Finite-time Nonlinear $\mathcal{H}_\infty$ Control of Robot Manipulators with Prescribed Performance

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***Abstract*—** **his letter addresses the problem of robust finite-time tracking control with prescribed performance for robot manipulators experiencing uncertain inertia, external disturbance, and actuator fault. We develop a control strategy that incorporates the nonlinear $\mathcal{H}_\infty$ concept into the backstepping approach, using a novel virtual control, to guarantee practical finite-time convergence to a trajectory, whilst the closed-loop $\mathcal{L}_2$ gain is less than a pre-specified value. We also use adaptive gains, instead of complex error transformations (common in prescribed performance controllers), to simultaneously impose constraints on the steady-state and transient response of the closed-loop, including maximum error, maximum overshoot, and minimum convergence ratehis letter addresses the problem of robust finite-time tracking control with prescribed performance for robot manipulators experiencing uncertain inertia, external disturbance, and actuator fault. We develop a control strategy that incorporates the nonlinear $\mathcal{H}_\infty$ concept into the backstepping approach, using a novel virtual control, to guarantee practical finite-time convergence to a trajectory, whilst the closed-loop $\mathcal{L}_2$ gain is less than a pre-specified value. We also use adaptive gains, instead of complex error transformations (common in prescribed performance controllers), to simultaneously impose constraints on the steady-state and transient response of the closed-loop, including maximum error, maximum overshoot, and minimum convergence rateT. The developed controller is not contingent on solving the Hamilton-Jacobi or Riccati equations and is free of the singularities associated with using fractional power in finite-time control. The performance and efficacy of the proposed control framework are demonstrated through simulation studies and comparisons with pertinent works.**

***Index Terms*—** **Nonlinear $\mathcal{H}_\infty$; backstepping; finite-time stability; fault-tolerant control; prescribed performance**

I. Introduction

ROBOT manipulators have been rapidly evolving during the past seven decades. They have increasingly received attention from many industrial sectors to provide fast, reliable, and safe solutions to involving problems. Robotic systems are inherently nonlinear and often face uncertainties and disturbances during a mission, which inevitably leads to known challenges in their control design. To provide accurate tracking performance of robot manipulators under realistic conditions, numerous classical control strategies have been explored, including robust control, [1] adaptive control [2], passivity-based control [3], variable structure control [4], backstepping control [5], PID control [6], observer-based control [7], just to name a few. More recently, finite-time control strategies have been introduced that can result in a quick transient response, improved steady-state accuracy, as well as guaranteed finite-time convergence [8], [9].

The research on the finite-time trajectory tracking control of robot manipulators is often divided into two categories: (i) the geometric homogeneity-based method [10], and (ii) the Lyapunov-based approach [11]. The main drawback of the geometric homogeneity approach is the knowledge required about the exact robot dynamics. Due to the inherent system uncertainties, the second technique has attracted the attention of many researchers. For instance, [12] presents an accurate sliding mode control-based trajectory tracking scheme with finite-time convergent for robot manipulators that is capable of coping with external disturbances and system uncertainties. To achieve finite-time convergence of trajectories of robot manipulators, an adaptive control law using neural networks has been developed in [13] without having to measure the joints' accelerations. In [14], a new Proportional-Derivative (PD) control law with a feedforward compensation term has been proposed to ensure global trajectory-tracking in a finite period of time for uncertain robot manipulators. To alleviate the undesired chattering problem in this controller, its discontinuous terms have been approximated by continuous functions. However, when there exists system uncertainty or nonvanishing perturbation, practical stability can only be guaranteed. To deal with actuator saturation in robot manipulators, an output feedback control that contains a continuous PD-like control plus a feed-forward compensator has been presented in [15]. By employing a first-order nonlinear velocity filter, the need to measure the velocity for the PD part of the controller has been removed. Despite demonstrating superior robust performance, this control scheme can only provide finite-time trajectory tracking if the desired velocity and acceleration are not changing fast.

