Explicit Recursive Track-to-Track Fusion Rules for Nonlinear Multi-Sensor Systems

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***Abstract*— This letter presents explicit sub-optimal track-to-track fusion algorithms for Multi-Sensor Systems (MSS) estimating nonlinear processes. The individual tracks in an MSS are correlated due to the presence of a common process noise in the track estimation errors. Herein, we propose recursive formulae for consistent correlation estimation in mildly and highly nonlinear systems that respectively use Extended Kalman Filters (EKF) and Unscented Kalman Filters (UKF) for track estimation. In mildly nonlinear systems, the EKF provides sufficiently accurate estimates based on the linearized model of the system at its latest estimate. This linear model offers a correlation propagation formula that will be coupled with the optimal track fusion rule to generate a sub-optimal fused estimate in an EKF-based MSS architecture. On the other hand in highly nonlinear systems, the UKF-based architectures are proven effective for track estimation. The UKF works based on the unscented transform of deterministic sigma points and it is accurate up to the third order of the Taylor Series expansion of the system. The unscented transform is equivalent to the Statistical Linearization Regression (SLR) process when using the sigma points. For UKF-based MSS architectures, we propose a consistent correlation propagation recursion according to the SLR technique that will be coupled with the optimal track fusion rule to generate a sub-optimal fused estimate. The performance of the developed fusion algorithms is demonstrated through conducting a statistical test and an average root mean square error analysis.**

***Index Terms*— Track-to-Track Fusion, Correlation Propagation, Unscented Kalman Filter, Statistical Linearization Regression**

I. Introduction

MULTI-Sensor Systems (MSS) have recently gained a great attention in robotics and automation. They play a central role in applications that demand state and parameter estimation from multiple sources such as cooperative motion estimation [1], distributed estimation and tracking [2], [3], and localization and mapping [4], [5]. In comparison to a single sensor, a well-designed MSS fuses data from multiple sensors with different visibility and capability to improve estimation.

Sensor-level and track-level fusions are two commonly used fusion strategies in an MSS (see Fig. 1). In sensor-level fusion, Fig. 1(a), the raw observations, obtained directly from the sensors, are combined for better visibility and coverage. However, in track-level fusion, Fig. 1(b), which is the strategy under study in this paper, multiple noise-corrupted estimates generated by local estimators are combined to obtain the so-called fused estimate. This process is also referred to as track-to-track fusion [6]. In track-to-track fusion the local estimates are correlated due to a common process noise that enters the estimation errors corresponding to all sensors [6]. To generate consistent fused tracks, the main challenge is to explicitly or implicitly account for such correlation terms. Explicit approaches keep track of correlation terms and use the known correlations in their fusion rules. A recursion for correlation propagation in linear systems was proposed using Kalman gains and the properties of recursive estimators [6]. This recursion along with the fusion rule proposed by the same author yields the optimum solution (minimum variance) for track-to-track fusion of correlated estimates [7]. However, the proposed recursion in [6] is limited to linear systems and under some strict conditions it can be extended to mildly nonlinear systems where Kalman filter and its variants are applicable. A simple but rough computation of correlations is proposed in [8], where the cross-covariance matrix between two tracks is approximated based on the product of sensors' covariance matrices and a constant correlation coefficient determined by numerical simulations. In a relatively recent paper, sample-based track fusion is proposed for linear systems in which the cross-correlation between the tracks are reconstructed by a set of deterministic sample points [9]. In implicit approaches, however, the correlation terms are assumed unknown and this lack of knowledge is compensated by ensuring that the resulting fused estimate is conservative comparing to the optimal solution. As the result, the implicit approaches provide an upper bound for the error covariance of the optimum solution. The well-known classic example of such approaches is the Covariance Intersection (CI) [10]. The CI's solutions are sub-optimal as they tend to yield conservative estimates. Alternative track-to-track fusion methods that produce less

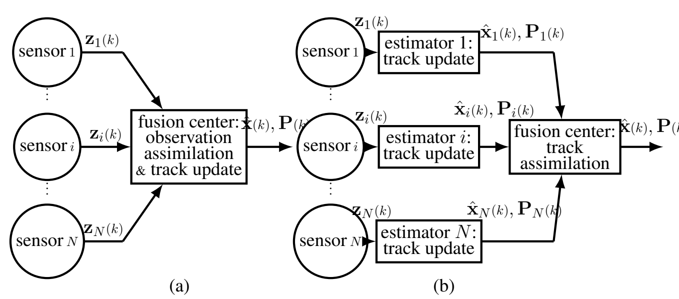

Fig. 1: Two fusion strategies. (a) sensor-level, (b) track-level

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conservative estimates have been proposed [11]–[14].

The main drawback of such approaches is that they may produce too optimistic estimates that can lead to performance degradation. In [15] and [16] the authors use optimization techniques to compute the cross-covariance matrix between the estimates that is not computationally efficient for large systems. In [17] and [18] two similar centralized fusion algorithms are proposed that are mainly based on the use of cross-covariance matrix of the tracks and minimizing the trace of the fused covariance. Indeed both methods produce unbiased fused tracks, however, since they work with the trace (scalar measure) of the joint covariance matrix instead of the full matrix, their consistency may be compromised. Furthermore, the proposed algorithms can undergo computational barriers when dealing with large systems, due to requirement of inverting large matrices. To remediate the drawbacks of implicit approaches, we propose an explicit sub-optimal track-to-track fusion rule for nonlinear systems that extends the existing explicit approaches limited to linear systems.

With the emphasis on nonlinear systems, this paper develops consistent methodologies to recursively calculate correlation terms in explicit track-to-track fusion of Kalman filter-based MSS architectures. The EKF and UKF are the two commonly used nonlinear recursive estimators. The UKF is accurate up to the third order of the Taylor Series expansion of the system; however, the EKF truncates the Taylor Series at first order. In mildly nonlinear systems, as suggested by the structure of the EKF, the linear approximation using the first order terms of the Taylor Series is sufficient to calculate a sub-optimal fusion rule. On the contrary in UKF-based MSS architectures, the Statistical Linearization Regression (SLR) technique [19] is employed to linearize a highly nonlinear system. The SLR is equivalent to the Unscented Transform (UT) that lies in the core of the UKF structure. It uses a set of deterministic sample points (sigma points in the UKF) to linearize a nonlinear function [19], [20]. The linearized models obtained in both EKF- and UKF-based MSS architectures are then used to propose consistent recursions for propagating the cross-covariance matrices. The obtained matrices are coupled with the optimal fusion rule to generate sub-optimal fused tracks that are computationally cost effective comparing to, for example, the CI or inverse CI relying on real-time numerical optimizations. As the result, the proposed methodologies offer a balance between speed and accuracy in the real-time estimation of nonlinear systems.

The problem of track-to-track fusion in nonlinear MSS architectures is described in Section II. In Section III, we review the fundamental equations governing the basic forms of the EKF and UKF, and the SLR technique. Section IV derives a recursion for EKF-based MSS architectures. Section V uses the SLR technique to propose a novel recursion for cross-covariance propagation in UKF-based MSS architectures. Through rigorous analytical error analysis in Section VI, we demonstrate the effectiveness of the proposed methodologies. Finally, Section VII includes some concluding remarks.

