is invertible.

4.2 Velocity Error

For Lie groups, one can use the tangent to left or right translation maps to trivialize the tangent bundle of the Lie group. Hence, we identify the tangent space at each element of the Lie group with the Lie algebra. In this section we use the right translation map of $\mathcal{P}_e$ to define an isomorphism between the tangent spaces of $\mathcal{R}$ to define the output velocity error of a controlled multi-body system.

For an element $r' \in \mathcal{P}_e$, let $TR_{r'}\colon T\mathcal{P}_e \to T\mathcal{P}_e$ be the tangent to the right translation map by r' on the Lie group $\mathcal{P}_e$. Recall that the canonical inclusion map and the projection map for the output manifold $\mathcal{R}$ are denoted by $\iota_{\mathcal{R}}\colon \mathcal{R} \to \mathcal{P}_e$ and $\mathfrak{r}\colon \mathcal{P}_e \to \mathcal{R}$, respectively. We define the linear isomorphism $\Gamma_{(r_1,r_2)}\colon T_{r_2}\mathcal{R} \to T_{r_1}\mathcal{R}$ by

$$\begin{aligned}\Gamma_{(r_1,r_2)} &= (T_{\iota_{\mathcal{R}}(r_1)}\mathfrak{r})(T_{\iota_{\mathcal{R}}(r_2)}R_{\iota_{\mathcal{R}}(r_2)^{-1}\iota_{\mathcal{R}}(r_1)})(T_{r_2}\iota_{\mathcal{R}}) \\ &= (T_{\iota_{\mathcal{R}}(r_1)}\mathfrak{r})(T_{\iota_{\mathcal{R}}(r_2)}R_{r_e^{-1}})(T_{r_2}\iota_{\mathcal{R}}),\end{aligned} \tag{4.25}$$

which is a well-defined map for any $r_2 \in \mathcal{R}$ in a neighbourhood of $r_1 \in \mathcal{R}$. Unlike the tangent map of the right translation, which is globally defined, the isomorphism defined in (4.25) can only make sense, locally. The size of the neighbourhood of r_1, in which the above map is a linear isomorphism, depends on the given projection map $\mathfrak{r}$.

Definition 3 Let $\gamma_1\colon \mathbb{R} \to \mathcal{R}$ and $\gamma_2\colon \mathbb{R} \to \mathcal{R}$ be two curves, and $t \in \mathbb{R}$ be their curve parameter. We call

$$V_e(t) := \dot{\gamma}_1(t) - \Gamma_{(\gamma_1(t),\gamma_2(t))}(\dot{\gamma}_2(t)) \tag{4.26}$$

the *output velocity error* of a system.

Note that unlike the case of systems on linear spaces, where the velocity error can be simply defined by subtracting the velocity of curves, in the case of systems on manifolds, we need an isomorphism Γ to define the velocity error. In the next section, we use this notion to design control laws for controlled multi-body systems with symmetry.

Definition 4 The isomorphism Γ is called *compatible* with the error function E_r, if $\forall r_1, r_2 \in \mathcal{R}$ the following equality holds [5]:

$$d_2E_r(r_1,r_2) = -\Gamma^*_{(r_1,r_2)}d_1E_r(r_1,r_2). \tag{4.27}$$

The map $\Gamma^*_{(r_1,r_2)}\colon T^*_{r_1}\mathcal{R} \to T^*_{r_2}\mathcal{R}$ is the dual map that naturally corresponds to the linear isomorphism $\Gamma_{(r_1,r_2)}$.

Lemma 1 ([5]) *Let $\gamma_1\colon \mathbb{R} \to \mathcal{R}$ and $\gamma_2\colon \mathbb{R} \to \mathcal{R}$ be two curves. If Γ is compatible with the error function, then*

$$\frac{d}{dt}E_r(\gamma_1(t),\gamma_2(t)) = \langle d_1E_r(\gamma_1(t),\gamma_2(t)), V_e(t)\rangle\,. \tag{4.28}$$

Proposition 1 ([9]) *If $\mathcal{R}$ is the right translation of a Lie subgroup of $\mathcal{P}_e$, then the linear isomorphism Γ in (4.25) is compatible with the error function E_r in (4.22).*

CON5) From now on, we assume that $\mathcal{R} \subseteq \mathcal{P}_e$ is a Lie subgroup of $\mathcal{P}_e$. Note that any statement in the rest of this paper also holds for any right translation of Lie subgroups of $\mathcal{P}_e$.

To simplify our notation, from now on, we do not use the inclusion map $\iota_{\mathcal{R}}$ to show elements of the Lie subgroup $\mathcal{R}$ in $\mathcal{P}_e$, whenever it does not result any confusion.

5 Input-output Linearization and Inverse Dynamics in the Reduced Phase Space

In this section, we present an input-output linearization process for the reduced dynamics of controlled open-chain multi-body systems with symmetry, based on left trivialization of the tangent bundles of $\mathcal{Q}$ and $SE(3)$. This process is useful for deriving an output tracking feed-forward PD (proportional-derivative)-like controller for such systems, which is the subject of the next section.

Consider the Jacobian maps for the extremities that map the joint velocities to the twist of the extremities with respect to the inertial coordinate frame and expressed in the body coordinate frames (attached to the extremities). We may use the fact that $\mathcal{Q}$ is a Lie group and left trivialize its tangent bundle. As a result, we define the Jacobian maps $J_i^0 \colon \mathcal{Q} \times \mathfrak{Q} \to se(3)$ $(i = 1, \cdots, n_e)$ by

$$
\begin{aligned}
J_1^0 &:= \left(T_{FK_1(q)} L_{FK_1(q)^{-1}}\right) T_q FK_1 \left(T_{e_1} L_{q_1}\right) \left(T_{e_1} \iota_1\right) = \mathrm{Ad}_{(r_{1,0}^0)^{-1}} \begin{bmatrix} T_{e_1}\iota_1 & 0 \end{bmatrix}, \\
J_i^0 &:= \left(T_{FK_i(q)} L_{FK_i(q)^{-1}}\right) T_q FK_i \left(T_{e_1} L_{q_1} \oplus \cdots \oplus T_{e_{i_0}} L_{q_{i_0}}\right) \left(T_{e_1}\iota_1 \oplus \cdots \oplus T_{e_{i_0}}\iota_{i_0}\right) \\
&= \mathrm{Ad}_{(r_{i_0,0}^0)^{-1}} \begin{bmatrix} \mathrm{Ad}_{(q_2 \cdots q_{i_0})^{-1}} T_{e_1}\iota_1 & \cdots & T_{e_{i_0}}\iota_{i_0} \end{bmatrix},
\end{aligned}
$$

where $\mathcal{Q}$ is the Lie algebra of $\mathcal{Q}$, $r_{i_0,0}^0 \in SE(3)$ $(i = 1, \cdots, n_e)$ is the initial pose of the coordinate frame attached to an extremity (with respect to the inertial coordinate frame), and $\iota_j \colon \mathcal{Q}_j \to SE(3)$ for $j = 1, \cdots, N$ are the canonical inclusion maps. We denote the collection of J_i^0's by

$$
J_q := \begin{bmatrix} (J_1^0)_q \\ \vdots \\ (J_{n_e}^0)_q \end{bmatrix} : \mathfrak{Q} \to \mathfrak{P}_e.
$$

Note that the Jacobian maps $q \mapsto (J_i^0)_q$'s and consequently $q \mapsto J_q$ are $\mathcal{Q}_1$ invariant, as was detailed in [10]. We now define the Jacobian maps whose images are projected to the Lie algebra of the output manifold $\mathfrak{R}$:

$$
\jmath_i^0 := \mathcal{E}_i \circ J_i^0 := T_{e_{\mathcal{P}}} \left(L_{\mathfrak{r}_i(FK_i(q))^{-1}} \circ \mathfrak{r}_i \circ L_{FK_i(q)}\right) J_i^0, \qquad i = 1, \cdots, n_e \tag{5.29}
$$

where the fibre-wise linear maps $(\mathcal{E}_i)_q\colon se(3) \to \mathfrak{R}_i$ are obtained from the derivative of the projection maps $\mathfrak{r}_i\colon SE(3) \to \mathcal{R}_i$ (see page 13) and the left translation map. We then denote the collection of $\mathcal{E}_i$'s by $\mathcal{E}_q := (\mathcal{E}_1)_q \oplus \cdots \oplus (\mathcal{E}_{n_e})_q\colon \mathfrak{P}_e \to \mathfrak{R}$. As a result, we introduce the fibre-wise linear (Jacobian) map $\jmath_q\colon \to \mathfrak{R}$ by

$$\jmath_q := \begin{bmatrix} (\jmath_1^0)_q \\ \vdots \\ (\jmath_{n_e}^0)_q \end{bmatrix} = \mathcal{E}_q \circ J_q.$$

In other words, in terms of the output function FK that is a function of the projection maps

$$\jmath_q = (T_{FK(q)} L_{FK(q)^{-1}})\ (T_q FK)\ (T_{(e_1,\cdots,e_N)} L_q)\ (T_{e_1}\iota_1 \oplus \cdots \oplus T_{e_{i_0}}\iota_N).$$

Consider an initial phase for the controlled multi-body system with symmetry $(q_0, p_0) \in T^*\mathcal{Q}$ that induces an initial phase for the reduced controlled multi-body system $(\check{q}_0, \check{p}_0) \in \check{\mathcal{S}}$. We denote the integral curves of the system and its reduced dynamics by $t \mapsto (q(t), p(t)) \in T^*\mathcal{Q}$ and $t \mapsto (\check{q}(t), \check{p}(t)) \in \check{\mathcal{S}}$, respectively.

CON6) We assume that the above initial conditions respect a pre-chosen constant (non-zero) momentum of the system, and respect the nonholonomic constraints.

We restrict the map $\jmath\colon \mathcal{Q} \times \mathfrak{Q} \to \mathfrak{R}$ to the curve

$$t \mapsto \left(q(t), \tau(t) := T_{q(t)} L_{q(t)^{-1}} \dot{q}(t) = T_{q(t)} L_{q(t)^{-1}} \mathbb{F}L_{q(t)}^{-1}(p(t))\right),$$

which is defined as the result of the integral curve of the system in the phase space $(q(t), p(t))$. This is the curve in the trivialized bundle $\mathcal{Q} \times \mathfrak{Q}$ that corresponds to the evolution of the relative twists of the extremities. Then the image of this curve under the map $\jmath$,

$$t \mapsto \jmath_{q(t)}(\tau(t)) = \mathcal{E}_{q(t)} \circ J_{q(t)}(\tau_1(t), \cdots, \tau_N(t)) := \mathcal{E}_{q(t)} \circ J_{q(t)}(\tau(t)) \in \mathfrak{R} \quad (5.30)$$

corresponds to the evolution of the output of the system in the Lie algebra of the output manifold $\mathfrak{R}$. In the holonomic case the first entry of this map $\tau_1(t) = T_{q_1(t)} L_{q(t)^{-1}} \dot{q}_1(t)$ is the relative twist of the first body with respect to the inertial coordinate frame and expressed in the body coordinate frame, which is the outcome of the reconstruction equation (3.18). That is,

$$\tau_1(t) = \overline{K}_{11}^{-1}(\overline{q}(t)) \mathrm{Ad}^*_{\widehat{q}_1(t)}(\mu) - A_{\overline{q}(t)} \dot{\overline{q}}(t). \quad (5.31)$$

Based on (3.19), for a nonholonomic open-chain multi-body system with symmetry we have

$$\begin{aligned} \tau_1(t) &= (\mathfrak{g}(t)\widehat{q}_1(t))^{-1} \left(\dot{\mathfrak{g}}(t)\widehat{q}_1(t) + \mathfrak{g}(t)\dot{\widehat{q}}_1(t)\right) = \mathrm{Ad}_{\widehat{q}_1(t)^{-1}} \left(\mathfrak{g}^{-1}(t)\dot{\mathfrak{g}}(t)\right) + \widehat{q}_1^{-1}(t)\dot{\widehat{q}}_1(t) \\ &= -\mathrm{Ad}_{\widehat{q}_1(t)^{-1}} \left(\widehat{A}_{\widehat{q}(t)} \dot{\widehat{q}}(t)\right) + \widehat{q}_1^{-1}(t)\dot{\widehat{q}}_1(t). \end{aligned} \quad (5.32)$$

Let $[\check{\Omega}]\colon T\check{\mathcal{S}} \to T^*\check{\mathcal{S}}$ be the vector bundle map corresponding to the 2-form $\check{\Omega} \in \Omega^2(\check{\mathcal{S}})$, in the dynamical equations of a reduced controlled holonomic or nonholonomic open-chain multi-body system. For a holonomic multi-body system with symmetry and with non-zero conserved momentum, this map is equal to $[\widetilde{\Omega}]$ as defined in Theorem 1, and it is easy to check that

$$[\widetilde{\Omega}]^{-1}(\widetilde{q}) = \begin{bmatrix} [\widetilde{\Omega}]_{11}^{-1} & 0 & -[\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12} \\ 0 & 0 & id \\ -[\widetilde{\Omega}]_{12}^{T}[\widetilde{\Omega}]_{11}^{-1} & -id & [\widetilde{\Omega}]_{22} + [\widetilde{\Omega}]_{12}^{T}[\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12} \end{bmatrix}.$$

For a nonholonomic multi-body system with symmetry, this map is equal to $[\widehat{\Omega}]$ as defined in Theorem 2, and its inverse is

$$[\widehat{\Omega}]^{-1}(\widehat{q}, \widehat{p}) = \begin{bmatrix} 0 & id \\ -id & [\widehat{\Omega}]_1 \end{bmatrix}.$$

Note that for the holonomic multi-body systems where $\mathcal{G}_\mu = \mathcal{G}$, the form of the vector bundle map $[\widetilde{\Omega}]$ and consequently its inverse is the same as the maps appearing in the nonholonomic case.