For uncertain nonlinear systems, designing robust $\mathcal{H}_\infty$ controllers using the energy dissipation notion and the $\mathcal{L}_2$ gain analysis is customary [16]. However, the challenge is to solve the resulting Hamilton-Jacobi equation. Based on output feedback and Riccati equations, the $\mathcal{H}_\infty$ control design for time-varying systems has been investigated in [17]. Applying the energy-shaped technique, an $\mathcal{H}_\infty$ control with finite-time convergence for nonlinear time-delay Hamiltonian systems has been developed in [18]. A nonlinear $\mathcal{H}_\infty$ control with finite-time convergence for uncertain robot manipulators has been proposed in [19]. The superiority of this method compared to its rivals is that it guarantees disturbance attenuation without any need for solving the Hamilton-Jacobi equality/inequality. A common drawback in the reviewed works is that they are not able to ensure certain performance in the transient and steady-state phases at the same time which is unacceptable in practice.

Prescribed Performance Control (PPC) is a widely used method to provide specific safety measures and performance specifications in the response of dynamical systems. This methodology has been widely employed not only in robot manipulators but also in spacecraft attitude controllers [20], surface vessels [21], active suspension systems [22], power systems [23], just to mention a few. By introducing a Prescribed Performance Function (PPF) to specify performance restrictions on tracking errors and a transformation of the constrained system to an unconstrained one, a neural network-based control with predetermined performance has been proposed [24]. The problem of fault-tolerant PPC for Euler-Lagrange systems subject to output constraints has been investigated in [25]. By virtue of the sliding mode control methodology, a disturbance observer-based control scheme for rigid robot manipulators has been designed [26], such that the transient and steady-state performance of the system is ensured. Despite their proven performance enhancement, the aforementioned control structures are quite complicated to design, since they contain partial derivative and intricate functions which are resulted from the

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stabilization of the transformed tracking error.

Through a backstepping approach, this letter develops a robust finite-time control strategy for robot manipulators that can further impose constraints on the transient and steady state response of the closed-loop system. The key contribution is the design of a novel virtual control that consists of:

- a term involving an adaptive time-varying gain to guarantee some predefined control performances. In contrast to the existing PPC strategies [20]–[27] which include intricate terms due to error transformations, our approach has a simple structure.
- a nowhere-singular term providing finite-time convergence of the tracking error to zero. Instead of filters or piecewise continuous functions, we use quadratic functions to avoid the singularities associated with the fractional powers appearing in the finite-time controls.

The resulting control strategy enjoys a straightforward design and stability analysis that facilitate its practical implementation. The robustness of the controller is investigated in the presence of actuator fault, uncertainty, and disturbance, based on the concept of finite-time nonlinear $\mathcal{H}_\infty$.

The rest of this letter is arranged as follows: in the next section, the dynamics of an uncertain n-link rigid robot manipulator is modelled and the control problem is explained. The principal results are given in Section III, where a novel constrained control is developed to achieve high accuracy in trajectory tracking control. Finally, some simulation results and concluding remarks are reported in Sections IV and V, respectively.

II. Preliminaries

A. Problem Statement

Consider an uncertain rigid n-Degree-Of-Freedom (DOF) serial-link robot manipulator system with single DOF joints whose dynamics subject to actuator fault is expressed as [28]:

$$\boldsymbol{H}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{d}, \tag{1}$$