II. Problem Statement

Without loss of generality, we consider the simplest MSS consisting of two sensors $\mathbf{s}_1$ and $\mathbf{s}_2$ that are observing a common process whose true state at the k^{th} time step is denoted by $\mathbf{x}(k) \in \mathbb{R}^n$. For multiple tracks, a sequential fusion method can be used [21]. The procedure starts by fusing two local tracks to obtain a fused track, which is then sequentially fused with the remaining tracks. Let the nonlinear process and measurement models be given as

$$\mathbf{x}(k) = \mathbf{f}\big(\mathbf{x}(k-1), \mathbf{u}(k-1)\big) + \mathbf{w}(k-1), \tag{1}$$

$$\mathbf{z}_m(k) = \mathbf{h}_m\big(\mathbf{x}(k)\big) + \mathbf{v}_m(k), \quad m \in \{1,2\}, \tag{2}$$

$$\mathbf{w}(k) \sim \mathcal{N}(0, \mathbf{Q}(k)), \tag{3}$$

$$\mathbf{v}_m(k) \sim \mathcal{N}(0, \mathbf{R}_m(k)), \quad m \in \{1,2\}, \tag{4}$$

where $\mathbf{u}(k) \in \mathbb{R}^p$ and $\mathbf{z}_m(k) \in \mathbb{R}^q$, for $m \in \{1,2\}$, respectively denote the control vector and measurement vector from $\mathbf{s}_m$, and $\mathbf{f}\colon \mathbb{R}^{n+p} \rightarrow \mathbb{R}^n$ and $\mathbf{h}_m\colon \mathbb{R}^n \rightarrow \mathbb{R}^q$ are nonlinear functions. Furthermore, $\mathbf{w}(k) \in \mathbb{R}^n$ and $\mathbf{v}_m(k) \in \mathbb{R}^q$, for $m \in \{1,2\}$, are assumed to be independent normal white noise sequences respectively corresponding to the dynamical process and the sensor $\mathbf{s}_m$, i.e.,

$$\mathbb{E}[\mathbf{w}(k)\mathbf{w}^\top(l)] = \mathbf{Q}(k)\delta(k-l), \tag{5}$$

$$\mathbb{E}[\mathbf{v}_m(k)\mathbf{v}_m^\top(l)] = \mathbf{R}_m(k)\delta(k-l), \quad m \in \{1,2\}, \tag{6}$$

$$\mathbb{E}[\mathbf{v}_m(k)\mathbf{w}^\top(l)] = 0, \quad m \in \{1,2\}, \tag{7}$$

$$\mathbb{E}[\mathbf{v}_1(k)\mathbf{v}_2^\top(l)] = 0. \tag{8}$$

Here, $\delta(k-l)$ is the Kronecker delta function, $\mathbb{E}[\cdot]$ denotes the expected value operator, and $\mathbf{Q}(k) \in \mathbb{R}^{n\times n}$ and $\mathbf{R}_m(k) \in \mathbb{R}^{q\times q}$ are the symmetric positive definite covariance matrices corresponding to the process and measurement noise, respectively.

For this system, the posterior estimates of the sensors $\mathbf{s}_1$ and $\mathbf{s}_2$ and their corresponding covariance matrices are assumed to be generated by two recursive filters and respectively denoted by $(\hat{\mathbf{x}}_1(k|k), \mathbf{P}_1(k|k))$ and $(\hat{\mathbf{x}}_2(k|k), \mathbf{P}_2(k|k))$. We also assume that the estimates of the two sensors are consistent and synchronized. Note that the common process noise $\mathbf{w}(k)$ enters the estimation error of both sensors and results in two correlated estimates of the system (1)-(4), i.e., the cross-covariance matrix $\mathbf{P}_{12}(k|k) \neq 0$. To obtain a consistent fused estimate $\hat{\mathbf{x}}_f(k|k)$ along with its covariance matrix $\mathbf{P}_f(k|k)$ this cross-correlation should be formally accounted for. The earlier works that carelessly rely on zero correlation assumption to compute the fused track suffer from double counting problem and typically produce optimistic results [22]. To avoid this problem, a fusion rule, called the optimal (minimum variance) track fusion, was proposed based on the known cross-covariance matrix [7]:

$$\begin{aligned}\hat{\mathbf{x}}_f(k|k) &= \hat{\mathbf{x}}_1(k|k) + \big(\mathbf{P}_1(k|k) - \mathbf{P}_{12}(k|k)\big)\\ &\big(\mathbf{P}_1(k|k)+\mathbf{P}_2(k|k)-\mathbf{P}_{12}(k|k)-\mathbf{P}_{12}^\top(k|k)\big)^{-1}\big(\hat{\mathbf{x}}_2(k|k)-\hat{\mathbf{x}}_1(k|k)\big),\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbf{P}_f(k|k) &= \mathbf{P}_1(k|k) - \big(\mathbf{P}_1(k|k) - \mathbf{P}_{12}(k|k)\big)\\ &\big(\mathbf{P}_1(k|k)+\mathbf{P}_2(k|k)-\mathbf{P}_{12}(k|k)-\mathbf{P}_{12}^\top(k|k)\big)^{-1}\big(\mathbf{P}_1(k|k)-\mathbf{P}_{12}^\top(k|k)\big).\end{aligned} \tag{10}$$

Reliably estimating the cross-covariance matrix is an identified challenge in fusing tracks of an MSS. Considering the above optimal fusion rule, this paper seeks a recursion to consistently approximate $\mathbf{P}_{12}(k|k)$ and propose sub-optimal track-to-track fusion rules for nonlinear systems. In the next section, we review the nonlinear filters and the SLR to pave the way leading to our proposed method.

III. Preliminaries

A. Extended Kalman Filter

Consider the discrete-time nonlinear system given in (1)-(4). In the standard EKF [23], the system is linearized at its most recent estimate assuming a perfectly known control input $\mathbf{u}(k)$. For the sensor $\mathbf{s}_m$

$$\mathbf{x}_m(k) = \mathbf{F}_m(k-1)\mathbf{x}(k-1) + \tilde{\mathbf{u}}_m(k-1) + \mathbf{w}(k-1), \tag{11}$$

$$\mathbf{z}_m(k) = \mathbf{H}_m(k)\mathbf{x}(k) + \tilde{\mathbf{y}}_m(k) + \mathbf{v}_m(k), \tag{12}$$

where

$$\mathbf{F}_m(k) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}|_{\hat{\mathbf{x}}_m(k|k)} \in \mathbb{R}^{n\times n}, \tag{13}$$

$$\mathbf{H}_m(k) = \frac{\partial \mathbf{h}_m}{\partial \mathbf{x}}|_{\hat{\mathbf{x}}_m(k|k-1)} \in \mathbb{R}^{n\times q}, \tag{14}$$

$$\tilde{\mathbf{u}}_m(k) = \mathbf{f}\big(\hat{\mathbf{x}}_m(k|k), \mathbf{u}(k)\big) - \mathbf{F}_m(k)\hat{\mathbf{x}}_m(k|k), \tag{15}$$

$$\tilde{\mathbf{y}}_m(k) = \mathbf{h}_m\big(\hat{\mathbf{x}}_m(k|k-1)\big) - \mathbf{H}_m(k)\hat{\mathbf{x}}_m(k|k-1). \tag{16}$$