Therefore, we can write the speed of the integral curve of a controlled holonomic open-chain multi-body system in the reduced phase space as:

$$\begin{aligned}
\begin{bmatrix} \dot{\widetilde{q}}_1 \\ \dot{\overline{q}} \\ \dot{\overline{p}} \end{bmatrix} &= [\widetilde{\Omega}]^{-1}(\widetilde{q}(t)) \begin{bmatrix} \frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} \\ \frac{\partial \widetilde{H}}{\partial \overline{q}} + \overline{u} \\ \frac{\partial \widetilde{H}}{\partial \overline{p}} \end{bmatrix} \\
&= \begin{bmatrix} [\widetilde{\Omega}]_{11}^{-1} & 0 & -[\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12} \\ 0 & 0 & id \\ -[\widetilde{\Omega}]_{12}^{T}[\widetilde{\Omega}]_{11}^{-1} & -id & [\widetilde{\Omega}]_{22} + [\widetilde{\Omega}]_{12}^{T}[\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12} \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} \\ \frac{\partial \widetilde{H}}{\partial \overline{q}} + \overline{u} \\ \frac{\partial \widetilde{H}}{\partial \overline{p}} \end{bmatrix} \\
&= \begin{bmatrix} [\widetilde{\Omega}]_{11}^{-1} \frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} - \left([\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12}\right) \frac{\partial \widetilde{H}}{\partial \overline{p}} \\ \frac{\partial \widetilde{H}}{\partial \overline{p}} \\ -\left([\widetilde{\Omega}]_{12}^{T}[\widetilde{\Omega}]_{11}^{-1}\right) \frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} - \frac{\partial \widetilde{H}}{\partial \overline{q}} - \overline{u} + \left([\widetilde{\Omega}]_{22} + [\widetilde{\Omega}]_{12}^{T}[\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12}\right) \frac{\partial \widetilde{H}}{\partial \overline{p}} \end{bmatrix}.
\end{aligned} \tag{5.33}$$

In the nonholonomic case, the above calculation is performed as follows:

$$\begin{aligned}
\begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} &= [\widehat{\Omega}]^{-1}(\widehat{q}(t), \widehat{p}(t)) \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} + \widehat{u} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix} = \begin{bmatrix} 0 & id \\ -id & [\widehat{\Omega}]_1 \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} + \widehat{u} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{p}} \\ -\frac{\partial \widehat{H}}{\partial \widehat{q}} - \widehat{u} + [\widehat{\Omega}]_1 \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix}.
\end{aligned} \tag{5.34}$$

Based on the first two equations in (5.33), in the holonomic case we define the functions

$$\widetilde{\mathfrak{q}}(\widetilde{q}(t),\overline{p}(t)) := \dot{\widetilde{q}}_1(t) = [\widetilde{\Omega}]_{11}^{-1}\frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} - \left([\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12}\right)\frac{\partial \widetilde{H}}{\partial \overline{p}}, \tag{5.35}$$

$$\overline{\mathfrak{q}}(\widetilde{q}(t),\overline{p}(t)) := \dot{\overline{q}}(t) = \frac{\partial \widetilde{H}}{\partial \overline{p}}, \tag{5.36}$$

and for a nonholonomic system, i.e., based on (5.34), we only have the inverse of the Legendre transformation in the reduced phase space:

$$\widehat{\mathfrak{q}}(\widehat{q}(t),\widehat{p}(t)) := \dot{\widehat{q}}(t) = \frac{\partial \widehat{H}}{\partial \widehat{p}}. \tag{5.37}$$

By substituting these equations in (5.31) and (5.32), we obtain the following relations for holonomic and nonholonomic cases, respectively:

$$\tau_1(t) = \overline{K}_{11}^{-1}(\overline{q}(t))\mathrm{Ad}^*_{\widetilde{q}_1(t)}(\mu) - A_{\overline{q}(t)}\overline{\mathfrak{q}}(\widetilde{q}(t),\overline{p}(t)).$$

and

$$\tau_1(t) = -\mathrm{Ad}_{\widehat{q}_1^{-1}(t)}\left(\widehat{A}_{\widehat{q}(t)}\widehat{\mathfrak{q}}((\widehat{q}(t),\widehat{p}(t)))\right) + \widehat{q}_1^{-1}(t)\widehat{\mathfrak{q}}_1(\widehat{q}(t),\widehat{p}(t)),$$

where $\widehat{\mathfrak{q}}_1 : T^*\widehat{\mathcal{Q}} \to T(\mathcal{Q}_1/\mathcal{G})$ specifies the components of the dynamical vector field in $T(\mathcal{Q}_1/\mathcal{G})$ as a portion of the components of $\widehat{\mathfrak{q}}$. To unify our notation, for a holonomic or nonholonomic open-chain multi-body system we write

$$\check{\mathfrak{q}}(\check{q}(t),\check{p}(t)) := \dot{\check{q}}(t), \tag{5.38}$$

where in the holonomic case this formula represents the combination of (5.35) and (5.36), and for a nonholonomic system it is (5.37).

We can then write $\check{\tau}(\check{q}(t),\check{p}(t)) := \tau(t)$, where the function $\check{\tau} : \check{\mathcal{S}} \to \mathfrak{Q}$ is defined based on (5.38) and the reconstruction equations. Therefore, the curve in (5.30) can be rewritten as

$$t \mapsto \jmath_{q(t)} \circ \check{\tau}(\check{q}(t),\check{p}(t))) \in \mathfrak{R},$$

where we have $\jmath \circ \check{\tau} : \mathcal{G} \times \check{\mathcal{S}} \to \mathfrak{R}$. Taking the derivative of this curve with respect to time, we obtain a curve in $T\mathfrak{R} \cong \mathfrak{R}$:

$$\begin{aligned} t \mapsto \frac{d}{dt}\left(\jmath_{q(t)} \circ \check{\tau}(\check{q}(t),\check{p}(t))\right) &= \left(\frac{\partial \jmath}{\partial q}\dot{q}(t)\right)\check{\tau} + \jmath\left(\frac{\partial \check{\tau}}{\partial \check{q}}\dot{\check{q}}(t) + \frac{\partial \check{\tau}}{\partial \check{p}}\dot{\check{p}}(t)\right) \\ &= \left(\frac{\partial \jmath}{\partial q}\left(T_e L_{q(t)}(\check{\tau})\right)\right)\check{\tau} + \jmath\left(\frac{\partial \check{\tau}}{\partial \check{q}}\dot{\check{q}}(t) + \frac{\partial \check{\tau}}{\partial \check{p}}\dot{\check{p}}(t)\right) \\ &= \left(\frac{\partial \jmath}{\partial q}\left(T_e L_{q(t)}(\check{\tau})\right)\right)\check{\tau} + \jmath\left(\frac{\partial \check{\tau}}{\partial \check{q}}\check{\mathfrak{q}} + \frac{\partial \check{\tau}}{\partial \check{p}}\left(\check{\mathfrak{p}} - \check{u}\right)\right) \in \mathfrak{R}, \end{aligned} \tag{5.39}$$

where the last line is the consequence of substituting (5.38) in the equation, $\check{u}$ is equal to $\overline{u}$ or $\widehat{u}$ in the holonomic or nonholonomic case, respectively.

Further, the map $\check{\mathfrak{p}}$ is defined based on the last equation of (5.33) or (5.34) for the holonomic or nonholonomic case, respectively:

$$\check{\mathfrak{p}}(\widetilde{q}(t),\overline{p}(t)) = -\left([\widetilde{\Omega}]_{12}^T[\widetilde{\Omega}]_{11}^{-1}\right)\frac{\partial \widetilde{H}}{\partial \widetilde{q}_1} - \frac{\partial \widetilde{H}}{\partial \overline{q}} + \left([\widetilde{\Omega}]_{22} + [\widetilde{\Omega}]_{12}^T[\widetilde{\Omega}]_{11}^{-1}[\widetilde{\Omega}]_{12}\right)\frac{\partial \widetilde{H}}{\partial \overline{p}},$$

or

$$\check{\mathfrak{p}}(\widehat{q}(t),\widehat{p}(t)) = -\frac{\partial \widehat{H}}{\partial \widehat{q}} + [\widehat{\Omega}]_1\frac{\partial \widehat{H}}{\partial \widehat{p}}.$$

Equation (5.39) is the input-output linearized form in the reduced phase space of a holonomic or nonholonomic open-chain multi-body system with multi-d.o.f. joints and non-zero momentum. The input-output linearization method presented in this section generalizes different approaches to the input-output linearization of underactuated, holonomic and nonholonomic multi-body systems used, e.g., in [1,3,12,14,21,22], to derive nonlinear control laws. Equation (5.39) holds for any holonomic open-chain multi-body system with non-abelian symmetry group and non-zero momentum, and also it holds for Chaplygin systems with underactuated joints.

In (5.39), $\check{u}$ is going to be designed such that the output of the controlled holonomic or nonholonomic open-chain multi-body system follows the desired trajectory $t \mapsto \gamma(t)$. As a result, we solve the inverse dynamics problem for a free-base, open-chain multi-body system with symmetry by equating the curve in (5.39) and $\frac{d}{dt}\left(\gamma^{-1}(t)\dot{\gamma}(t)\right)$:

$$\frac{d}{dt}(r^{-1}(t)\dot{r}(t)) = \frac{d}{dt}\left(\jmath_{q(t)} \circ \check{\tau}(\check{q}(t),\check{p}(t))\right) = \frac{d}{dt}\left(\gamma^{-1}(t)\dot{\gamma}(t)\right),$$

where $t \mapsto r(t) := FK(q(t))$ is the output of the system. Now, we use the last line of (5.39) to find the solution for the inverse dynamics problem, by solving for $\check{u}$ in the reduced phase space.

$$\begin{aligned}\check{u}(\mathfrak{g}(t),&\check{q}(t),\check{p}(t),\gamma(t),\dot{\gamma}(t),\ddot{\gamma}(t))\\ &= \left(\jmath\frac{\partial\check{\tau}}{\partial\check{p}}\right)^{-1}\left(\left(\frac{\partial \jmath}{\partial q}\left(T_eL_{q(t)}(\check{\tau})\right)\right)\check{\tau} + \jmath\frac{\partial\check{\tau}}{\partial\check{q}}\check{\mathfrak{q}} - \frac{d}{dt}\left(\gamma^{-1}(t)\dot{\gamma}(t)\right)\right) + \check{\mathfrak{p}},\end{aligned} \tag{5.40}$$

where $(\mathfrak{g}(t),\check{q}(t)) = q(t)$ in the local trivialization. Note that (5.40) matches with the equation (17) in [12] in a special case where the total momentum of the system is equal to zero and there is no nonholonomic constraints. Also, the formulation in [12] is based on a specific parametrization of the output manifold of the system. Therefore, (5.40) can be considered as a generalization of the inverse dynamics solution of a free-base, open-chain multi-body system, in the reduced phase space, and subject to holonomic (possibly with non-zero total momentum) or nonholonomic constraints.

Remark 5 The matrix $\jmath\frac{\partial\check{\tau}}{\partial\check{p}}$ is square if the number of control inputs n_c is equal to the dimension of the output manifold, i.e., $n_o = n_c$. In case $n_o < n_c$, we can either choose $\breve{u}$ amongst all possible solutions by, for example, optimizing a function along the trajectories of the system, or using a pseudo inverse matrix in the above equation.