in which $\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}} \in \mathbb{R}^n$ denote the joint angles, their velocity and acceleration, respectively, $\boldsymbol{H}(\boldsymbol{q}) = \boldsymbol{H}_0(\boldsymbol{q}) + \Delta\boldsymbol{H}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the uncertain symmetric positive-definite inertia matrix, $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}}) + \Delta\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^n$ denotes the uncertain matrix of Coriolis and centrifugal forces, and $\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{G}_0(\boldsymbol{q}) + \Delta\boldsymbol{G}(\boldsymbol{q}) \in \mathbb{R}^n$ is the uncertain vector of gravitational forces. The index 0 corresponds to the nominal model of the system and Δ indicates the bounded uncertain terms. Further, $\boldsymbol{d}(t) \in \mathbb{R}^n$ serves as the vector of external disturbance which is bounded but unknown. Moreover, $\boldsymbol{\tau}(t) = \boldsymbol{L}(t)\boldsymbol{\tau}_n(t) + \bar{\boldsymbol{\tau}}(t) \in \mathbb{R}^n$ is the vector of the applied control torque in which $\boldsymbol{\tau}_n(t) \in \mathbb{R}^n$ represents the control signal to be designed, $\boldsymbol{L}(t) = \text{diag}[L_1(t), \ldots, L_n(t)]^T \in \mathbb{R}^{n\times n}$ denotes a health index for each actuator and $\bar{\boldsymbol{\tau}}(t) \in \mathbb{R}^n$ is the additive actuator fault.

Let $\tilde{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}_d$ be the tracking error where $\boldsymbol{q}_d$ denotes the reference trajectory for the robot manipulator. Defining $\boldsymbol{x}_1 = \tilde{\boldsymbol{q}}$ and $\boldsymbol{x}_2 = \dot{\tilde{\boldsymbol{q}}}$, the error dynamics of the robot manipulator could be described as

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2, \\ \dot{\boldsymbol{x}}_2 &= -\boldsymbol{H}_0^{-1}(\boldsymbol{q})(\boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}_0(\boldsymbol{q})) + \boldsymbol{u} + \boldsymbol{w} - \ddot{\boldsymbol{q}}_d, \end{aligned} \tag{2}$$

where $\boldsymbol{u} = \boldsymbol{H}_0^{-1}\boldsymbol{\tau}_n$, $\boldsymbol{w} = -\boldsymbol{H}_0^{-1}(\Delta\boldsymbol{H}\ddot{\boldsymbol{q}} + \Delta\boldsymbol{C}\dot{\boldsymbol{q}} + \Delta\boldsymbol{G} - (\boldsymbol{L} - \boldsymbol{I}_n)\boldsymbol{\tau}_n - \bar{\boldsymbol{\tau}} - \boldsymbol{d})$ and $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

Problem 1: The principal control objective is to develop a nonsingular finite-time nonlinear H_∞-based control framework for a robot manipulator with error dynamics (2) to track a time-varying reference trajectory $\boldsymbol{q}_d(t)$, such that

1) The closed-loop robot manipulator system is finite-time stable in spite of actuator fault, system uncertainty and external disturbance.
2) The closed-loop robot manipulator system has an $\mathcal{L}_2$ gain not greater than γ.
3) The prescribed performance for the error trajectory is satisfied and the proposed constrained controller possesses a simple structure.

B. Finite-time Nonlinear $\mathcal{H}_\infty$ Control

Definition 1: Consider an uncertain nonlinear system as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{w}, \\ \boldsymbol{z} &= \boldsymbol{h}(\boldsymbol{x}), \end{aligned} \tag{3}$$

in which $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, $\boldsymbol{w} \in \mathbb{R}^r$ and $\boldsymbol{z} \in \mathbb{R}^r$ represent the system state, the control input, the external disturbance and a performance vector, respectively. A dynamic controller in the form of

$$\boldsymbol{u} = \boldsymbol{\psi}(\boldsymbol{x}, t), \tag{4}$$

is called a finite-time $\mathcal{H}_\infty$ control provided that the conditions below hold [19]:

1) The robot manipulator system in (3) and (4) is finite-time stable for the case $\boldsymbol{w} = \boldsymbol{0}$.
2) Given a positive scalar γ, the $\mathcal{L}_2$ gain is not greater than γ provided that the performance vector $\boldsymbol{z}$ satisfies

$$\int_{t_0}^{t_1} \|\boldsymbol{z}\|^2 dt \le \gamma^2 \int_{t_0}^{t_1} \|\boldsymbol{w}\|^2 dt, \tag{5}$$

for $t_0 < t_1$ and all nonlinear disturbances $\boldsymbol{w}(t) \in \mathscr{W}$ which belongs to the set $\mathscr{W} \subset \mathcal{L}_2[t_0, t_1]$.