The operator $\frac{\partial(\cdot)}{\partial \mathbf{x}}$ denotes the Jacobian of a vector function with respect to $\mathbf{x}$. In this framework, the prior estimate for the m^{th} sensor is given by

$$\hat{\mathbf{x}}_m(k|k-1) = \mathbf{F}_m(k-1)\hat{\mathbf{x}}_m(k-1|k-1) + \tilde{\mathbf{u}}_m(k-1), \tag{17}$$

$$\mathbf{P}_m(k|k-1) = \mathbf{F}_m(k-1)\mathbf{P}_m(k-1|k-1)\mathbf{F}_m^\top(k-1) + \mathbf{Q}(k-1), \tag{18}$$

and the posterior estimate is calculated as

$$\hat{\mathbf{x}}_m(k|k) = \hat{\mathbf{x}}_m(k|k-1) + \mathbf{K}_m(k)\big(\mathbf{z}_m(k) - \mathbf{H}_m(k)\hat{\mathbf{x}}_m(k|k-1) - \tilde{\mathbf{y}}_m(k)\big), \tag{19}$$

$$\mathbf{P}_m(k|k) = \big(\mathbf{I} - \mathbf{K}_m(k)\mathbf{H}_m(k)\big)\mathbf{P}_m(k|k-1), \tag{20}$$

where $\mathbf{I} \in \mathbb{R}^{n\times n}$ is the identity matrix and the EKF gain matrix for the m^{th} sensor is $\mathbf{K}_m(k) = \mathbf{P}_m(k|k)\mathbf{H}_m^\top(k)\mathbf{R}_m^{-1}(k) \in \mathbb{R}^{n\times q}$.

B. Unscented Kalman Filter

The UKF first proposed in [24] is an alternative nonlinear estimator with performance superiority comparing to the EKF. In the UKF a set of deterministic sample points called sigma points are used to propagate mean and covariance using UT. It has been shown that the UKF is accurate up to the third order of the Taylor Series expansion [24], and to reduce the forth-order errors some techniques have been suggested, e.g., in [25], [26]. In the following, we present the basic form of the UKF for the nonlinear system described in (1)-(4). It is noted that our proposed methodology in Section V works for all variants of the UKF. But for simplicity, we only focus on its basic form.

The $2n$ sigma points for the m^{th} sensor are defined as

$$\boldsymbol{\chi}_m^{(i)}(k-1|k-1) = \hat{\mathbf{x}}_m(k-1|k-1) + \tilde{\boldsymbol{\chi}}_m^{(i)}(k-1|k-1),\ i=1,...,2n \tag{21}$$

$$\tilde{\boldsymbol{\chi}}_m^{(j)}(k-1|k-1) = \Big(\sqrt{n\mathbf{P}_m(k-1|k-1)}\Big)_j^\top,$$
$$\tilde{\boldsymbol{\chi}}_m^{(n+j)}(k-1|k-1) = -\Big(\sqrt{n\mathbf{P}_m(k-1|k-1)}\Big)_j^\top, \quad j=1,...,n \tag{22}$$

where $(\sqrt{n\mathbf{P}_m(k-1|k-1)})_j$ denotes the j^{th} row of matrix square root of $n\mathbf{P}_m(k-1|k-1)$. The known nonlinear function $\mathbf{f}$ is used to transform the sigma points

$$\boldsymbol{\chi}_m^{(i)}(k|k-1) = \mathbf{f}\big(\boldsymbol{\chi}_m^{(i)}(k-1|k-1), \mathbf{u}(k-1)\big), \tag{23}$$

which are then combined to yield the prior estimate

$$\hat{\mathbf{x}}_m(k|k-1) = \frac{1}{2n}\sum_{i=1}^{2n}\boldsymbol{\chi}_m^{(i)}(k|k-1), \tag{24}$$

$$\mathbf{P}_m(k|k-1) = \frac{1}{2n}\sum_{i=1}^{2n}\big(\boldsymbol{\chi}_m^{(i)}(k|k-1) - \hat{\mathbf{x}}_m(k|k-1)\big)\big(.\big)^\top + \mathbf{Q}(k-1), \tag{25}$$

where $(.)$ implies the quantity between parenthesis is the same as the quantity between the previous parenthesis.

The propagated sigma points $\boldsymbol{\chi}_m^{(i)}(k|k-1)$ are updated based on (21)-(22) using the mean $\hat{\mathbf{x}}_m(k|k-1)$ and covariance $\mathbf{P}_m(k|k-1)$. Then, the new sigma points define the predicted measurement vector

$$\boldsymbol{\mathcal{Z}}_m^{(i)}(k|k-1) = \mathbf{h}_m\big(\boldsymbol{\chi}_m^{(i)}(k|k-1)\big), \tag{26}$$

$$\hat{\mathbf{z}}_m(k) = \frac{1}{2n}\sum_{i=1}^{2n}\boldsymbol{\mathcal{Z}}_m^{(i)}(k|k-1). \tag{27}$$

The covariance matrix of measurements for the sensor $\mathbf{s}_m$ is

$$\mathbf{P}_{zz_m}(k) = \frac{1}{2n}\sum_{i=1}^{2n}\big(\boldsymbol{\mathcal{Z}}_m^{(i)}(k|k-1) - \hat{\mathbf{z}}_m(k)\big)\big(.\big)^\top + \mathbf{R}_m(k). \tag{28}$$

In addition, the cross-covariance matrix between measurements and estimated states need to be determined based on

$$\mathbf{P}_{xz_m}(k) = \frac{1}{2n}\sum_{i=1}^{2n}\big(\boldsymbol{\chi}_m^{(i)}(k|k-1) - \hat{\mathbf{x}}_m(k|k-1)\big)\big(\boldsymbol{\mathcal{Z}}_m^{(i)}(k|k-1) - \hat{\mathbf{z}}_m(k)\big)^\top \tag{29}$$

to complete the estimation recursion. Finally, the posterior estimate of the states is computed by the standard recursive filter equations

$$\hat{\mathbf{x}}_m(k|k) = \hat{\mathbf{x}}_m(k|k-1) + \mathbf{K}_m(k)\big(\mathbf{z}_m(k) - \hat{\mathbf{z}}_m(k)\big), \tag{30}$$

$$\mathbf{P}_m(k|k) = \mathbf{P}_m(k|k-1) - \mathbf{K}_m(k)\mathbf{P}_{zz_m}(k)\mathbf{K}_m^\top(k), \tag{31}$$

where $\mathbf{K}_m(k) = \mathbf{P}_{xz_m}(k)\mathbf{P}_{zz_m}^{-1}(k)$ denotes the UKF gain for $\mathbf{s}_m$.