Remark 6 Note that since the Legendre transformation is invertible for the reduced open-chain multi-body system with symmetry, the matrix $\frac{\partial \breve{\tau}}{\partial \breve{p}}$ is always full rank. Furthermore, for a feasible desired trajectory $t \mapsto \gamma(t)$ the Jacobian $\jmath_q$ is also full rank. Therefore, the inverse dynamics problem in the reduced phase space has a unique solution $\breve{u}$ (or the matrix $\jmath \frac{\partial \breve{\tau}}{\partial \breve{p}}$ is invertible), if $n_o = n_c$ and the desired trajectory $t \mapsto \gamma(t)$ is feasible.

CON7) In the next section, we assume that the dimension of the output manifold n_o is equal to the number of control inputs n_c of a controlled holonomic or nonholonomic open-chain multi-body system.

6 An Output-tracking Feed-forward Servo-like Controller

In this section, under the dimensional assumption CON7 and the feasibility of the desired trajectory $t \mapsto \gamma(t)$, we develop an output tracking feed-forward servo-like controller for a open-chain multi-body system with symmetry. The system can include multi-d.o.f. joints and can be subject to holonomic or non-holonomic constraints. Also, for the holonomic case the total momentum of the system can be non-zero. In this process, we use the definition of the error function and velocity error of the system output introduced in Section 4. Consequently, we show that the developed controller exponentially stabilizes the closed-loop system using a Lyapunov function $t \mapsto \mathcal{V}_L(r(t), \dot{r}(t), \gamma(t), \dot{\gamma}(t)) \in \mathbb{R}$.

Definition 5 ([5]) Let $t \mapsto r(t) = FK(q(t)) \in \mathcal{R}$ denote the output of a controlled holonomic or nonholonomic, open-chain multi-body system, and let $t \mapsto \gamma(t) \in \mathcal{R}$ be a feasible desired output trajectory. The desired trajectory γ

i) is *Lyapunov stable* with Lyapunov function $t \mapsto \mathcal{V}_L(r(t), \dot{r}(t), \gamma(t), \dot{\gamma}(t)) \in \mathbb{R}$, if $\mathcal{V}_L(t) \leq \mathcal{V}_L(0)$ from all initial conditions $(r(0), \dot{r}(0))$.
ii) is *exponentially stable* with Lyapunov function $t \mapsto \mathcal{V}_L(r(t), \dot{r}(t), \gamma(t), \dot{\gamma}(t)) \in \mathbb{R}$, if there exist two positive constants δ_1 and δ_2 such that $\mathcal{V}_L(t) \leq \delta_1 \mathcal{V}_L(0) e^{-\delta_2 t}$ from all initial conditions $(r(0), \dot{r}(0))$.

Theorem 3 *Consider the controlled holonomic or nonholonomic, open-chain multi-body system in (3.20) or (3.21), and let the curve $t \mapsto \gamma(t) \in \mathcal{R}$ be a twice differentiable feasible trajectory in the output manifold that satisfies the assumption CON3. Also, let $E_r \colon \mathcal{R} \times \mathcal{R} \to \mathbb{R}^{\geq 0}$ be the error function in (4.22) and Γ be its compatible linear isomorphism (assuming CON5), defined by (4.25). Let $\mathcal{K}_P \colon \mathfrak{R} \to \mathfrak{R}^*$, $\mathcal{K}_D \colon \mathfrak{R} \to \mathfrak{R}^*$ and $\mathcal{I} \colon \mathfrak{R} \to \mathfrak{R}^*$ be self-adjoint positive-definite linear maps, such that the induced norm of $\mathcal{I}$ on $\mathfrak{R}$ is denoted* *by* $\| \cdot \|_{\mathcal{I}}$. *Under the condition CON7 the control input in the reduced phase space is*

$$\check{u}(q, \check{p}, \gamma, \dot{\gamma}, \ddot{\gamma}) = \left(\jmath \frac{\partial \check{\tau}}{\partial \check{p}} \right)^{-1} \left(\left(\frac{\partial \jmath}{\partial q} \left(T_e L_q(\check{\tau}) \right) \right) \check{\tau} + \jmath \frac{\partial \check{\tau}}{\partial \check{q}} \check{\mathfrak{q}} - \check{\nu} \right) + \overline{\mathfrak{p}}, \quad (6.41)$$

where we have the control law as:

$$\check{\nu} = \check{\nu}_{PD} + \check{\nu}_{FF}, \quad (6.42)$$

$$\check{\nu}_{PD} = -\mathcal{I}^{-1} T^*_{e_{\mathcal{P}}} L_{r(t)} \left(d_1 E_r(r, \gamma) \right) - \mathcal{I}^{-1} \mathcal{K}_D \mathfrak{v}_e(r, \gamma, \dot{r}, \dot{\gamma}), \quad (6.43)$$

$$\check{\nu}_{FF} = ad_{(r^{-1}\dot{r})} Ad_{(r^{-1}\gamma)} \left(\gamma^{-1}\dot{\gamma} \right) + Ad_{(r^{-1}\gamma)} \left(\frac{d}{dt} \left(\gamma^{-1}\dot{\gamma} \right) \right) \quad (6.44)$$

where $r(t) = FK(q(t)) \in \mathcal{R}$ *is the output of the system, and*

$$\mathfrak{v}_e(r, \gamma, \dot{r}, \dot{\gamma}) := r^{-1}\dot{r} - r^{-1} \Gamma_{(r,\gamma)}(\dot{\gamma}) = r^{-1}\dot{r} - Ad_{(r^{-1}\gamma)}(\gamma^{-1}\dot{\gamma}) \quad (6.45)$$

is the left translated output velocity error to $\mathfrak{R}$. *Then, the desired trajectory* $t \mapsto \gamma(t)$ *is Lyapunov stable with the Lyapunov function* $\mathcal{V}_L : \mathbb{R} \to \mathbb{R}^{\geq 0}$:

$$\mathcal{V}_L(t) = E_r(r, \gamma) + \frac{1}{2} \| \mathfrak{v}_e(r, \gamma, \dot{r}, \dot{\gamma}) \|^2_{\mathcal{I}} . \quad (6.46)$$

Further, the desired trajectory $t \mapsto \gamma(t)$ *is exponentially stable with Lyapunov function* $\mathcal{V}_L$ *from all initial conditions, such that we have* $\mathcal{V}_L(0) < W^2_{\mathcal{R}}$. *Here,* $W_{\mathcal{R}}$ *is the length of the radius of an open ball in* $\mathfrak{R}$ *with respect to the norm induced by* $\mathcal{I}$, *where the exponential map is a diffeomorphism.*

Proof In order to show Lyapunov stability of the desired trajectory, we have to show that the time derivative of a candidate Lyapunov function is always less than or equal to zero. We choose the Lyapunov function to be (6.46), and we start with the time derivative of the error function. Based on CON5 and Lemma 1, we have

$$\frac{d}{dt} E_r(r, \gamma) = \langle d_1 E_r(r, \gamma), V_e \rangle = \left\langle T^*_{e_{\mathcal{P}}} L_{r(t)} \left(d_1 E_r(r, \gamma) \right), \mathfrak{v}_e \right\rangle . \quad (6.47)$$

The time derivative of the second term in (6.46) is also calculated as follows:

$$\begin{aligned}
\frac{d}{dt} \left(\frac{1}{2} \langle \mathcal{I}\mathfrak{v}_e, \mathfrak{v}_e \rangle \right) &= \left\langle \mathcal{I}\mathfrak{v}_e, \frac{d}{dt} \mathfrak{v}_e \right\rangle \\
&= \left\langle \mathcal{I}\mathfrak{v}_e, \frac{d}{dt} \left(r^{-1}\dot{r} - r^{-1} \Gamma_{(r,\gamma)}(\dot{\gamma}) \right) \right\rangle \\
&= \left\langle \mathcal{I}\mathfrak{v}_e, \frac{d}{dt} \left(\jmath_q \check{\tau} \right) - \frac{d}{dt} \left(r^{-1} \Gamma_{(r,\gamma)}(\dot{\gamma}) \right) \right\rangle \\
&= \left\langle \mathcal{I}\mathfrak{v}_e, \check{\nu} - \frac{d}{dt} \left(r^{-1}(\dot{\gamma})\gamma^{-1} r \right) \right\rangle \\
&= \left\langle \mathcal{I}\mathfrak{v}_e, \check{\nu}_{PD} + \check{\nu}_{FF} - \frac{d}{dt} \left(\mathrm{Ad}_{(r^{-1}\gamma)}(\gamma^{-1}\dot{\gamma}) \right) \right\rangle \\
&= \left\langle \mathcal{I}\mathfrak{v}_e, \check{\nu}_{PD} + \check{\nu}_{FF} - \mathrm{ad}_{(r^{-1}\dot{r})} \mathrm{Ad}_{(r^{-1}\gamma)} \left(\gamma^{-1}\dot{\gamma} \right) + \mathrm{Ad}_{(r^{-1}\gamma)} \left(\frac{d}{dt} \left(\gamma^{-1}\dot{\gamma} \right) \right) \right\rangle
\end{aligned}$$

$$= \langle \mathcal{I}\mathfrak{v}_e, \check{\nu}_{PD} \rangle$$
$$= \left\langle \mathcal{I}\mathfrak{v}_e, -\mathcal{I}^{-1} T^*_{e_\mathcal{P}} L_{r(t)} (d_1 E_r) - \mathcal{I}^{-1}\mathcal{K}_D \mathfrak{v}_e \right\rangle$$
$$= \left\langle -T^*_{e_\mathcal{P}} L_{r(t)} (d_1 E_r) - \mathcal{K}_D \mathfrak{v}_e, \mathfrak{v}_e \right\rangle . \tag{6.48}$$

By adding (6.47) and (6.48), we calculate the time derivative of $\mathcal{V}_L$ as:

$$\frac{d\mathcal{V}_L}{dt}(t) = -\langle \mathcal{K}_D \mathfrak{v}_e, \mathfrak{v}_e \rangle ,$$

which is always less than or equal to zero due to the fact that K_D is positive definite. This proves the Lyapunov stability of the feasible desired trajectory.

In order to show the exponential stability, we need to add a term to $\mathcal{V}_L$ and define a new Lyapunov function $\check{\mathcal{V}}_L : \mathbb{R} \to \mathbb{R}^{\geq 0}$ [6]:

$$\check{\mathcal{V}}_L(t) := E_r(r,\gamma) + \frac{1}{2} \parallel \mathfrak{v}_e(r,\gamma,\dot{r},\dot{\gamma}) \parallel^2_\mathcal{I} + \epsilon \frac{d}{dt} E_r(r,\gamma),$$

where we have to find $\epsilon > 0$ such that $\check{\mathcal{V}}_L$ is positive definite, i.e., $\check{\mathcal{V}}_L$ is greater than or equal to zero and it is zero if and only if $E_r(r,\gamma) = 0$ and $\mathfrak{v}_e(r,\gamma,\dot{r},\dot{\gamma}) = 0$. Let $r_e(t) = r(t)^{-1}\gamma(t)$:

$$\begin{aligned}\check{\mathcal{V}}_L(t) &= E_r + \frac{1}{2} \langle \mathcal{I}\mathfrak{v}_e, \mathfrak{v}_e \rangle + \epsilon \left\langle T^*_{e_\mathcal{P}} L_r (d_1 E_r), \mathfrak{v}_e \right\rangle \\ &= \frac{1}{2} \left\langle \mathcal{K}_P \exp^{-1}(r_e), \exp^{-1}(r_e) \right\rangle + \frac{1}{2} \langle \mathcal{I}\mathfrak{v}_e, \mathfrak{v}_e \rangle + \epsilon \left\langle T^*_{e_\mathcal{P}} L_r (d_1 E_r), \mathfrak{v}_e \right\rangle \\ &= \frac{1}{2} \left\langle \mathcal{K}_P \exp^{-1}(r_e), \exp^{-1}(r_e) \right\rangle + \frac{1}{2} \langle \mathcal{I}\mathfrak{v}_e, \mathfrak{v}_e \rangle - \epsilon \left\langle \mathrm{Ad}^*_{r_e^{-1}} (\Theta(r_e)^{-1})^* \mathcal{K}_P \exp^{-1}(r_e), \mathfrak{v}_e \right\rangle .\end{aligned}$$

From the proof of Lyapunov stability of the closed loop for $t \mapsto \gamma(t)$, we have

$$\mathcal{V}_L(t) \leq W_0 := \mathcal{V}_L(0) \quad \Longrightarrow \quad E_r(r,\gamma) \leq W_0, \ \parallel \mathfrak{v}_e(r,\gamma,\dot{r},\dot{\gamma}) \parallel_\mathcal{I} \leq W_0.$$