Lemma 1: For the uncertain nonlinear system (3), assume that there exist $c > 0$, $0 < \alpha < 1$ and a positive-definite Lyapunov function $V(x)$ defined in a neighborhood $\hat{U} \subset \mathbb{R}^n$ of the origin so that [19]

- $V(x) > 0$ in $\hat{U}$.
- $\dot{V}(x) + cV^\alpha(x) \le \frac{1}{2}\left(\gamma^2\|\boldsymbol{w}\|^2 - \|\boldsymbol{z}\|^2\right), \forall x \in \hat{U}\backslash\{0\}$.

The uncertain nonlinear system (3) is finite-time stable and its $\mathcal{L}_2$ gain is less than or equal to γ.

Remark 1: The inequality (5) indicates that disturbances and uncertainties with bounded energy, i.e. signals which are square-integrable on $[t_0, t_1]$, can be handled by the nonlinear $\mathcal{H}_\infty$ control approach. It is, therefore, possible to consider actuator faults as long as they have bounded energy [29]. □

Lemma 2: For all $a_i > 0, (i = 1, \ldots, n)$ and $0 < c < 1$, the following inequality holds [28]

$$\left(\sum_{i=1}^{n} a_i\right)^c \le \sum_{i=1}^{n} a_i^c. \tag{6}$$

Lemma 3: For all $b \in \mathbb{R}$ and $\beta \ge 0$, the following inequality holds [30]

$$0 \le |b| \le \beta + \frac{b^2}{\sqrt{b^2 + \beta^2}}. \tag{7}$$

Lemma 4: For $\varrho_1, \varrho_2 \in \mathbb{R}^n$, we have:

$$\varrho_1^T(\varrho_1 \circ \varrho_2 \circ \varrho_2) = (\varrho_1 \circ \varrho_2)^T(\varrho_1 \circ \varrho_2), \tag{8}$$

where $\circ$ denotes the Hadamard product that is the element-wise product of vectors ϱ_1 and ϱ_2 [31].

C. Prescribed Performance Control

The associated prescribed performance of the tracking error $x_{1i}(t)$ in transient (convergence rate and overshoot) and steady state (ultimate tracking error) is accomplished provided that the following condition holds [32]

$$-\rho_i(t) < x_{1i}(t) < \rho_i(t), \tag{9}$$

in which $\rho_i(t)$ is a Finite-Time Prescribed Performance Function (FTPPF). In general, an FTPPF is a positive and non-increasing function such that $\lim_{t\to T_s}\rho_i(t) = \rho_{iT} > 0$ and $\rho_i(t) = \rho_{iT}$ for any $t \geq T_s$, where ρ_{iT} and T_s are arbitrarily small positive scalar and convergence time, respectively [33]. In our development, we employ an FTPPF in the following form:

$$\rho_i(t)=\begin{cases}\left(\rho_{i0}-\rho_{iT}(\frac{T_s+t}{T_s})\right)\exp\left(\frac{\kappa_i t}{t-T_s}\right)+\rho_{iT}, & 0\leq t<T_s\\ \rho_{iT}, & t\geq T_s\end{cases} \tag{10}$$

in which ρ_{iT}, ρ_{i0} and κ_i are positive constants. Based on [32], the predefined performance for the tracking error is acquired if it is kept in the predefined region (9). Designing a control scheme that takes into account the performance constraints (9) is a non-trivial task. To deal with this issue, conventionally an error transformation is conducted in the form $x_{1i}(t) = \rho_i(t)S_i(\varepsilon_i(t))$, such that

1) $S_i(\cdot)$ is a smooth and strictly increasing function,
2) $-1 < S_i(\varepsilon_i) < 1$,
3) $\lim_{\varepsilon_i\to\pm\infty} S_i(\varepsilon_i) = \pm 1$.