C. Statistical Linear Regression of Nonlinear Functions

The SLR is a method to linearize a nonlinear function using a set of deterministic sample points. Let $\mathbf{y} = \mathbf{g}(\mathbf{x})$ be an arbitrary nonlinear function of the system states. The SLR evaluates this function in r sample points $(\boldsymbol{\chi}^{(i)}, \boldsymbol{\mathcal{Y}}^{(i)})$, where $\boldsymbol{\mathcal{Y}}^{(i)} = \mathbf{g}(\boldsymbol{\chi}^{(i)})$ and $i = 1, ..., r$. The objective is to find the linear model $\mathbf{y} = \mathbf{A}\mathbf{x}+\mathbf{b}$ based on the following minimization problem:

$$\underset{\mathbf{A},\mathbf{b}}{\operatorname{argmin}} \sum_{i=1}^{r} (\boldsymbol{\mathcal{E}}^{(i)})^\top \boldsymbol{\mathcal{E}}^{(i)}, \tag{32}$$

where the error $\boldsymbol{\mathcal{E}}^{(i)}$ is calculated as

$$\boldsymbol{\mathcal{E}}^{(i)} = \boldsymbol{\mathcal{Y}}^{(i)} - (\mathbf{A}\boldsymbol{\chi}^{(i)} + \mathbf{b}). \tag{33}$$

The solution of the optimization problem (32) as developed in [19] is

$$\mathbf{A} = \mathbf{P}_{xy}^\top \mathbf{P}_{xx}^{-1}, \tag{34}$$

$$\mathbf{b} = \hat{\mathbf{y}} - \mathbf{A}\hat{\mathbf{x}}, \tag{35}$$

where $\hat{\mathbf{x}} = \frac{1}{r}\sum_{i=1}^{r} \boldsymbol{\chi}^{(i)}$, $\hat{\mathbf{y}} = \frac{1}{r}\sum_{i=1}^{r} \boldsymbol{\mathcal{Y}}^{(i)}$ and

$$\mathbf{P}_{xx} = \frac{1}{r}\sum_{i=1}^{p} (\boldsymbol{\chi}^{(i)} - \hat{\mathbf{x}})(.)^\top, \tag{36}$$

$$\mathbf{P}_{yy} = \frac{1}{r}\sum_{i=1}^{r} (\boldsymbol{\mathcal{Y}}^{(i)} - \hat{\mathbf{y}})(.)^\top, \tag{37}$$

$$\mathbf{P}_{xy} = \frac{1}{r}\sum_{i=1}^{r} (\boldsymbol{\chi}^{(i)} - \hat{\mathbf{x}})(\boldsymbol{\mathcal{Y}}^{(i)} - \hat{\mathbf{y}})^\top. \tag{38}$$

The statistical linearized model is therefore

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} + \boldsymbol{\mathcal{E}}, \tag{39}$$

where for any nonlinear function $\mathbf{y}$, the linearization error $\boldsymbol{\mathcal{E}}$ will be treated as an independent zero mean random variable with the covariance matrix given by

$$\mathbf{P}_{\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}} = \frac{1}{r}\sum_{i=1}^{r} \boldsymbol{\mathcal{E}}^{(i)}(\boldsymbol{\mathcal{E}}^{(i)})^\top = \mathbf{P}_{yy} - \mathbf{A}\mathbf{P}_{xx}\mathbf{A}^\top. \tag{40}$$

Note that the matrix $\mathbf{A}$ is not a Jacobian and the linearization error can be estimated in the process. The key point is that the SLR equations are equivalent to those of the UT using the sigma points generated by (21)-(22) as reported in [19], [20].

IV. Correlation Propagation in EKF-based MSS Architectures

Bar-shalom in [6] has proposed a method to propagate cross-covariance matrix in linear track-to-track fusion problems. Herein, following a similar procedure, we derive a recursion for EKF-based MSS architectures and discuss the conditions under which this recursion can be used effectively.

The posterior cross-covariance matrix between the two estimates at time k is

$$\mathbf{P}_{12(k|k)} = \mathbb{E}[\tilde{\mathbf{x}}_{1(k|k)}\tilde{\mathbf{x}}_{2}^\top{}_{(k|k)}], \tag{41}$$

where $\tilde{\mathbf{x}}_{m(k|k)},\ m \in \{1, 2\}$, denotes the posterior error of the m^{th} sensor and can be computed by

$$\begin{aligned}\tilde{\mathbf{x}}_{m(k|k)} &= \mathbf{x}_{(k)} - \hat{\mathbf{x}}_{m(k|k)} \\ &= \big(\mathbf{I} - \mathbf{K}_{m(k)}\mathbf{H}_{m(k)}\big)\tilde{\mathbf{x}}_{m(k|k-1)} - \mathbf{K}_{m(k)}\mathbf{v}_{m(k)}.\end{aligned} \tag{42}$$

Here, $\hat{\mathbf{x}}_{m(k|k)}$ is obtained from (19) and the prior estimation error

$$\begin{aligned}\tilde{\mathbf{x}}_{m(k|k-1)} &= \mathbf{x}_{(k)} - \hat{\mathbf{x}}_{m(k|k-1)} \\ &= \mathbf{F}_{m(k-1)}\tilde{\mathbf{x}}_{m(k-1|k-1)} + \mathbf{w}_{(k-1)},\end{aligned} \tag{43}$$

with the prior estimate $\hat{\mathbf{x}}_{m(k|k-1)}$ substituted from (17).

Theorem 1: In an EKF-based MSS architecture, let $\tilde{\mathbf{x}}_{m(k|k-1)}$ and $\tilde{\mathbf{x}}_{m(k|k)}$ be respectively the prior and posterior estimation errors for the m^{th} sensor given by (43) and (42). Assuming the independence of noise signals given in (5)-(8) , the consistent cross-covariance estimation recursion for the nonlinear system (1)-(4) is

$$\mathbf{P}_{12(k|k)} = \big(\mathbf{I}-\mathbf{K}_{1(k)}\mathbf{H}_{1(k)}\big)\mathbf{P}_{12(k|k-1)}\big(\mathbf{I}-\mathbf{K}_{2(k)}\mathbf{H}_{2(k)}\big)^\top, \tag{44}$$

$$\mathbf{P}_{12(k|k-1)} = \mathbf{F}_{1(k-1)}\mathbf{P}_{12(k-1|k-1)}\mathbf{F}_{2}^\top{}_{(k-1)} + \mathbf{Q}_{(k-1)}. \tag{45}$$

Proof: The posterior cross-covariance matrix is calculated as

$$\begin{aligned}\mathbf{P}_{12(k|k)} &= \mathbb{E}[\tilde{\mathbf{x}}_{1(k|k)}\tilde{\mathbf{x}}_{2}^\top{}_{(k|k)}] \\ &= (\mathbf{I} - \mathbf{K}_{1(k)}\mathbf{H}_{1(k)})\mathbb{E}[\tilde{\mathbf{x}}_{1(k|k-1)}\tilde{\mathbf{x}}_{2}^\top{}_{(k|k-1)}](\mathbf{I} - \mathbf{K}_{2(k)}\mathbf{H}_{2(k)})^\top \\ &- (\mathbf{I} - \mathbf{K}_{1(k)}\mathbf{H}_{1(k)})\mathbb{E}[\tilde{\mathbf{x}}_{1(k|k-1)}\mathbf{v}_{2}^\top{}_{(k)}]\mathbf{K}_{2}^\top{}_{(k)} \\ &- \mathbf{K}_{1(k)}\mathbb{E}[\mathbf{v}_{1(k)}\tilde{\mathbf{x}}_{2}^\top{}_{(k|k-1)}](\mathbf{I} - \mathbf{K}_{2(k)}\mathbf{H}_{2(k)})^\top \\ &+ \mathbf{K}_{1(k)}\mathbb{E}[\mathbf{v}_{1(k)}\mathbf{v}_{2}^\top{}_{(k)}]\mathbf{K}_{2}^\top{}_{(k)}\end{aligned} \tag{46}$$