We consider the induced norm by $\mathcal{K}_{e_\mathcal{P}}$ and $\mathcal{I}$ on the space of all automorphisms of $\mathfrak{R}$, as a vector space. For any linear map $\Im : \mathfrak{R} \to \mathfrak{R}$, this induced norm is defined by

$$\parallel \Im \parallel_\mathcal{I} := \max \{ \parallel \Im\xi \parallel_\mathcal{I} | \, \xi \in \mathfrak{R}, \ \parallel \xi \parallel_\mathcal{I} = 1 \} ,$$

for $\mathcal{I}$, and similarly we can define the norm, which is induced by $\mathcal{K}_{e_\mathcal{P}}$. Since the error function is bounded by the Lyapunov stability and $r_e \in \mathcal{R}$ is assumed to be in a neighbourhood of the identity where $\Theta(r_e)$ is invertible, $\forall t \in \mathbb{R}$ we have the following bounds:

$$\parallel \mathrm{Ad}_{r_e^{-1}} \parallel_\mathcal{I} \leq \sup \left\{ \parallel \mathrm{Ad}_{r_e^{-1}} \parallel_\mathcal{I} \Big| \, r_e \in \mathcal{R}, \ \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \leq \sqrt{W_0} \right\} = W_1,$$
$$\parallel \Theta(r_e)^{-1} \parallel_\mathcal{I} \leq \sup \left\{ \parallel \Theta(r_e)^{-1} \parallel_\mathcal{I} \Big| \, r_e \in \mathcal{R}, \ \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \leq \sqrt{W_0} \right\} = W_2.$$

As the result of these bounds,

$$\begin{aligned}\check{\mathcal{V}}_L(t) &\geq \frac{1}{2} \parallel \exp^{-1}(r_e) \parallel^2_{\mathcal{K}_{e_\mathcal{P}}} + \frac{1}{2} \parallel \mathfrak{v}_e \parallel^2_\mathcal{I} - \epsilon \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \parallel \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \parallel_\mathcal{I} \\ &\geq \frac{1}{2} \parallel \exp^{-1}(r_e) \parallel^2_{\mathcal{K}_{e_\mathcal{P}}} + \frac{1}{2} \parallel \mathfrak{v}_e \parallel^2_\mathcal{I} \\ &\qquad - \epsilon \parallel \mathrm{Ad}_{r_e^{-1}} \parallel_\mathcal{I} \parallel \Theta(r_e)^{-1} \parallel_\mathcal{I} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \parallel \mathfrak{v}_e \parallel_\mathcal{I}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2} \parallel \exp^{-1}(r_e) \parallel^2_{\mathcal{K}_{e_\mathcal{P}}} + \frac{1}{2} \parallel \mathfrak{v}_e \parallel^2_{\mathcal{I}} - \epsilon W_1 W_2 \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \\
&= \frac{1}{2} \begin{bmatrix} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \end{bmatrix}^T \check{\mathcal{W}} \begin{bmatrix} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \end{bmatrix} \\
&:= \frac{1}{2} \begin{bmatrix} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \end{bmatrix}^T \begin{bmatrix} 1 & -\epsilon W_1 W_2 \\ -\epsilon W_1 W_2 & 1 \end{bmatrix} \begin{bmatrix} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \end{bmatrix},
\end{aligned}
$$

where, $\check{\mathcal{W}}$ is a positive definite matrix if $0 < \epsilon < \frac{1}{\sqrt{W_1 W_2}}$. Therefore, for any ϵ in this range, $\check{\mathcal{V}}_L$ is a well-defined Lyapunov function.

Now, to calculate the time derivative of $\check{\mathcal{V}}_L(t)$ we only need to take the derivative of the term $\frac{d}{dt} E_r(r, \gamma)$:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{d}{dt} E_r(r,\gamma)\right) &= \frac{d}{dt}\left\langle T^*_{e_\mathcal{P}} L_r \left(d_1 E_r\right), \mathfrak{v}_e\right\rangle = -\frac{d}{dt}\left\langle \mathrm{Ad}^*_{r_e^{-1}} (\Theta(r_e)^{-1})^* \mathcal{K}_P \exp^{-1}(r_e), \mathfrak{v}_e\right\rangle \\
&= -\frac{d}{dt}\left\langle \mathcal{K}_P \exp^{-1}(r_e), \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \right\rangle \\
= -\left\langle \mathcal{K}_P \frac{d}{dt}\left(\exp^{-1}(r_e)\right), \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \right\rangle &- \left\langle \mathcal{K}_P \exp^{-1}(r_e), \frac{d}{dt}\left(\Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e\right)\right\rangle.
\end{aligned}
$$

In the following, we calculate the terms appeared in the above equation.

$$
\begin{aligned}
\left\langle \mathcal{K}_P \frac{d}{dt}\left(\exp^{-1}(r_e)\right), \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \right\rangle &= \left\langle \mathcal{K}_P \Theta(r_e)^{-1} T_{e_\mathcal{P}} L_{r_e^{-1}} \dot{r}_e, \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \right\rangle \\
&= -\left\langle \mathcal{K}_P \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e, \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \right\rangle \\
&= - \parallel \Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e \parallel^2_{\mathcal{K}_{e_\mathcal{P}}}, \quad (6.49)
\end{aligned}
$$

since we have

$$
\begin{aligned}
\dot{r}_e &= \frac{d}{dt}\left(r^{-1}\gamma)\right) = -r^{-1}\dot{r} r^{-1}\gamma + r^{-1}\dot{\gamma} = -\left(r^{-1}\dot{r} - r^{-1}\dot{\gamma}\gamma^{-1} r\right) r_e \\
&= -T_{e_\mathcal{P}} R_{r_e}\left(r^{-1}\dot{r} - r^{-1}\Gamma_{(r,\gamma)}(\dot{\gamma})\right) = -T_{e_\mathcal{P}} R_{r_e}(\mathfrak{v}_e).
\end{aligned}
$$

And,

$$
\begin{aligned}
\frac{d}{dt}\left(\Theta(r_e)^{-1} \mathrm{Ad}_{r_e^{-1}} \mathfrak{v}_e\right) &= \left(\frac{\partial \Theta^{-1}}{\partial r_e} \dot{r}_e\right) \mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e) - \Theta^{-1} \mathrm{Ad}_{r_e^{-1}} \mathrm{ad}_{\dot{r}_e r_e^{-1}}(\mathfrak{v}_e) + \Theta^{-1} \mathrm{Ad}_{r_e^{-1}}(\dot{\mathfrak{v}}_e) \\
&= \left(\frac{\partial \Theta^{-1}}{\partial r_e}(-\mathfrak{v}_e r_e)\right) \mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e) - \Theta^{-1} \mathrm{Ad}_{r_e^{-1}} \mathrm{ad}_{\mathfrak{v}_e}(\mathfrak{v}_e) + \Theta^{-1} \mathrm{Ad}_{r_e^{-1}}(\check{\nu}_{PD}) \\
&= -\left(\frac{\partial \Theta^{-1}}{\partial r_e}(\mathfrak{v}_e r_e)\right) \mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e) + \Theta^{-1} \mathrm{Ad}_{r_e^{-1}}(\check{\nu}_{PD}) \\
&= -\left(\frac{\partial \Theta^{-1}}{\partial r_e}(\mathfrak{v}_e r_e)\right) \mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e) - \Theta^{-1} \mathrm{Ad}_{r_e^{-1}}\left(\mathcal{I}^{-1} T^*_{e_\mathcal{P}} L_r \left(d_1 E_r\right) + \mathcal{I}^{-1} \mathcal{K}_D \mathfrak{v}_e\right) \\
&= -\left(\frac{\partial \Theta^{-1}}{\partial r_e}(\mathfrak{v}_e r_e)\right) \mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e) \\
&\qquad + \Theta^{-1} \mathrm{Ad}_{r_e^{-1}}\left(\mathcal{I}^{-1} \mathrm{Ad}^*_{r_e^{-1}} (\Theta(r_e)^{-1})^* \mathcal{K}_P \exp^{-1}(r_e) - \mathcal{I}^{-1} \mathcal{K}_D \mathfrak{v}_e\right), \quad (6.50)
\end{aligned}
$$

which is the result of the following equalities:

$$\begin{aligned}
\dot{r}_e &= -\mathfrak{v}_e r_e, \\
\dot{\mathfrak{v}}_e &= \check{\nu}_{PD}, \\
\frac{d}{dt}\mathrm{Ad}_{r_e^{-1}}(\eta) &= \frac{d}{dt}\left(r_e^{-1}\eta r_e\right) = -r_e^{-1}\dot{r}_e r_e^{-1}\eta r_e + r_e^{-1}\eta \dot{r}_e = r_e^{-1}\left(\eta\dot{r}_e r_e^{-1} - \dot{r}_e r_e^{-1}\eta\right) r_e \\
&= -\mathrm{Ad}_{r_e^{-1}}\mathrm{ad}_{\dot{r}_e r_e^{-1}}(\eta),
\end{aligned}$$

where in the last equality η is an element of $\mathfrak{R}$. Therefore, by (6.49) and (6.50) we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{d}{dt}E_r(r,\gamma)\right) = &\parallel \Theta(r_e)^{-1}\mathrm{Ad}_{r_e^{-1}}\mathfrak{v}_e \parallel^2_{\mathcal{K}_{e_\mathcal{P}}} \\
&+ \left\langle \mathcal{K}_P \exp^{-1}(r_e), \left(\frac{\partial \Theta^{-1}}{\partial r_e}(\mathfrak{v}_e r_e)\right)\mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e)\right\rangle \\
&- \left\langle \mathrm{Ad}^*_{r_e^{-1}}(\Theta(r_e)^{-1})^*\mathcal{K}_P\exp^{-1}(r_e), \mathcal{I}^{-1}\mathrm{Ad}^*_{r_e^{-1}}(\Theta(r_e)^{-1})^*\mathcal{K}_P\exp^{-1}(r_e)\right\rangle \\
&+ \left\langle \mathcal{K}_P\exp^{-1}(r_e), \Theta^{-1}\mathrm{Ad}_{r_e^{-1}}\mathcal{I}^{-1}\mathcal{K}_D\mathfrak{v}_e\right\rangle.
\end{aligned}$$

Based on the norm equivalence inequality and since we have the bounds $\parallel \mathrm{Ad}_{r_e^{-1}} \parallel_{\mathcal{I}} \leq W_1$ and $\parallel \Theta(r_e)^{-1} \parallel_{\mathcal{I}} \leq W_2$, there exists $W_3 > 0$ such that

$$\begin{aligned}
&\parallel \mathrm{Ad}^*_{r_e^{-1}}(\Theta(r_e)^{-1})^*\mathcal{K}_P\exp^{-1}(r_e) \parallel^2_{\mathcal{I}} \\
&\qquad \geq W_3 \parallel \mathcal{K}_P\exp^{-1}(r_e) \parallel^2_{\mathcal{K}_{e_\mathcal{P}}} = W_3 \parallel \exp^{-1}(r_e) \parallel^2_{\mathcal{K}_{e_\mathcal{P}}}, \qquad (6.51)
\end{aligned}$$

wherever we have an element of $\mathfrak{R}^*$ inside the norm we mean the naturally induced norm by a metric on $\mathfrak{R}^*$. Also, we have the following inequalities:

$$\parallel \Theta(r_e)^{-1}\mathrm{Ad}_{r_e^{-1}}\mathfrak{v}_e \parallel^2_{\mathcal{K}_{e_\mathcal{P}}} \leq W_4 \parallel \mathfrak{v}_e \parallel^2_{\mathcal{I}}, \qquad (6.52)$$

$$\left\langle \mathcal{K}_P\exp^{-1}(r_e), \left(\frac{\partial\Theta^{-1}}{\partial r_e}(\mathfrak{v}_e r_e)\right)\mathrm{Ad}_{r_e^{-1}}(\mathfrak{v}_e)\right\rangle \leq W_5 \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}}, \qquad (6.53)$$

$$\left\langle \mathcal{K}_P\exp^{-1}(r_e), \Theta^{-1}\mathrm{Ad}_{r_e^{-1}}\mathcal{K}_P^{-1}\mathcal{K}_D\mathfrak{v}_e\right\rangle \leq W_6 \parallel \mathfrak{v}_e \parallel_{\mathcal{I}} \parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}}, \qquad (6.54)$$

where $W_4, W_5, W_6 > 0$ are three positive real numbers. The inequalities in (6.51), (6.52), (6.53) and (6.54) yields to

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{d}{dt}E_r(r,\gamma)\right) \leq \\
&-\frac{1}{2}\begin{bmatrix}\parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}}\end{bmatrix}^T \widetilde{\mathcal{W}} \begin{bmatrix}\parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}}\end{bmatrix} \\
&:= -\frac{1}{2}\begin{bmatrix}\parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}}\end{bmatrix}^T \begin{bmatrix} 2W_3 & -(W_5+W_6) \\ -(W_5+W_6) & -2W_4\end{bmatrix} \begin{bmatrix}\parallel \exp^{-1}(r_e) \parallel_{\mathcal{K}_{e_\mathcal{P}}} \\ \parallel \mathfrak{v}_e \parallel_{\mathcal{I}}\end{bmatrix},
\end{aligned}$$

where $\widetilde{\mathcal{W}}$ is a symmetric matrix. As a result, we have

$$\begin{aligned}
&\frac{d}{dt}\check{\mathcal{V}}_L(t) \leq \\
&-\frac{1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \widehat{\mathcal{W}} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix} \\
&:= -\frac{1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \begin{bmatrix} 2\epsilon W_3 & -\epsilon(W_5+W_6) \\ -\epsilon(W_5+W_6) & -2(\epsilon W_4 - W_6)\end{bmatrix} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}.
\end{aligned} \tag{6.55}$$

It is easy to check that if

$$0 < \epsilon < \frac{4W_3W_6}{(W_5+W_6)^2 + 4W_3W_4},$$

then $\widehat{\mathcal{W}}$ is a symmetric positive-definite matrix. Therefore, for any positive ϵ less than the min $\left\{\frac{1}{\sqrt{W_1W_2}}, \frac{4W_3W_6}{(W_5+W_6)^2+4W_3W_4}\right\}$, $\check{\mathcal{V}}_L$ is a Lyapunov function and (6.55) holds. Until this step, we have proved the asymptotic stability of the desired feasible trajectory $t \mapsto \gamma(t) \in \mathcal{R}$.