The new variable $\varepsilon_i(t)$ is called the transformed error. It is obvious that due to the defined error transformation, the second property of $S_i(\varepsilon_i)$, and since $\rho_i(t) > 0$, we have

$$-\rho_i(t) < \underbrace{\rho_i(t)S_i(\varepsilon_i)}_{=x_{1i}(t)} < \rho_i(t).$$

The inverse transformation is then expressed as

$$\varepsilon_i(t) = S_i^{-1}\left(\frac{x_{1i}(t)}{\rho_i(t)}\right), \tag{11}$$

which is well-defined if (9) is satisfied for $\varepsilon(t) \in \mathcal{L}_\infty$. The objective would be to maintain $\varepsilon(t)$ bounded $\forall t \geq 0$ in order to meet the constraint (9). For this purpose, the dynamics of the transformed error $\varepsilon(t)$ must be described in terms of the tracking error $x_1(t)$. The resulting controller contains complex terms involving partial derivatives of the inverse of $S^{-1}(\cdot)$ in (11) [32].

Remark 2: To alleviate the explained complexity of the conventional PPC, we present a novel straightforward methodology that uses non-dynamic adaptive gains instead of an error transformation to provide prescribed performance. □

III. Main Results

To accomplish the control objectives mentioned in Problem 1, a nonsingular finite-time nonlinear $\mathcal{H}_\infty$ control with prescribed performance for the tracking error $\boldsymbol{x}_1$ is developed in the backstepping framework. To start, we define the time-dependent matrices $\boldsymbol{\sigma} = \mathrm{diag}_{i=1}^n(\sigma_i)$ and $\boldsymbol{\eta} = \mathrm{diag}_{i=1}^n(\eta_i)$, such that $\sigma_i(t) := \frac{1}{\rho_i(t)-|x_{1i}(t)|}$ and $\eta_i(t) := \int_0^t \sigma_i(s)ds$. Note that $\rho_i(t)$ must depend on the initial condition $x_1(0)$; otherwise, $\eta_i(t)$ may not be positive definite. We define the following nonsingular constrained virtual control

$$\boldsymbol{\varphi} := -(\boldsymbol{\eta}+k_0\boldsymbol{I}_n)\boldsymbol{x}_1 - \frac{(\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}}\circ\bar{\boldsymbol{\varphi}})}{\sqrt{(\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}})^T(\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}})+\beta^2}}, \tag{12}$$

where $\beta > 0$ is an arbitrary small constant and

$$\bar{\boldsymbol{\varphi}} = k_1\mathrm{sig}^\alpha(\boldsymbol{x}_1). \tag{13}$$

Here, $k_0 > \frac{1}{2} + \frac{w_1^2}{2}$, $k_1 > 0$, and $w_1 > 0$. Moreover, $\mathrm{sig}^\alpha(\boldsymbol{x}_1) := [|x_{11}|^\alpha\mathrm{sgn}(x_{11}), \ \cdots, \ |x_{1n}|^\alpha\mathrm{sgn}(x_{1n})]^T$, in which $0 < \alpha < 1$ and $\mathrm{sgn}(\cdot)$ is the sign function.

Defining the error variable $\boldsymbol{\xi} := \boldsymbol{x}_2 - \boldsymbol{\varphi}$, the robot manipulator dynamics (2) can be expressed as

$$\begin{cases}\dot{\boldsymbol{x}}_1=\boldsymbol{\xi}+\boldsymbol{\varphi}\\ \dot{\boldsymbol{\xi}}=-\boldsymbol{H}_0^{-1}(\boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}+\boldsymbol{G}_0(\boldsymbol{q}))+\boldsymbol{u}+\boldsymbol{w}-\ddot{\boldsymbol{q}}_d-\dot{\boldsymbol{\varphi}}.\end{cases} \tag{14}$$

Remark 3: Note that the conventional backstepping control may only involve $\bar{\boldsymbol{\varphi}}$ as the second term of the virtual control $\boldsymbol{\varphi}$. In this case, the time derivative of the virtual control contains the terms $|x_{1i}|^{\alpha-1}x_{2i}$. Then, singularity happens when $x_{1i} = 0$ and $x_{2i} \neq 0$, since the fractional power $\alpha - 1$ is negative. While in the time derivative of the virtual control (12), the terms $|x_{1i}|^{\alpha-1}x_{2i}$ existed in $\dot{\bar{\varphi}}_i$ are always multiplied by x_{1i}, and the singularities due to negative fractional powers are avoided.