Assuming independence of noise signals, $\mathbb{E}\big[\tilde{\mathbf{x}}_{1(k|k-1)}\mathbf{v}_{2}^\top{}_{(k)}\big] = 0$, $\mathbb{E}\big[\mathbf{v}_{1(k)}\mathbf{v}_{2}^\top{}_{(k)}\big] = 0$, and $\mathbb{E}\big[\mathbf{v}_{1(k)}\tilde{\mathbf{x}}_{2}^\top{}_{(k|k-1)}\big] = 0$, we obtain (44). The prior cross-covariance matrix is

$$\begin{aligned}\mathbf{P}_{12(k|k-1)} &= \mathbb{E}[\tilde{\mathbf{x}}_{1(k|k-1)}\tilde{\mathbf{x}}_{2}^\top{}_{(k|k-1)}] \\ &= \mathbf{F}_{1(k-1)}\mathbb{E}[\tilde{\mathbf{x}}_{1(k-1|k-1)}\tilde{\mathbf{x}}_{2}^\top{}_{(k-1|k-1)}]\mathbf{F}_{2}^\top{}_{(k-1)} \\ &+ \mathbf{F}_{1(k-1)}\mathbb{E}[\tilde{\mathbf{x}}_{1(k-1|k-1)}\mathbf{w}^\top{}_{(k-1)}] \\ &+ \mathbb{E}[\mathbf{w}_{(k-1)}\tilde{\mathbf{x}}_{2}^\top{}_{(k-1|k-1)}]\mathbf{F}_{2}^\top{}_{(k-1)} \\ &+ \mathbb{E}[\mathbf{w}_{(k-1)}\mathbf{w}^\top{}_{(k-1)}].\end{aligned} \tag{47}$$

By the assumptions, we have $\mathbb{E}[\tilde{\mathbf{x}}_{1(k-1|k-1)}\mathbf{w}^\top{}_{(k-1)}] = 0$ and $\mathbb{E}[\mathbf{w}_{(k-1)}\tilde{\mathbf{x}}_{2}^\top{}_{(k-1|k-1)}] = 0$, which yields (45). ■

It is noted that at the beginning of the estimation the two estimates are not correlated [6]. Thus, we can assume that the initial condition of (45) is zero, i.e. $\mathbf{P}_{12(0|0)} = 0$. In linear systems, a Gaussian distribution is linearly transformed to another Gaussian distribution. However, in nonlinear systems the transformed distributions may not remain Gaussian. When a system is mildly nonlinear, the transformed distributions are slightly distorted. As a result, mean and covariance can still be approximated under linear assumptions. The EKF, is an estimation technique that relies on this assumption. Obviously, the linearity assumption in EKF is not necessarily valid for all nonlinear systems, particularly those with high curvatures.

V. Correlation Propagation in UKF-Based MSS Architectures

To derive a recursion for estimating the cross-covariance matrix in UKF-based MSS architectures, we linearize the system defined in (1)-(4) for the m^{th} sensor, in the first step. The need for lineariztion is that the nonlinear terms cannot be computed recursively in the procedure of correlation propagation. The accuracy of the linearization technique used in this step must be consistent with that of the estimations provided by the UKF. The SLR technique is equivalent to the UT in the UKF, when using sigma points to calculate an estimate. Therefore, we use the SLR technique to linearize the process and measurement models in (1)-(4) based on (39):

$$\mathbf{x}(k) = \boldsymbol{\mathcal{F}}_m(k-1)\mathbf{x}(k-1) + \mathbf{b}^f_m(k-1) + \mathbf{w}(k-1) + \boldsymbol{\mathcal{E}}^f_m(k-1), \tag{48}$$

$$\mathbf{z}_m(k) = \boldsymbol{\mathcal{H}}_m(k)\mathbf{x}(k) + \mathbf{b}^h_m(k) + \mathbf{v}_m(k) + \boldsymbol{\mathcal{E}}^h_m(k), \tag{49}$$

The symbols are defined based on Table I that reports the correspondence between the linearized model described in section III-C and the system in (48)-(49). The linearization

TABLE I: The symbol correspondence in the linearization process

SLR Method	UKF Process Model	UKF Measurement Model
$\mathbf{x}$	$\mathbf{x}(k-1)$	$\mathbf{x}(k)$
$\mathbf{y}$	$\mathbf{x}(k)$	$\mathbf{z}_m(k)$
$\hat{\mathbf{x}}$	$\hat{\mathbf{x}}_m(k-1\|k-1)$	$\hat{\mathbf{x}}_m(k\|k-1)$
$\hat{\mathbf{y}}$	$\hat{\mathbf{x}}_m(k\|k-1)$	$\hat{\mathbf{z}}_m(k)$
$\mathbf{g}(.)$	$\mathbf{f}(.)$	$\mathbf{h}(.)$
$\mathbf{A}$	$\boldsymbol{\mathcal{F}}_m(k-1)$	$\boldsymbol{\mathcal{H}}_m(k)$
$\mathbf{b}$	$\mathbf{b}^f_m(k-1)$	$\mathbf{b}^h_m(k)$
$\boldsymbol{\chi}^{(i)}$	$\boldsymbol{\chi}^{(i)}_m(k-1\|k-1)$	$\boldsymbol{\chi}^{(i)}_m(k\|k-1)$
$\boldsymbol{\mathcal{Y}}^{(i)}$	$\boldsymbol{\chi}^{(i)}_m(k\|k-1)$	$\boldsymbol{\mathcal{Z}}^{(i)}_m(k\|k-1)$
$\boldsymbol{\mathcal{E}}$	$\boldsymbol{\mathcal{E}}^f_m(k-1)$	$\boldsymbol{\mathcal{E}}^h_m(k)$
r	$2n$	$2n$

errors for the functions $\mathbf{f}$ and $\mathbf{h}$ corresponding to the m^{th} sensor are denoted by $\boldsymbol{\mathcal{E}}^f_m(k)$ and $\boldsymbol{\mathcal{E}}^h_m(k)$, respectively. All linearization errors are zero mean random variables with the following covariance matrices

$$\mathbb{E}[\boldsymbol{\mathcal{E}}^\alpha_m(k)(\boldsymbol{\mathcal{E}}^\beta_{m'}(l))^\top] = \mathbf{P}^{\alpha\beta}_{\boldsymbol{\mathcal{E}}_{mm'}}(k)\delta(k-l),\ m,m' \in \{1,2\}, \tag{50}$$