In the final step, we show that in fact this trajectory is exponential stable. Consider the Lyapunov function $\check{\mathcal{V}}_L$ for an appropriate ϵ. We have

$$\begin{aligned}
\check{\mathcal{V}}_L(t) &\leq \frac{1}{2} \| \exp^{-1}(r_e) \|^2_{\mathcal{K}_{e_\mathcal{P}}} + \frac{1}{2} \| \mathfrak{v}_e \|^2_{\mathcal{I}} + \epsilon W_1 W_2 \| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \| \mathfrak{v}_e \|_{\mathcal{I}} \\
&= \frac{1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \mathcal{W}' \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix} \\
&:= \frac{1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \begin{bmatrix} 1 & \epsilon W_1 W_2 \\ \epsilon W_1 W_2 & 1\end{bmatrix} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix},
\end{aligned}$$

where $\mathcal{W}'$ is a symmetric positive definite matrix. Using the norm equivalence inequality, there exists a positive number δ_2 such that

$$\begin{aligned}
\frac{d}{dt}\check{\mathcal{V}}_L(t) &\leq -\frac{1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \widehat{\mathcal{W}} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix} \\
&\leq \frac{-\delta_2}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \widehat{\mathcal{W}'} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix} \leq -\delta_2 \check{\mathcal{V}}_L(t).
\end{aligned}$$

Based on this inequality, $\check{\mathcal{V}}_L(t) \leq \check{\mathcal{V}}_L(0)e^{-\delta_2 t}$. Consequently, there exists a positive number δ_1 such that

$$\begin{aligned}
\mathcal{V}_L(t) &= \frac{1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix} \\
&\leq \frac{\delta_1}{2}\begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix}^T \check{\mathcal{W}} \begin{bmatrix}\| \exp^{-1}(r_e) \|_{\mathcal{K}_{e_\mathcal{P}}} \\ \| \mathfrak{v}_e \|_{\mathcal{I}}\end{bmatrix} \\
&\leq \delta_1 \check{\mathcal{V}}_L(t) \leq \delta_1 \check{\mathcal{V}}_L(0)e^{-\delta_2 t} \leq 2\delta_1 \mathcal{V}_L(0)e^{-\delta_2 t},
\end{aligned}$$

where the last inequality holds due to the fact that $\check{\mathcal{V}}_L(0) \leq 2\mathcal{V}_L(0)$. □

Remark 7 Theorem 3 presents an output tracking, feed-forward PD-like controller in the reduced phase space that exponentially stabilizes the closed-loop holonomic or nonholonomic multi-body system. The control input is a function of the joint displacements, including those of the unactuated joints eliminated in the reduction process, and the velocities of the actuated joints.

Remark 8 The controller input in (6.41) is in the reduced space. In order to find the control input for the original system in (3.16), we have to first express $\check{u}$ in the reduced basis for the control directions, i.e. $\{\check{\mathcal{U}}_i\}_{i=1}^{n_c}$, then we have to lift the result to the original phase space of the system. That is,

$$\sum_{i=1}^{n_c} \mathfrak{u}_i \check{\mathcal{U}}_i := \check{u} = \sum_{i=1}^{n_c} u_i \mathcal{U}_i,$$

where $\mathcal{U}_i$'s are the original control directions of the controlled multi-body system.

Figures 1 and 2 depict the block diagrams of the proposed control scheme for free-base, holonomic and nonholonomic controlled multi-body systems, respectively. In these diagrams, $\mathfrak{g}$ is an element of the symmetry group $\mathcal{G}$, and $r', \dot{r}'$ correspond to the actual motion of the robot. Also, we have

$$\overline{p}(\overline{q},\overline{p}) = \begin{bmatrix} \overline{K}_{22}(\overline{q}) & \cdots & \overline{K}_{2N}(\overline{q}) \\ \vdots & \ddots & \vdots \\ \overline{K}_{N2}(\overline{q}) & \cdots & \overline{K}_{NN}(\overline{q}) \end{bmatrix} \dot{\overline{q}} - \left[\overline{K}_{12}(\overline{q}) \quad \cdots \quad \overline{K}_{1N}(\overline{q})\right]^* A_{\overline{q}}\dot{\overline{q}},$$

$$\widehat{p}(\widehat{q},\widehat{p}) = \mathbb{F}\widehat{L}_{\widehat{q}}\dot{\widehat{q}},$$

$$\check{\nu}_P = -\mathcal{I}^{-1} T^*_{e_{\mathcal{P}}} L_{r(t)} \left(d_1 E_r(r,\gamma)\right) = \mathcal{I}^{-1} \mathrm{Ad}^*_{r_e^{-1}} (\Theta(r_e)^{-1})^* \mathcal{K}_P \exp^{-1}(r_e),$$

$$\check{\nu}_D = -\mathcal{I}^{-1}\mathcal{K}_D \mathfrak{v}_e(r,\gamma,\dot{r},\dot{\gamma}) = -\mathcal{I}^{-1}\mathcal{K}_D \left(r^{-1}\dot{r} - \mathrm{Ad}_{r_e}(\gamma^{-1}\dot{\gamma})\right),$$

$$\begin{aligned}\check{\nu}_{FF} &= \mathrm{ad}_{(r^{-1}\dot{r})}\mathrm{Ad}_{r_e}\left(\gamma^{-1}\dot{\gamma}\right) + \mathrm{Ad}_{r_e}\left(\frac{d}{dt}\left(\gamma^{-1}\dot{\gamma}\right)\right) \\ &= \mathrm{ad}_{(r^{-1}\dot{r})}\mathrm{Ad}_{r_e}\left(\gamma^{-1}\dot{\gamma}\right) + \mathrm{Ad}_{r_e}\left(\gamma^{-1}\ddot{\gamma} - (\gamma^{-1}\dot{\gamma})(\gamma^{-1}\dot{\gamma})\right)\end{aligned}$$

In Theorem 3, we introduce a feed-forward PD-like controller at the output of a controlled open-chain multi-body system. We used the group structure of the output manifold to define the pose and velocity error for the extremities, and consequently, to construct this controller. As a result, the controller, i.e., $\check{\nu}$, is dependent on the group structure of the output manifold. For this controller, the behaviour of the closed-loop system can be presented in the form of the following set of coupled differential equations:

$$\begin{aligned}\frac{d}{dt}(r^{-1}\dot{r}) &= \check{\nu}_{FF} + \check{\nu}_P + \check{\nu}_D \\ &= \frac{d}{dt}(\mathrm{Ad}_{r_e}(\gamma^{-1}\dot{\gamma})) - \mathrm{Ad}^*_{r_e^{-1}}(\Theta(r_e)^{-1})^*\mathcal{K}_P \exp^{-1}(r_e) - \mathcal{K}_D\left(r^{-1}\dot{r} - \mathrm{Ad}_{r_e}(\gamma^{-1}\dot{\gamma})\right),\end{aligned}$$

where we assume that the linear map $\mathcal{I}\colon \mathfrak{R} \to \mathfrak{R}^*$, which is used to define Lyapunov function, is the identity matrix. This simplification helps illustrating

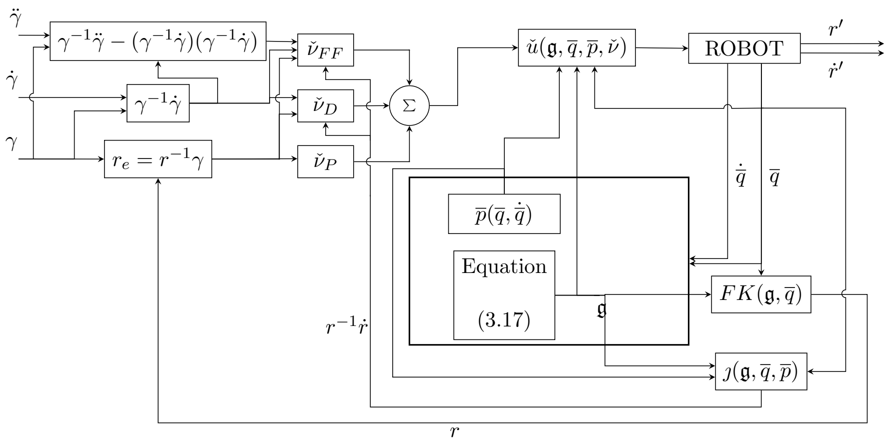


Fig. 1 Feed-forward servo-like control for a free-base, holonomic open-chain multi-body system with non-zero momentum

the behaviour of the closed-loop system. By some manipulation, we get the following set of coupled differential equations for the output error ($r_e = r^{-1}\gamma$):

$$\frac{d}{dt}(\dot{r}_e r_e^{-1}) + \mathcal{K}_D(\dot{r}_e r_e^{-1}) + \mathrm{Ad}^*_{r_e^{-1}}(\Theta(r_e)^{-1})^* \mathcal{K}_P \exp^{-1}(r_e) = 0.$$

Now by appropriately choosing the self-adjoint linear maps $\mathcal{K}_P$ and $\mathcal{K}_D$, we can achieve a desired performance of the closed-loop system.

If the output manifold of the system is an abelian subgroup of $\mathcal{P}_e$, the above differential equation can be simplified to the familiar second order linear differential equation. Then by choosing diagonal matrices for $\mathcal{K}_P$ and $\mathcal{K}_D$

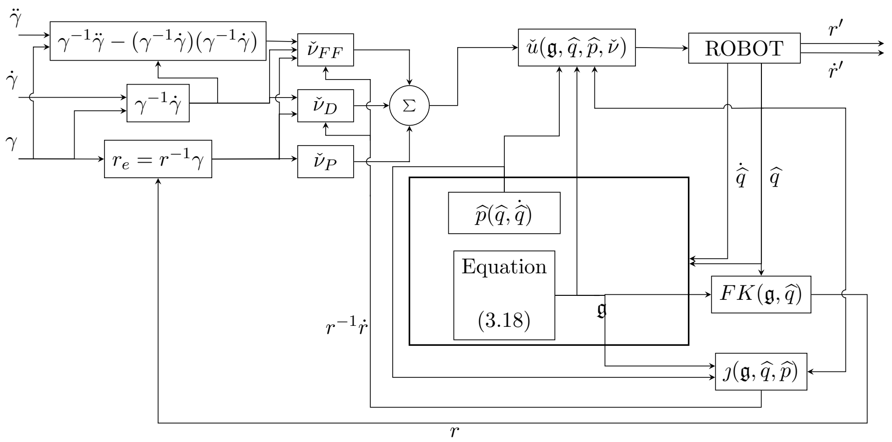


Fig. 2 Feed-forward servo-like control for a free-base, nonholonomic open-chain multi-body system

we can decouple the differential equations representing the behaviour of the closed-loop system, and we can explicitly design the controller to achieve any desired performance of the system. In this case, the controller is simply a PD controller, the output error is the difference between the desired and actual output of the system, and the maps Ad_{r_e}, $\exp^{-1}$ and $\Theta(r_e)$ are the identity maps. Therefore, we have

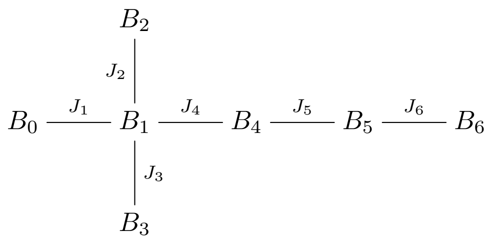
$$\frac{d^2}{dt^2}(r-\gamma) + \mathcal{K}_D \frac{d}{dt}(r-\gamma) + \mathcal{K}_P(r-\gamma) = 0.$$

This situation occurs, for example, when we want to control the position of the extremities without considering their orientation. This idea is illustrated in the next section. Note that if we are interested in controlling the orientation of the extremities, e.g., orientation of the base body of a free-floating manipulator, then the controller design is not as simple.