Theorem 1: For any given $\gamma > 0$, the following control law

$$\begin{aligned}\boldsymbol{u} =&\boldsymbol{H}_0^{-1}(\boldsymbol{q})(\boldsymbol{C}_0(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - k_2\mathrm{sig}^\alpha(\boldsymbol{\xi}) + \ddot{\boldsymbol{q}}_d + \dot{\boldsymbol{\varphi}}\\ &-\left(\frac{1}{2\gamma^2}+\frac{w_2^2}{2}+\frac{1}{2}\right)\boldsymbol{\xi}\end{aligned} \tag{15}$$

guarantees that the robot manipulator (14) is practically finite-time stable and its $\mathcal{L}_2$ gain is not greater than the predetermined value γ, where $w_2 > 0$ and $k_2 > 0$.

Proof: Substituting control law (15) into (14), one has

$$\begin{cases}\dot{\boldsymbol{x}}_1 = \boldsymbol{\xi}+\boldsymbol{\varphi}(\boldsymbol{x}_1)\\ \dot{\boldsymbol{\xi}} = -k_2\mathrm{sig}^\alpha(\boldsymbol{\xi})+\boldsymbol{w}-\left(\frac{1}{2\gamma^2}+\frac{w_2^2}{2}+\frac{1}{2}\right)\boldsymbol{\xi}.\end{cases} \tag{16}$$

Let us construct a Lyapunov function as

$$V(\boldsymbol{x}_1,\boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{x}_1^T\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi}. \tag{17}$$

Based on Lemmas 3 and 4, the following inequality for the time derivative of $V(\boldsymbol{x}_1,\boldsymbol{\xi})$ is obtained

$$\begin{aligned}\dot{V}(\boldsymbol{x}_1,\boldsymbol{\xi}) \leq& -\boldsymbol{x}_1^T(\boldsymbol{\eta}+k_0\boldsymbol{I}_n)\boldsymbol{x}_1 - k_1\|\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}}\|_2 + \beta + \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi}\\ &+\frac{1}{2}\boldsymbol{x}_1^T\boldsymbol{x}_1 - k_2\boldsymbol{\xi}^T\mathrm{sig}^\alpha(\boldsymbol{\xi}) + \boldsymbol{\xi}^T\boldsymbol{w}\\ &-\boldsymbol{\xi}^T\left(\frac{1}{2\gamma^2}+\frac{w_2^2}{2}+\frac{1}{2}\right)\boldsymbol{\xi}.\end{aligned} \tag{18}$$

According to Cauchy-Schwarz, one has the following inequality involving the 1-norm and 2-norm:

$$\|\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}}\|_1 \leq \sqrt{n}\ \|\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}}\|, \tag{19}$$

Then, (18) can be rewritten as

$$\begin{aligned}\dot{V}(\boldsymbol{x}_1,\boldsymbol{\xi}) \leq& -\boldsymbol{x}_1^T\boldsymbol{\eta}\boldsymbol{x}_1 - \left(k_0-\frac{1}{2}\right)\boldsymbol{x}_1^T\boldsymbol{x}_1 - \frac{k_1}{\sqrt{n}}\|\boldsymbol{x}_1\circ\bar{\boldsymbol{\varphi}}\|_1\\ &+\beta+\frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi} - k_2\boldsymbol{\xi}^T\mathrm{sig}^\alpha(\boldsymbol{\xi})+\boldsymbol{\xi}^T\boldsymbol{w}\\ &-\boldsymbol{\xi}^T\left(\frac{1}{2\gamma^2}+\frac{w_2^2}{2}+\frac{1}{2}\right)\boldsymbol{\xi}.\end{aligned} \tag{20}$$