Here, for $m,m' \in \{1,2\}$ and $\alpha,\beta \in \{f,h\}$ the covariance matrix

$$\mathbf{P}^{\alpha\beta}_{\boldsymbol{\mathcal{E}}_{mm'}}(k) = \frac{1}{2n}\sum_{i=1}^{2n}\boldsymbol{\mathcal{E}}^{\alpha\,(i)}_m(k)(\boldsymbol{\mathcal{E}}^{\beta\,(i)}_{m'}(k))^\top, \tag{51}$$

where the error terms $\boldsymbol{\mathcal{E}}^{\alpha\,(i)}_m(k)$ and $\boldsymbol{\mathcal{E}}^{\beta\,(i)}_{m'}(k)$ are obtained based on (33) with appropriate parameters and sigma points defined in Table I for process and measurement models. The assumption in (50) is valid due to the independence of the noise signals in different time steps described in (5)-(8). For the system (48)-(49), the posterior error of the m^{th} sensor is

$$\begin{aligned}\tilde{\mathbf{x}}_m(k|k) &= \mathbf{x}(k) - \hat{\mathbf{x}}_m(k|k)\\ &= \tilde{\mathbf{x}}_m(k|k-1) - \mathbf{K}_m(k)\big(\mathbf{z}_m(k) - \hat{\mathbf{z}}_m(k)\big),\end{aligned} \tag{52}$$

where $\tilde{\mathbf{x}}_m(k|k-1) = \mathbf{x}(k) - \hat{\mathbf{x}}_m(k|k-1)$ is the prior estimation error, $\mathbf{z}_m(k)$ is substituted from (49), and knowing that the signal $\boldsymbol{\mathcal{E}}^h_m(k)$ is zero mean $\hat{\mathbf{z}}_m(k)$ is calculated by

$$\hat{\mathbf{z}}_m(k) = \boldsymbol{\mathcal{H}}_m(k)\hat{\mathbf{x}}(k|k-1) + \mathbf{b}^h_m(k), \tag{53}$$

Therefore,

$$\begin{aligned}\tilde{\mathbf{x}}_m(k|k) = &\big(\mathbf{I}-\mathbf{K}_m(k)\boldsymbol{\mathcal{H}}_m(k)\big)\tilde{\mathbf{x}}_m(k|k-1)\\ &-\mathbf{K}_m(k)(\mathbf{v}_m(k)+\boldsymbol{\mathcal{E}}^h_m(k)).\end{aligned} \tag{54}$$

The prior estimation error $\tilde{\mathbf{x}}_m(k|k-1)$ is determined in the following. Under the assumption of $\mathbf{w}(k)$ and $\boldsymbol{\mathcal{E}}^f_m(k)$ being zero mean signals,

$$\hat{\mathbf{x}}_m(k|k-1) = \boldsymbol{\mathcal{F}}_m(k-1)\hat{\mathbf{x}}_m(k-1|k-1) + \mathbf{b}^f_m(k-1). \tag{55}$$

Substituting (48) and (55) in the definition of the prior estimation error,

$$\tilde{\mathbf{x}}_m(k|k-1) = \boldsymbol{\mathcal{F}}_m(k-1)\tilde{\mathbf{x}}_m(k-1|k-1) + \mathbf{w}(k-1) + \boldsymbol{\mathcal{E}}^f_m(k-1). \tag{56}$$

Theorem 2: In a UKF-based MSS architecture, let $\tilde{\mathbf{x}}_m(k|k-1)$ and $\tilde{\mathbf{x}}_m(k|k)$ be respectively the prior and posterior estimation errors for the m^{th} sensor given by (56) and (54). Assuming the independence of random signals given in (5)-(8) and considering the relationship defined in (50), the consistent cross-covariance estimation recursion for the nonlinear system (1)-(4) is

$$\begin{aligned}\mathbf{P}_{12}(k|k) = &(\mathbf{I}-\mathbf{K}_1(k)\boldsymbol{\mathcal{H}}_1(k))\mathbf{P}_{12}(k|k-1)(\mathbf{I}-\mathbf{K}_2(k)\boldsymbol{\mathcal{H}}_2(k))^\top\\ &+\mathbf{K}_1(k)\mathbf{P}^{hh}_{\boldsymbol{\mathcal{E}}_{12}}(k)\mathbf{K}_2^\top(k),\end{aligned} \tag{57}$$

$$\begin{aligned}\mathbf{P}_{12}(k|k-1) = &\boldsymbol{\mathcal{F}}_1(k-1)\mathbf{P}_{12}(k-1|k-1)\boldsymbol{\mathcal{F}}_2^\top(k-1) + \mathbf{Q}(k-1)\\ &-\boldsymbol{\mathcal{F}}_1(k-1)\mathbf{K}_1(k-1)\mathbf{P}^{hf}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)\\ &-\mathbf{P}^{fh}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)\mathbf{K}_2^\top(k-1)\boldsymbol{\mathcal{F}}_2^\top(k-1) + \mathbf{P}^{ff}_{\boldsymbol{\mathcal{E}}_{12}}(k-1),\end{aligned} \tag{58}$$

where $\mathbf{P}^{hh}_{\boldsymbol{\mathcal{E}}_{12}}(k)$, $\mathbf{P}^{fh}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$, $\mathbf{P}^{hf}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$ and $\mathbf{P}^{ff}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$ are determined in (51)

Proof: The posterior cross-covariance between the two estimates is calculated by

$$\begin{aligned}&\mathbf{P}_{12}(k|k) = \mathbb{E}[\tilde{\mathbf{x}}_1(k|k)\tilde{\mathbf{x}}_2^\top(k|k)]\\ &= (\mathbf{I}-\mathbf{K}_1(k)\boldsymbol{\mathcal{H}}_1(k))\mathbb{E}[\tilde{\mathbf{x}}_1(k|k-1)\tilde{\mathbf{x}}_2^\top(k|k-1)](\mathbf{I}-\mathbf{K}_2(k)\boldsymbol{\mathcal{H}}_2(k))^\top\\ &-(\mathbf{I}-\mathbf{K}_1(k)\boldsymbol{\mathcal{H}}_1(k))\mathbb{E}[\tilde{\mathbf{x}}_1(k|k-1)\tilde{\mathbf{v}}_2^\top(k)]\mathbf{K}_2^\top(k)\\ &-(\mathbf{I}-\mathbf{K}_1(k)\boldsymbol{\mathcal{H}}_1(k))\mathbb{E}[\tilde{\mathbf{x}}_1(k|k-1)(\boldsymbol{\mathcal{E}}^h_2(k))^\top]\mathbf{K}_2^\top(k)\\ &-\mathbf{K}_1(k)\mathbb{E}[\mathbf{v}_1(k)\tilde{\mathbf{x}}_2^\top(k|k-1)](\mathbf{I}-\mathbf{K}_2(k)\boldsymbol{\mathcal{H}}_2(k))^\top\\ &+\mathbf{K}_1(k)\mathbb{E}[\mathbf{v}_1(k)\mathbf{v}_2^\top(k)]\mathbf{K}_2^\top(k) + \mathbf{K}_1(k)\mathbb{E}[\mathbf{v}_1(k)(\boldsymbol{\mathcal{E}}^h_2(k))^\top]\mathbf{K}_2^\top(k)\\ &-\mathbf{K}_1(k)\mathbb{E}[\boldsymbol{\mathcal{E}}^h_1(k)\tilde{\mathbf{x}}_2^\top(k|k-1)](\mathbf{I}-\mathbf{K}_2(k)\boldsymbol{\mathcal{H}}_2(k))^\top\\ &+\mathbf{K}_1(k)\mathbb{E}[\boldsymbol{\mathcal{E}}^h_1(k)\mathbf{v}_2^\top(k)]\mathbf{K}_2^\top(k) + \mathbf{K}_1(k)\mathbb{E}[\boldsymbol{\mathcal{E}}^h_1(k)(\boldsymbol{\mathcal{E}}^h_2(k))^\top]\mathbf{K}_2^\top(k).\end{aligned}$$