7 Case Study

In this section, we derive the developed controller for the example of a three-d.o.f. manipulator mounted on top of a two-wheeled differential rover. Note that the manipulator may move out of the rover's plane of motion.

We first identify the bodies as depicted in Figure 3, and the following graph introduces the joints of the nonholonomic open-chain multi-body system.



We then identify the relative configuration manifolds corresponding to the joints of the robotic system. The relative pose of B_1 with respect to the inertial coordinate frame is identified by $SE(2)$.

$$Q_1^0 = \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x \\ \sin(\theta) & \cos(\theta) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| x, y \in \mathbb{R}, \theta \in \mathbb{S}^1 \right\};$$

Here, (x, y) is the position of C (see Figure 3) with respect to the inertial coordinate frame and θ is the angle between the X_1-axis and X_0-axis (see Figure 4). The second joint is a one-d.o.f. revolute joint between B_2 and B_1,

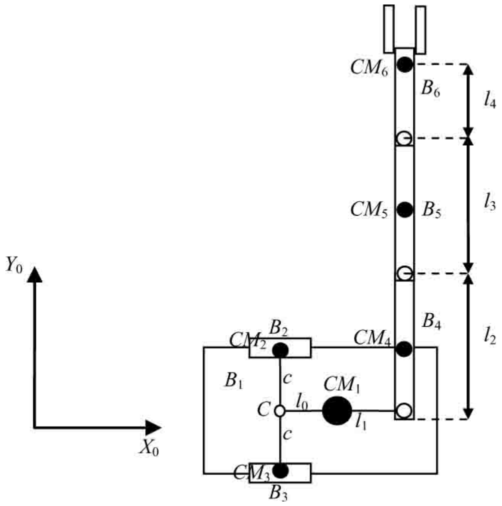


Fig. 3 An example of a mobile manipulator

and its corresponding relative configuration manifold is given by

$$Q_2^1 = \left\{ \begin{bmatrix} \cos(\psi_1) & 0 & \sin(\psi_1) & 0 \\ 0 & 1 & 0 & c \\ -\sin(\psi_1) & 0 & \cos(\psi_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \;\middle|\; \psi_1 \in \mathbb{S}^1 \right\},$$

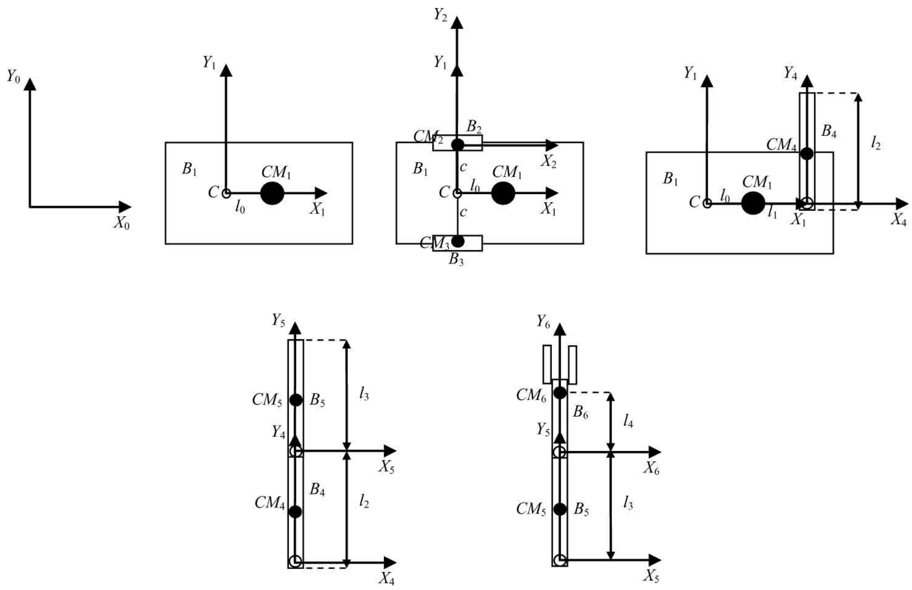


Fig. 4 The coordinate frames attached to the bodies of the mobile manipulator (Note that, the Z_i-axis ($i = 0, \cdots, 6$) is normal to the plane)

where c is the distance between the point C and the wheels. Similarly, for the third joint we have

$$Q_3^1 = \left\{ \begin{bmatrix} \cos(\psi_2) & 0 & \sin(\psi_2) & 0 \\ 0 & 1 & 0 & -c \\ -\sin(\psi_2) & 0 & \cos(\psi_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_2 \in \mathbb{S}^1 \right\}.$$

The fourth, fifth and sixth joints are one-d.o.f. revolute joints whose axes of revolution are the Z_4, X_5 and X_6 axes, respectively. The relative configuration manifolds of these joints are identified by

$$Q_4^1 = \left\{ \begin{bmatrix} \cos(\varphi_1) & -\sin(\varphi_1) & 0 & l_0 + l_1 \\ \sin(\varphi_1) & \cos(\varphi_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_1 \in \mathbb{S}^1 \right\},$$

$$Q_5^4 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi_2) & -\sin(\varphi_2) & l_2 \\ 0 & \sin(\varphi_2) & \cos(\varphi_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_2 \in \mathbb{S}^1 \right\},$$

$$Q_6^5 = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi_3) & -\sin(\varphi_3) & l_3 \\ 0 & \sin(\varphi_3) & \cos(\varphi_3) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_3 \in \mathbb{S}^1 \right\}.$$

We assume that the initial pose of B_1 with respect to the inertial coordinate frame $r_{1,0}^0$ is the identity element of $SE(3)$. Therefore based on Figure 3 and 4, we have the initial relative poses of the bodies:

$$r_{2,0}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_{3,0}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_{4,0}^1 = \begin{bmatrix} 1 & 0 & 0 & l_0 + l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_{5,0}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_{6,0}^5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As a result, we have the initial relative pose of the centre of mass of the bodies with respect to B_0:

$$r_{cm,1} = \begin{bmatrix} 1 & 0 & 0 & l_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_{cm,2} = r_{2,0}^1, r_{cm,3} = r_{3,0}^1, r_{cm,4} = \begin{bmatrix} 1 & 0 & 0 & l_0 + l_1 \\ 0 & 1 & 0 & l_2/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_{cm,5} = \begin{bmatrix} 1 & 0 & 0 & l_0 + l_1 \\ 0 & 1 & 0 & l_2 + l_3/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, r_{cm,6} = \begin{bmatrix} 1 & 0 & 0 & l_0 + l_1 \\ 0 & 1 & 0 & l_2 + l_3 + l_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With the above specifications of the system we identify the configuration manifold of the nonholonomic open-chain multi-body system in this case study by $\mathcal{Q} = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_6$, where

$$\mathcal{Q}_1 = \left\{ q_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x \\ \sin(\theta) & \cos(\theta) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| x, y \in \mathbb{R}, \theta \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_2 = \left\{ q_2 = \begin{bmatrix} \cos(\psi_1) & 0 & \sin(\psi_1) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\psi_1) & 0 & \cos(\psi_1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_1 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_3 = \left\{ q_3 = \begin{bmatrix} \cos(\psi_2) & 0 & \sin(\psi_2) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\psi_2) & 0 & \cos(\psi_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_2 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_4 = \left\{ q_4 = \begin{bmatrix} \cos(\varphi_1) & -\sin(\varphi_1) & 0 & 2(l_0 + l_1)\sin^2(\varphi_1/2) \\ \sin(\varphi_1) & \cos(\varphi_1) & 0 & -(l_0 + l_1)\sin(\varphi_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_1 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_5 = \left\{ q_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi_2) & -\sin(\varphi_2) & 2l_2\sin^2(\varphi_2/2) \\ 0 & \sin(\varphi_2) & \cos(\varphi_2) & -l_2\sin(\varphi_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_2 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_6 = \left\{ q_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi_3) & -\sin(\varphi_3) & 2(l_2 + l_3)\sin^2(\varphi_3/2) \\ 0 & \sin(\varphi_3) & \cos(\varphi_3) & -(l_2 + l_3)\sin(\varphi_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_3 \in \mathbb{S}^1 \right\}.$$

The pose of the coordinate frames attached to the centres of mass of B_i $(i = 1, ..., 6)$ with respect to B_0 is a function $F\colon \mathcal{Q} \to \mathcal{P} = SE(3) \times \cdots \times SE(3)$ $(6 - times)$, such that

$$F(q_1, \cdots, q_6) = (q_1 r_{cm,1}, q_1 q_2 r_{cm,2}, q_1 q_3 r_{cm,3}, q_1 q_4 r_{cm,4}, q_1 q_4 q_5 r_{cm,5}, q_1 q_4 q_5 q_6 r_{cm,6})$$

the tangent map $T_q(L_{F(q)^{-1}} F)\colon T_q\mathcal{Q} \to \mathfrak{P}$ in (2.2) is as follows:

$$T_q(L_{F(q)^{-1}} F) = \begin{bmatrix} \mathrm{Ad}_{r_{cm,1}^{-1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{Ad}_{r_{cm,6}^{-1}} \end{bmatrix} \begin{bmatrix} id_6 & 0_6 & 0_6 & 0_6 & 0_6 & 0_6 \\ \mathrm{Ad}_{q_2^{-1}} & id_6 & 0_6 & 0_6 & 0_6 & 0_6 \\ \mathrm{Ad}_{q_3^{-1}} & 0_6 & id_6 & 0_6 & 0_6 & 0_6 \\ \mathrm{Ad}_{q_4^{-1}} & 0_6 & 0_6 & id_6 & 0_6 & 0_6 \\ \mathrm{Ad}_{(q_4 q_5)^{-1}} & 0_6 & 0_6 & \mathrm{Ad}_{q_5^{-1}} & id_6 & 0_6 \\ \mathrm{Ad}_{(q_4 q_5 q_6)^{-1}} & 0_6 & 0_6 & \mathrm{Ad}_{(q_5 q_6)^{-1}} & \mathrm{Ad}_{q_6^{-1}} & id_6 \end{bmatrix}$$

$$\begin{bmatrix} T_{q_1}(L_{q_1^{-1}} \circ \iota_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_{q_6}(L_{q_6^{-1}} \circ \iota_6) \end{bmatrix},$$

where

$$T_{q_1}(L_{q_1^{-1}} \circ \iota_1) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0\,0\,0\,0 \\ \sin(\theta) & \cos(\theta) & 0\,0\,0\,0 \\ 0 & 0 & 0\,0\,0\,1 \end{bmatrix}^T, \qquad T_{q_2}(L_{q_2^{-1}} \circ \iota_2) = \begin{bmatrix} 0\;0\;0\;0\;1\;0 \end{bmatrix}^T,$$

$$T_{q_3}(L_{q_3^{-1}} \circ \iota_3) = \begin{bmatrix} 0\;0\;0\;0\;1\;0 \end{bmatrix}^T, \qquad T_{q_4}(L_{q_4^{-1}} \circ \iota_4) = \begin{bmatrix} 0\; -l_0 - l_1\; 0\;0\;0\;1 \end{bmatrix}^T,$$

$$T_{q_5}(L_{q_5^{-1}} \circ \iota_5) = \begin{bmatrix} 0\;0\; -l_2\; 1\;0\;0 \end{bmatrix}^T, \qquad T_{q_6}(L_{q_6^{-1}} \circ \iota_6) = \begin{bmatrix} 0\;0\; -l_2 - l_3\; 1\;0\;0 \end{bmatrix}^T.$$

Note that $\forall r_0 = \begin{bmatrix} R_0 & p_0 \\ 0_{1\times 3} & 1 \end{bmatrix} \in SE(3)$, we calculate the Ad_{r_0} operator by $\mathrm{Ad}_{r_0} = \begin{bmatrix} R_0 & \widetilde{p}_0 R_0 \\ 0_{3\times 3} & R_0 \end{bmatrix}$, where $\widetilde{p}_0$ is the antisymmetric matrix corresponding to p_0.