By the assumptions of the theorem, i.e. (5)-(8), we have $\mathbb{E}[\tilde{\mathbf{x}}_1(k|k)\mathbf{v}_2^\top(k)] = 0$, $\mathbb{E}[\mathbf{v}_1(k)\mathbf{v}_2^\top(k)] = 0$, $\mathbb{E}[\boldsymbol{\mathcal{E}}^h_1(k)\mathbf{v}_2^\top(k)] = 0$, $\mathbb{E}[\mathbf{v}_1(k)\tilde{\mathbf{x}}_2^\top(k|k)] = 0$, $\mathbb{E}[\mathbf{v}_1(k)(\boldsymbol{\mathcal{E}}^h_2(k))^\top] = 0$, and due to (50) we conclude that $\mathbb{E}[\boldsymbol{\mathcal{E}}^h_1(k)\tilde{\mathbf{x}}_2^\top(k|k-1)] = 0$ and $\mathbb{E}[\tilde{\mathbf{x}}_1(k|k-1)(\boldsymbol{\mathcal{E}}^h_2(k))^\top] = 0$. Therefore, we obtain

$$\begin{aligned}\mathbf{P}_{12}(k|k) = &(\mathbf{I}-\mathbf{K}_1(k)\boldsymbol{\mathcal{H}}_1(k))\mathbf{P}_{12}(k|k-1)(\mathbf{I}-\mathbf{K}_2(k)\boldsymbol{\mathcal{H}}_2(k))^\top\\ &+\mathbf{K}_1(k)\mathbb{E}[\boldsymbol{\mathcal{E}}^h_1(k)(\boldsymbol{\mathcal{E}}^h_2(k))^\top]\mathbf{K}_2^\top(k),\end{aligned} \tag{59}$$

where $\mathbb{E}[\boldsymbol{\mathcal{E}}_1^h(k)(\boldsymbol{\mathcal{E}}_2^h(k))^\top] = \mathbf{P}^{hh}_{\boldsymbol{\mathcal{E}}_{12}}(k)$ and it is calculated in (51).

The prior cross-covariance matrix $\mathbf{P}_{12}(k|k-1)$ is calculated as

$$\begin{aligned}\mathbf{P}_{12}(k|k-1) &= \mathbb{E}[\tilde{\mathbf{x}}_1(k|k-1)\tilde{\mathbf{x}}_2^\top(k|k-1)]\\ &= \boldsymbol{\mathcal{F}}_1(k-1)\mathbb{E}[\tilde{\mathbf{x}}_1(k-1|k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-1)]\boldsymbol{\mathcal{F}}_2^\top(k-1)\\ &+ \boldsymbol{\mathcal{F}}_1(k-1)\mathbb{E}[\tilde{\mathbf{x}}_1(k-1|k-1)\mathbf{w}^\top(k-1)]\\ &+ \boldsymbol{\mathcal{F}}_1(k-1)\mathbb{E}[\tilde{\mathbf{x}}_1(k-1|k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top]\\ &+ \mathbb{E}[\mathbf{w}(k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-1)]\boldsymbol{\mathcal{F}}_2^\top(k-1)\\ &+ \mathbb{E}[\mathbf{w}(k-1)\mathbf{w}^\top(k-1)] + \mathbb{E}[\mathbf{w}(k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top]\\ &+ \mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-1)]\boldsymbol{\mathcal{F}}_2^\top(k-1)\\ &+\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\mathbf{w}(k-1)]+\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top],\end{aligned}$$

where it is assumed that $\mathbb{E}[\tilde{\mathbf{x}}_1(k-1|k-1)\mathbf{w}^\top(k-1)] = 0$, $\mathbb{E}[\mathbf{w}(k-1)\tilde{\mathbf{x}}_2^\top(k|k-1)] = 0$ $\mathbb{E}[\mathbf{w}(k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top] = 0$, and $\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1))\mathbf{w}^\top(k-1)] = 0$. The term $\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-1)]$ is

$$\begin{aligned}\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-1)] &= -\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\mathbf{v}_2^\top(k-1)]\mathbf{K}_2^\top(k-1)\\ &+ \mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-2)](\mathbf{I}-\mathbf{K}_m(k-1)\boldsymbol{\mathcal{H}}_m(k-1))^\top\\ &- \mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\boldsymbol{\mathcal{E}}_2^h(k-1)]\mathbf{K}_2^\top(k-1),\end{aligned} \tag{60}$$

where the first two terms are zero respectively due to the independence of the noise signals and the time lag between the random variables defined in (50). As the result, $\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)\tilde{\mathbf{x}}_2^\top(k-1|k-1)] = -\mathbf{P}^{fh}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)\mathbf{K}_2^\top(k-1)$. A similar reasoning would result in $\mathbb{E}[\tilde{\mathbf{x}}_1(k-1|k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top] = -\mathbf{K}_1(k-1)\mathbf{P}^{hf}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$, where the matrices $\mathbf{P}^{fh}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$ and $\mathbf{P}^{hf}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$ are calculated in (51). Thus,

$$\begin{aligned}\mathbf{P}_{12}(k|k-1) &= \boldsymbol{\mathcal{F}}_1(k-1)\mathbf{P}_{12}(k-1|k-1)\boldsymbol{\mathcal{F}}_2^\top(k-1) + \mathbf{Q}(k-1)\\ &- \boldsymbol{\mathcal{F}}_1(k-1)\mathbf{K}_1(k-1)\mathbf{P}^{hf}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)\\ &- \mathbf{P}^{fh}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)\mathbf{K}_2^\top(k-1)\boldsymbol{\mathcal{F}}_2^\top(k-1)\\ &+ \mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top],\end{aligned} \tag{61}$$

where $\mathbb{E}[\boldsymbol{\mathcal{E}}_1^f(k-1)(\boldsymbol{\mathcal{E}}_2^f(k-1))^\top] = \mathbf{P}^{ff}_{\boldsymbol{\mathcal{E}}_{12}}(k-1)$ is determined in (51).