The left-invariant metric $h = h_1 \oplus \cdots \oplus h_6$ on $\mathcal{P}$ is identified by the metrics $(h_i)_e$ on the copies of $se(3)$ represented in the standard basis of $se(3)$:

$$(h_i)_e = \begin{bmatrix} m_i id_3 & 0_3 \\ 0_3 & \begin{bmatrix} j_{x,i} & 0 & 0 \\ 0 & j_{y,i} & 0 \\ 0 & 0 & j_{z,i} \end{bmatrix} \end{bmatrix},$$

where e corresponds to the identity element of $SE(3)$, $i = 1, \cdots, 6$, m_i is the mass of B_i, and $(j_{x,i}, j_{y,i}, j_{z,i})$ are the moments of inertia of B_i about the X, Y and Z axes of the coordinate frame attached to the centre of mass of B_i whose axes coincide with the principal axes of B_i. Therefore, we have

$$\mathbb{F}L_q = T_q^*(L_{F(q)^{-1}}F) \begin{bmatrix} (h_1)_e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (h_6)_e \end{bmatrix} T_q(L_{F(q)^{-1}}F) = \begin{bmatrix} K_{11}(q) & \cdots & K_{16}(q) \\ \vdots & \ddots & \vdots \\ K_{61}(q) & \cdots & K_{66}(q) \end{bmatrix},$$

and the kinetic energy is calculated by

$$K_q(\dot{q}, \dot{q}) = \frac{1}{2}\dot{q}^T \mathbb{F}L_q \dot{q},$$

where $\dot{q}$ is the vector corresponding to the speed of the joint parameters.

The potential energy of the nonholonomic open-chain multi-body system for a constant potential field $[0\;0\;g]^T$ is calculated by

$$V(q) = g(l_4 m_6 \sin(\varphi_2 + \varphi_3) + l_3(\frac{m_5}{2} + m_6)\sin(\varphi_2)).$$

And, the Hamiltonian of the system is defined as

$$H(q,p) = \frac{1}{2}p^T \mathbb{F}L_q^{-1} p + V(q),$$

where $p = \mathbb{F}L_q\dot{q}$is the vector of generalized momenta.

The nonholonomic constraints of the system are non-slipping conditions of the wheels, i.e., B_2 and B_3. The linearly independent constraint 1-forms are

$$\begin{aligned}\omega_1 &= -\sin(\theta)dx + \cos(\theta)dy,\\ \omega_2 &= \cos(\theta)dx + \sin(\theta)dy - cd\theta - bd\psi_1,\\ \omega_3 &= \cos(\theta)dx + \sin(\theta)dy + cd\theta - bd\psi_2,\end{aligned}$$

where b is the radius of each wheel. The distribution $\mathcal{D} \subset T\mathcal{Q}$ is the annihilator of these constraint 1-forms, and it is the span of the following vector fields:

$$\left\{\frac{\partial}{\partial\psi_1} + \frac{b}{2}\left(\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} - \frac{1}{c}\frac{\partial}{\partial\theta}\right)\right.$$
$$\left., \frac{\partial}{\partial\psi_2} + \frac{b}{2}\left(\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} + \frac{1}{c}\frac{\partial}{\partial\theta}\right), \frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2}, \frac{\partial}{\partial\varphi_3}\right\}.$$

In this example, base of the multi-body system consists of three bodies whose configuration manifold $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3$ is isomorphic to $SE(2) \times SO(2) \times SO(2)$, as a Lie group. The kinetic and potential energy of the system are invariant under the action of this group by left translation on $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3$. Also, the distribution $\mathcal{D}$ is invariant under this action. Consider the action of $\mathcal{G} = SE(2)$, which satisfies the dimensional assumption (2.8) for Chaplygin systems. Using the joint parameters, $\forall(x_0, y_0, \theta_0) \in \mathcal{G}$ we have

$$\widehat{\Phi}_{(x_0,y_0,\theta_0)}(q) = (x\cos(\theta_0) - y\sin(\theta_0) + x_0, x\sin(\theta_0) + y\cos(\theta_0) + y_0, \theta + \theta_0, \widehat{q}),$$

where $\widehat{q} = (\psi_1, \psi_2, \varphi_1, \varphi_2, \varphi_3)$. We have the principal $\mathcal{G}$-bundle $\widehat{\pi}\colon \mathcal{Q} \to \widehat{\mathcal{Q}} = \mathcal{Q}_2 \times \cdots \times \mathcal{Q}_6$, and using the joint parameters its corresponding principal connection $\widehat{\mathcal{A}}\colon T\mathcal{Q} \to se(2)$ is defined by

$$\widehat{\mathcal{A}}_q = \overbrace{\begin{bmatrix}\cos(\theta) & -\sin(\theta) & y\\ \sin(\theta) & \cos(\theta) & -x\\ 0 & 0 & 1\end{bmatrix}}^{\mathrm{Ad}_{\mathfrak{g}}}\left[\overbrace{\begin{bmatrix}\cos(\theta) & \sin(\theta) & 0\\ -\sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}}^{T_{\mathfrak{g}}L_{\mathfrak{g}^{-1}}}\ \overbrace{\underbrace{\begin{bmatrix}-b/2 & -b/2\\ 0 & 0\\ b/(2c) & -b/(2c)\end{bmatrix}}_{\widehat{A}_1}\ 0_3}^{\widehat{A}_{\widehat{q}}}\right],$$

where $\mathfrak{g} = (x, y, \theta)$ is an element of $\mathcal{Q}_1$. Consequently, the horizontal lift map $\widehat{\mathrm{hl}}_q\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to T_q\mathcal{Q}$ and the Legendre transformation $\mathbb{F}\widehat{L}_{\widehat{q}}\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to T^*_{\widehat{q}}\widehat{\mathcal{Q}}$ are

$$\widehat{\mathrm{hl}}_q = \begin{bmatrix}\begin{bmatrix}b\cos(\theta)/2 & b\cos(\theta)/2\\ b\sin(\theta)/2 & b\sin(\theta)/2\\ -b/(2c) & b/(2c)\end{bmatrix} & 0_3\\ & id_5\end{bmatrix}, \mathbb{F}\widehat{L}_{\widehat{q}} = \widehat{\mathrm{hl}}_q^T\mathbb{F}L_q\widehat{\mathrm{hl}}_q = \begin{bmatrix}\widehat{K}_{11}(\widehat{q}) & \cdots & \widehat{K}_{14}(\widehat{q})\\ \vdots & \ddots & \vdots\\ \widehat{K}_{41}(\widehat{q}) & \cdots & \widehat{K}_{44}(\widehat{q})\end{bmatrix},$$

where the following equalities hold:

$$\widehat{K}_{11}(\widehat{q}) = \widehat{A}_1^T K_{11}((e_1,\widehat{q}))\widehat{A}_1 - \widehat{A}_1^T \begin{bmatrix} K_{21}((e_1,\widehat{q})) \\ K_{31}((e_1,\widehat{q})) \end{bmatrix}^T - \begin{bmatrix} K_{21}((e_1,\widehat{q})) \\ K_{31}((e_1,\widehat{q})) \end{bmatrix} \widehat{A}_1 + \begin{bmatrix} K_{22}((e_1,\widehat{q})) & K_{23}((e_1,\widehat{q})) \\ K_{32}((e_1,\widehat{q})) & K_{33}((e_1,\widehat{q})) \end{bmatrix},$$

$$\widehat{K}_{1\mathrm{j}}(\widehat{q}) = -\widehat{A}_1^T K_{1(\mathrm{j}+2)}((e_1,\widehat{q})) + \begin{bmatrix} K_{2(\mathrm{j}+2)}((e_1,\widehat{q})) \\ K_{3(\mathrm{j}+2)}((e_1,\widehat{q})) \end{bmatrix}, \quad \forall \mathrm{j} = 2,3,4$$

$$\widehat{K}_{\mathrm{j}1}(\widehat{q}) = \widehat{K}_{1\mathrm{j}}(\widehat{q})^T, \quad \forall \mathrm{j} = 2,3,4$$

$$\widehat{K}_{\mathrm{ij}}(\widehat{q}) = K_{\mathrm{ij}}((e_1,\widehat{q})). \quad \forall \mathrm{i},\mathrm{j} = 2,3,4$$

As a result, we can calculate the 2-form $\widehat{\Xi}$ by (2.9)

$$\widehat{\Omega} = -d\widehat{p} \wedge d\widehat{q} - \widehat{p}^T \mathbb{F}\widehat{L}_{\widehat{q}}^{-1} \begin{bmatrix} -\widehat{A}_1^T K_{11}((e_1,\widehat{q})) + \begin{bmatrix} K_{21}((e_1,\widehat{q})) \\ K_{31}((e_1,\widehat{q})) \end{bmatrix} \\ K_{41}((e_1,\widehat{q})) \\ \vdots \\ K_{61}((e_1,\widehat{q})) \end{bmatrix} \begin{bmatrix} 0 \\ b^2/(2c) \\ 0 \end{bmatrix} d\psi_1 \wedge d\psi_2$$
$$= -d\widehat{p} \wedge d\widehat{q} - \varUpsilon(\widehat{q},\widehat{p}) d\psi_1 \wedge d\psi_2,$$

where $\widehat{p} = \mathbb{F}\widehat{L}_{\widehat{q}}\dot{\widehat{q}}$ is the vector of generalized momenta in the reduced phase space. And, the reduced equations of motion of the nonholonomic system are

$$\begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} = \begin{bmatrix} 0_{5\times5} & id_5 \\ -id_5 & \begin{bmatrix} 0 & \varUpsilon(\widehat{q},\widehat{p}) & 0 & 0 & 0 \\ -\varUpsilon(\widehat{q},\widehat{p}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix},$$

where $\widehat{H}$ is calculated by (2.10), with $\widehat{V}(\widehat{q}) = V((e_1,\widehat{q}))$.

Now, we derive the control law presented in Theorem 3 for the example under study. We assume that the two wheels of the rover and the three joints of the manipulator are actuated. The output manifold is considered to be $\mathcal{R} = \mathbb{R}^2 \times \mathbb{R}^3 \subset SE(3) \times SE(3)$ corresponding to the position of the centre of mass of the rover and the centre of mass of the end-effector. Note that $\mathcal{R}$ is a subgroup of $\mathcal{P}_e = SE(3) \times SE(3)$. The forward kinematics maps for the extremities of the system are

$$FK_1(x,y,\theta) = q_1 r_{cm,1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x + l_0\cos(\theta) \\ \sin(\theta) & \cos(\theta) & 0 & y + l_0\sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$FK_2(x,y,\theta,\varphi_1,\varphi_2,\varphi_3) = \begin{bmatrix} R_E & p_E \\ 0_{1\times3} & 1 \end{bmatrix} := q_1 q_4 q_5 q_6 r_{cm,6},$$

where $R_E(x, y, \theta, \varphi_1, \varphi_2, \varphi_3) \in SO(3)$ and $p_E(x, y, \theta, \varphi_1, \varphi_2, \varphi_3) \in \mathbb{R}^3$ specify the pose of the end-effector with respect to the inertial coordinate frame and expressed in the same coordinate frame. The projection maps $\mathfrak{r}_1 : SE(3) \to \mathbb{R}^2$ and $\mathfrak{r}_2 : SE(3) \to \mathbb{R}^3$ are simply projection to the position components of the poses of the rover and the end-effector, respectively. The output of the system is calculated in the local coordinates as

$$\begin{aligned} FK(x, y, \theta, \varphi_1, \varphi_2, \varphi_3) &= (\mathfrak{r}_1 \circ FK_1(q), \mathfrak{r}_2 \circ FK_2(q)) \\ &= (x + l_0 \cos(\theta), y + l_0 \sin(\theta), p_E(x, y, \theta, \varphi_1, \varphi_2, \varphi_3)). \end{aligned}$$

Denote the output trajectory of the system by $t \mapsto r(t) = FK(x(t), y(t), \theta(t), \varphi_1(t), \varphi_2(t), \varphi_3(t))$, and consider a desired feasible trajectory $t \mapsto \gamma(t) \in \mathcal{R}$. Since the output manifold is an abelian subgroup of $\mathcal{P}_e$, the output pose error is just $r_e(t) = \gamma(t) - r(t)$, and considering $\mathcal{K}_P = diag(\mathcal{K}_P^1, \cdots, \mathcal{K}_P^5)$, the error function is defined by $E_r(r(t), \gamma(t)) = \frac{1}{2} \langle \mathcal{K}_P r_e(t), r_e(t) \rangle$. Note that the exponential map restricted to the abelian subgroup $\mathcal{R} \subset \mathcal{P}_e$ is the identity map, and it is everywhere invertible. The compatible linear isomorphism with E_r is the identity map, and the output velocity error is simply calculated by $V_e = \mathfrak{v}_e = \dot{r}(t) - \dot{\gamma}(t)$. We denote the derivative gain by $\mathcal{K}_D = diag(\mathcal{K}_D^1, \cdots, \mathcal{K}_D^5)$, with positive diagonal elements. Next, we calculate the Jacobian maps:

$$J_1^0 = \begin{bmatrix} id_3 - \widetilde{\begin{bmatrix} l_0 \\ 0 \\ 0 \end{bmatrix}} \\ 0_3 \quad id_3 \end{bmatrix} \begin{bmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0\,0\,0\,0 \\ 0 & 1 & 0\,0\,0\,0 \\ 0 & l_0 & 0\,0\,0\,1 \end{bmatrix}^T$$

$$J_2^0 = \mathrm{Ad}_{r_{cm,6}^{-1}} \left[\mathrm{Ad}_{(q_4 q_5 q_6)^{-1}} T_{e_1} \iota_1 \; \mathrm{Ad}_{(q_5 q_6)^{-1}} T_{e_4} \iota_4 \; \mathrm{Ad}_{q_6^{-1}} T_{e_5} \iota_5 \; T_{e_6} \iota_6 \right],$$

where

$$T_{e_1} \iota_1 = \begin{bmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{bmatrix}^T, T_{e_2} \iota_2 = \begin{bmatrix} 0\,0\,0\,0\,1\,0 \end{bmatrix}^T, T_{e_3} \iota_3 = \begin{bmatrix} 0\,0\,0\,0\,1\,0 \end{bmatrix}^T,$$

$$T_{e_4} \iota_4 = \begin{bmatrix} 0 \; -l_0 - l_1 \; 0\,0\,0\,1 \end{bmatrix}^T, T_{e_5} \iota_5 = \begin{bmatrix} 0\,0 \; -l_2 \; 1\,0\,0 \end{bmatrix}^T, T_{e_6} \iota_6 = \begin{bmatrix} 0\,0 \; -l_2 - l_3 \; 1\,0\,0 \end{bmatrix}^T.$$

And accordingly, we have

$$\jmath_q = \begin{bmatrix} [(\jmath_1^0)_q \; 0_{2\times 3}] \\ (\jmath_2^0)_q \end{bmatrix} \qquad \text{such that} \qquad \jmath_1^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & -l_0 \sin(\theta) \\ \sin(\theta) & \cos(\theta) & l_0 \cos(\theta) \end{bmatrix},$$

$$\jmath_2^0 = \begin{bmatrix} R_E \; -R_E \widetilde{\begin{bmatrix} l_0 + l_1 \\ l_2 + l_3 + l_4 \\ 0 \end{bmatrix}} \end{bmatrix} \left[\mathrm{Ad}_{(q_4 q_5 q_6)^{-1}} T_{e_1} \iota_1 \; \mathrm{Ad}_{(q_5 q_6)^{-1}} T_{e_4} \iota_4 \; \mathrm{Ad}_{q_6^{-1}} T_{e_5} \iota_5 \; T_{e_6} \iota_6 \right].$$

Consider an initial phase $(\widehat{q}(0), \widehat{p}(0))$ in the reduced phase space $T^* \widehat{\mathcal{Q}}$ that satisfies CON6. We denote the integral curve of the reduced system by $t \mapsto$

$(\widehat{q}(t), \widehat{p}(t))$. Hence, the reduced controlled Hamilton's equation is:

$$\begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} = \begin{bmatrix} 0_{5\times 5} & id_5 \\ -id_5 & \begin{bmatrix} \begin{bmatrix} 0 & \Upsilon(\widehat{q},\widehat{p}) \\ -\Upsilon(\widehat{q},\widehat{p}) & 0 \end{bmatrix} & 0_{2\times 3} \\ 0_{3\times 2} & 0_3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} + \widehat{u} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix}.$$

From this equation and the reconstruction equation we have

$$\begin{bmatrix} \widehat{\mathfrak{q}}_1(\widehat{q},\widehat{p}) \\ \check{\mathfrak{q}}(\widehat{q},\widehat{p}) \end{bmatrix} := \widehat{\mathfrak{q}}(\widehat{q},\widehat{p}) := \dot{\widehat{q}} = \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{p}_1} \\ \frac{\partial \widehat{H}}{\partial \check{p}} \end{bmatrix} = \mathbb{F}\widehat{L}_{\widehat{q}}^{-1}\widehat{p},$$

$$\widehat{\mathfrak{p}}(\widehat{q},\widehat{p}) := \begin{bmatrix} \begin{bmatrix} 0 & \Upsilon(\widehat{q},\widehat{p}) \\ -\Upsilon(\widehat{q},\widehat{p}) & 0 \end{bmatrix} & 0_{2\times 3} \\ 0_{3\times 2} & 0_{3\times 3} \end{bmatrix} \mathbb{F}\widehat{L}_{\widehat{q}}^{-1}\widehat{p} - \frac{\partial \widehat{H}}{\partial \widehat{q}}, \quad \tau_1 = -\widehat{A}_1 \widehat{\mathfrak{q}}_1(\widehat{q},\widehat{p}),$$

where $\widehat{p} = (\widehat{p}_1, \check{p}) = ((\widehat{p}_{\psi_1}, \widehat{p}_{\psi_2}), (\widehat{p}_{\varphi_1}, \widehat{p}_{\varphi_2}, \widehat{p}_{\varphi_3}))$ are the momenta corresponding to $\widehat{q} = (\widehat{q}_1, \check{q}) = ((\psi_1, \psi_2), (\varphi_1, \varphi_2, \varphi_3))$ in the reduced phase space, and

$$\widehat{A}_1 = \begin{bmatrix} -b/2 & -b/2 \\ 0 & 0 \\ b/(2c) & -b/(2c) \end{bmatrix}.$$

In the following, we calculate the components of the control law stated in (6.41). Since $\widehat{\mathcal{Q}}$ is an abelian group, $T_{\widehat{q}} L_{\widehat{q}^{-1}}$ is the identity map, and we have

$$\widehat{\tau}(\widehat{q}(t), \widehat{p}(t)) := \tau(t) = \begin{bmatrix} -\widehat{A}_{\widehat{q}_1(t)} \widehat{\mathfrak{q}}_1(\widehat{q}(t), \widehat{p}(t)) \\ \check{\mathfrak{q}}(\widehat{q}(t), \widehat{p}(t)) \end{bmatrix} = \begin{bmatrix} -\widehat{A}_{\widehat{q}_1(t)} & 0_3 \\ 0_{3\times 2} & id_3 \end{bmatrix} \mathbb{F}\widehat{L}_{\widehat{q}(t)}^{-1}\widehat{p},$$

$$\frac{\partial \widehat{\tau}}{\partial \widehat{p}} = \begin{bmatrix} -\widehat{A}_{\widehat{q}_1(t)} & 0_3 \\ 0_{3\times 2} & id_3 \end{bmatrix} \mathbb{F}\widehat{L}_{\widehat{q}}^{-1},$$

$$\begin{aligned} \frac{\partial \jmath_q}{\partial q} (T_e L_q \widehat{\tau}(\widehat{q},\widehat{p})) = &-\frac{\partial \jmath_q}{\partial x} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \end{bmatrix} \widehat{A}_1 \widehat{\mathfrak{q}}_1(\widehat{q},\widehat{p}) \\ &- \frac{\partial \jmath_q}{\partial y} \begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \widehat{A}_1 \widehat{\mathfrak{q}}_1(\widehat{q},\widehat{p}) - \frac{\partial \jmath_q}{\partial \theta} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \widehat{A}_1 \widehat{\mathfrak{q}}_1(\widehat{q},\widehat{p}) \\ &+ \frac{\partial \jmath_q}{\partial \varphi_1} \frac{\partial \widehat{H}}{\partial \widehat{p}_{\varphi_1}} + \frac{\partial \jmath_q}{\partial \varphi_2} \frac{\partial \widehat{H}}{\partial \widehat{p}_{\varphi_2}} + \frac{\partial \jmath_q}{\partial \varphi_3} \frac{\partial \widehat{H}}{\partial \widehat{p}_{\varphi_3}}. \end{aligned}$$

As a result, the following control law exponentially stabilizes the output of the system for any feasible desired trajectory $t \mapsto \gamma(t)$:

$$\begin{aligned} \widehat{u}(q, \widehat{p}, \gamma, \dot{\gamma}, \ddot{\gamma}) = \mathbb{F}\widehat{L}_{\widehat{q}(t)} \left(\jmath_q \begin{bmatrix} -\widehat{A}_1 & 0_3 \\ 0_{3\times 2} & id_3 \end{bmatrix} \right)^{-1} &\left(\left(\frac{\partial \jmath_q}{\partial q} (T_e L_q \widehat{\tau}) \right) \begin{bmatrix} -\widehat{A}_1 & 0_3 \\ 0_{3\times 2} & id_3 \end{bmatrix} \mathbb{F}\widehat{L}_{\widehat{q}}^{-1}\widehat{p} \right. \\ &\left. + \jmath_q \frac{\partial \widehat{\tau}}{\partial \widehat{q}} \mathbb{F}\widehat{L}_{\widehat{q}}^{-1}\widehat{p} - \widehat{\nu} \right) + \widehat{\mathfrak{p}}, \end{aligned}$$

such that for this case study we have

$$\begin{aligned}
\widehat{\nu} &= \widehat{\nu}_{PD} + \widehat{\nu}_{FF}, \\
\widehat{\nu}_{PD} &= \widehat{\nu}_P + \widehat{\nu}_D = -\mathcal{K}_P(r(t) - \gamma(t)) - \mathcal{K}_D(\dot{r}(t) - \dot{\gamma}(t)), \\
\widehat{\nu}_{FF} &= \ddot{\gamma}(t)
\end{aligned}$$

where $r(t) := FK(q(t)) \in \mathcal{R}$ is the output of the system. Here, we have chosen $\mathcal{I}$ to be an identity matrix for the standard basis of $\mathfrak{R} \cong \mathbb{R}^5$ and $\mathfrak{R}^* \cong \mathbb{R}^5$, and $\mathcal{K}_D = diag(\mathcal{K}_D^1, \cdots, \mathcal{K}_D^5)$ is a diagonal matrix with positive diagonal elements, as defined above. In this case study, we can explicitly write the differential equation that governs the behaviour of the closed-loop system:

$$\frac{d^2}{dt^2}(r - \gamma) + \mathcal{K}_D \frac{d}{dt}(r - \gamma) + \mathcal{K}_P(r - \gamma) = 0,$$

where $\gamma, r \in \mathbb{R}^5$, and by choosing $\mathcal{K}_P$ and $\mathcal{K}_D$ diagonal we decouple this differential equation. As a result, we can choose the diagonal elements of $\mathcal{K}_P$ and $\mathcal{K}_D$ such that the closed-loop system becomes decoupled with a desired performance. Consequently, the gains $\mathcal{K}_P^i$'s and $\mathcal{K}_D^i$'s can be design so that the system error dynamics will have a desired behaviour in each channel. Also, note that the feed-forward function, in this case, becomes gain one. The complete block diagram of the closed-loop system is shown in Figure 5. In this figure, $R_Z(\theta) \in SO(3)$ corresponds to the principal rotation about the Z axis for θ radian.

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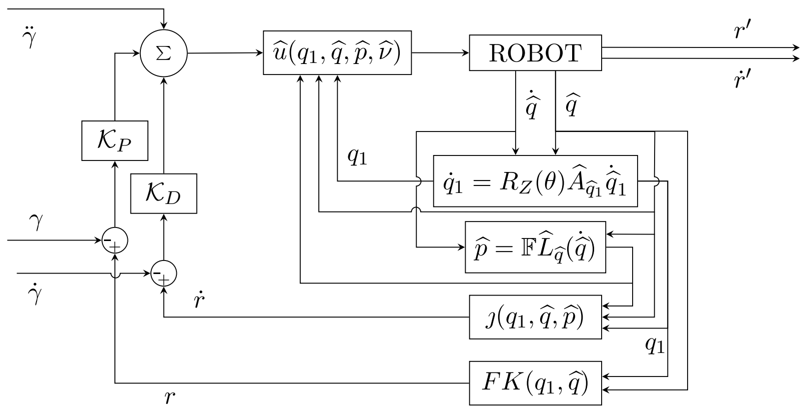

Fig. 5 Servo controller for concurrent control of a three-d.o.f. manipulator mounted on a two-wheeled rover

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