■

In comparison to EKF-based recursions, the matrices $\boldsymbol{\mathcal{F}}_m(k-1)$ and $\boldsymbol{\mathcal{H}}_m(k)$ are calculated based on the sigma points with higher precision and the new terms $\mathbf{P}^{hh}_{\boldsymbol{\mathcal{E}}_{12}}(k)$, $\mathbf{P}^{ff}_{\boldsymbol{\mathcal{E}}_{12}}(k)$,$\mathbf{P}^{fh}_{\boldsymbol{\mathcal{E}}_{12}}(k)$, and $\mathbf{P}^{hf}_{\boldsymbol{\mathcal{E}}_{12}}(k)$ are added to the recursion. These terms are respectively resulted from the errors in the linearization of the measurement and the process models of the system. For a linear system, these terms are always zero, and in an EKF-based architecture the contribution of these terms is assumed negligible.

VI. Simulations

The performance of the proposed fusion rules are evaluated in a numerical study of tracking a robot moving in a circle [18]. The case study considers the problem of estimating position and orientation of the robot, i.e. $\mathbf{x}(k) = [x(k)\ y(k)\ \theta(k)]^\top$ (x and y are in meters and θ is in radians), that moves with constant linear velocity V and angular velocity Ω. Furthermore, two range sensors along with two angle sensors take noisy measurements from the process. The nonlinear system for the two sensors $m \in \{1, 2\}$ is described as:

$$\mathbf{x}(k+1) = \begin{bmatrix} x(k+1)\\ y(k+1)\\ \theta(k+1)\end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V\cos(\theta(k))\\ y(k) + \Delta t V\sin(\theta(k))\\ \theta(k) + \Delta t\Omega\end{bmatrix} + \mathbf{w}(k) \tag{62}$$

$$\mathbf{z}_m(k) = \begin{bmatrix} \sqrt{(x_m(k) - x_{c_m}) + (y_m(k) - y_{c_m})}\\ tan^{-1}(\frac{y_m(k)}{x_m(k)})\end{bmatrix} + \mathbf{v}_m(k), \tag{63}$$

where $\Delta t = 0.5$(s) is time increment, $V = 1\ (m/s)$, and $\Omega = 0.15\ (rad/s)$. The covariance of the process noise $\mathbf{w}(k)$ is $\mathbf{Q} = 0.001 \times \mathrm{Diag}(0.1, 0.1, 0.01)$. Furthermore, the covariance of noise sequences $\mathbf{v}_1(k)$ and $\mathbf{v}_2(k)$ are respectively considered to be $R_1 = \mathrm{Diag}(0.1, 0.001)$ and $R_2 = \mathrm{Diag}(0.1, 0.001)$. We also assumed that the two range measuring sensors are located at the origin, i.e., $(x_{c_m}, y_{c_m}) = 0$, $m \in \{1, 2\}$. The initial state vector for all filters is $\mathbf{x}(0) = [50\ 50\ 0]^\top$ with the initial covariance given by $\mathbf{P}(0) = 0.01 \times \mathrm{Diag}(0.1, 0.1, 0.01)$.

The Average Normalised Estimation Error Squared (ANEES) and Average Root Mean Square Error (ARMSE) are two measures to evaluate the performance of the estimators. The ANEES is a numerical test to measure the consistency of estimations as proposed in [27]. The ANEES at time step k for the total number of N runs is obtained according to

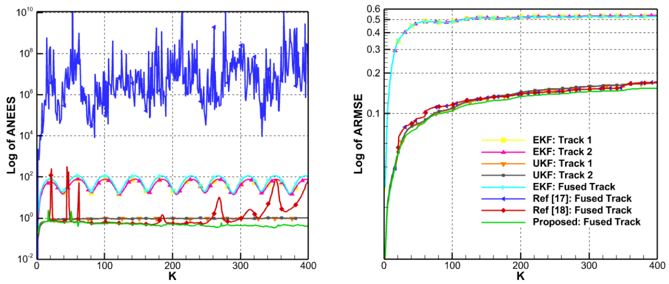
$$\mathrm{ANEES}(k) = \frac{1}{nN}\sum_{i=1}^{N}\epsilon_i(k), \tag{64}$$

where $\epsilon_i(k) = (\mathbf{x}_i(k) - \hat{\mathbf{x}}_i(k|k))^\top\mathbf{P}_i^{-1}(k|k)(\mathbf{x}_i(k) - \hat{\mathbf{x}}_i(k|k))$, and n denotes the dimension of state vector $\mathbf{x}_i(k)$. According to this test, an estimator is consistent if and only if the value of ANEES remains close to 1. If the ANEES is larger or smaller than 1, the estimator is evaluated as optimistic or pessimistic, respectively [27]. Moreover, we use ARMSE to compare the accumulated error of different estimators. The ARMSE resulted by an estimator at time step k for $i = 1, ..., N$ independent runs is calculated as follows (see [26]):

$$\boldsymbol{\pi}_{x^l}(k) = \frac{1}{N}\sum_{i=1}^{N}\sqrt{\frac{1}{k}\sum_{j=1}^{k}(x_i^l(j) - \hat{x}_i^l(j|j))^2}, \tag{65}$$

where $x_i^l(j)$, and $\hat{x}_i^l(j|j)$ are, respectively, the i^{th} true and posterior estimate of the l^{th} state $x^l(j)$ at time j.

The estimation task is conducted using an EKF-based and three UKF-based architectures in an MSS with two sensors. Fig. 2 shows the ANEES and ARMSE performance of the case study averaged over 1000 runs, for both individual tracks and fused tracks. It is observed that the performance of the proposed UKF-based fusion algorithm is significantly better than that of the EKF-based fusion rule and the methods proposed in [17] and [18], due to the following reasons: (i) the superiority of the UKF in handling nonlinearities in comparison to the EKF, and (ii) the novel design of a sub-optimal fusion rule based on the SLR method. As discussed, the SLR method



(a) (b)

Fig. 2: Estimation performanceof the moving robot, 1000 runs: (a) Log of ANEES and (b) Log of total ARMSE of the states

is accurate up to the third order of Taylor Series and it is compatible with UKF-based architectures. Fig. 2(a) demonstrates the ANEES performance of all methods. It is seen that the proposed architecture provides consistent tracks, however, the EKF-based architecture and the remaining UKF-based architectures generate optimistic results. Moreover, one can notice that the ANEES performance of EKF-based architecture is much better than the UKF-based architecture presented in [17]. In fact the proposed methods in [17] and [18] fail to generate consistent tracks due to the optimization scheme they use, i.e. trace optimization instead of full matrix optimization, and poor cross-covariance approximation, i.e. not including linearization error of SLR in [17] and rough estimation of the cross-covariance matrix based on sigma points in [18]. From Fig 2(b), it is concluded that the fused track in the proposed UKF-based MSS demonstrates the smallest average error among all tracks.

VII. Conclusion

The problem of track-to-track fusion using known correlation terms between tracks was considered in this letter. The focus was on nonlinear systems and development of a novel straightforward recursion for propagating cross-covariance matrix, introduced for EKF- and UKF-based architectures. For EKF-based MSS architectures, a natural extension of the fusion rule for linear systems was derived. For UKF-based MSS architectures, the developed formulation employed statistical linearization regression technique and the characteristics of deterministic sample points in unscented transform to propagate the cross-covariance matrix through time. In a numerical example, the superiority of the proposed fusion algorithm was demonstrated through comparing its consistency with two existing techniques, recently reported in the literature. Further, the average root mean square error was used to show the effectiveness of the developed fusion method, when fusing two tracks.

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