To illustrate the $\mathcal{L}_2$ gain of the closed-loop robot manipulator system is not greater than γ, let us define

$$H := \dot{V}(\boldsymbol{x}_1,\boldsymbol{\xi}) + \frac{1}{2}\left(\|\boldsymbol{z}\|^2 - \gamma^2\|\boldsymbol{w}\|^2\right). \tag{21}$$

where $\boldsymbol{z} = \left[w_1\boldsymbol{x}_1^T, w_2\boldsymbol{\xi}^T\right]^T$ is a performance vector and w_1 and w_2 are positive weight parameters. Substituting (20) into (21), one has

$$H \le -\boldsymbol{x}_1^T\boldsymbol{\eta}\boldsymbol{x}_1 - \left(k_0 - \frac{1}{2}\right)\boldsymbol{x}_1^T\boldsymbol{x}_1 - \frac{k_1}{\sqrt{n}}\|\boldsymbol{x}_1 \circ \bar{\boldsymbol{\varphi}}\|_1 + \beta + \frac{1}{2}\boldsymbol{\xi}^T\boldsymbol{\xi} - k_2\boldsymbol{\xi}^T\mathrm{sig}^\alpha(\boldsymbol{\xi}) + \boldsymbol{\xi}^T\boldsymbol{w} - \boldsymbol{\xi}^T\left(\frac{1}{2\gamma^2} + \frac{w_2^2}{2} + \frac{1}{2}\right)\boldsymbol{\xi} + \frac{1}{2}\left(\|\boldsymbol{z}\|^2 - \gamma^2\|\boldsymbol{w}\|^2\right)$$

$$\le -\left(k_0 - \frac{1}{2}\right)\boldsymbol{x}_1^T\boldsymbol{x}_1 - \frac{k_1}{\sqrt{n}}\|\boldsymbol{x}_1 \circ \bar{\boldsymbol{\varphi}}\|_1 + \beta - k_2\boldsymbol{\xi}^T\mathrm{sig}^\alpha(\boldsymbol{\xi}) + \boldsymbol{\xi}^T\boldsymbol{w} - \frac{1}{2\gamma^2}\|\boldsymbol{\xi}\|^2 + \frac{w_1^2}{2}\|\boldsymbol{x}_1\|^2 - \frac{\gamma^2}{2}\|\boldsymbol{w}\|^2$$

$$\le -\boldsymbol{x}_1^T\left(k_0 - \frac{1}{2}\right)\boldsymbol{x}_1 - \frac{k_1}{\sqrt{n}}\|\boldsymbol{x}_1 \circ \bar{\boldsymbol{\varphi}}\|_1 + \beta - k_2\boldsymbol{\xi}^T\mathrm{sig}^\alpha(\boldsymbol{\xi}) + \frac{w_1^2}{2}\|\boldsymbol{x}_1\|^2 - \left(\frac{1}{\sqrt{2}\gamma}\boldsymbol{\xi} - \frac{\gamma}{\sqrt{2}}\boldsymbol{w}\right)^T\left(\frac{1}{\sqrt{2}\gamma}\boldsymbol{\xi} - \frac{\gamma}{\sqrt{2}}\boldsymbol{w}\right)$$

$$\le -\bar{k}_1\left(\frac{1}{2}\sum_{i=1}^{n}|x_{1i}|^2\right)^{\frac{\alpha+1}{2}} - \bar{k}_2\left(\frac{1}{2}\sum_{i=1}^{n}|\xi_i|^2\right)^{\frac{\alpha+1}{2}} + \beta$$

$$\le -\tilde{k}V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu + \beta, \tag{22}$$

where $\bar{k}_1 = \frac{2^\mu k_1}{\sqrt{n}}$, $\bar{k}_2 = 2^\mu k_2$, $\mu = \frac{\alpha+1}{2}$, and $\tilde{k} = \min\{\bar{k}_1, \bar{k}_2\}$. Then, we have

$$\dot{V}(\boldsymbol{x}_1, \boldsymbol{\xi}) + \tilde{k}V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu - \beta \le \frac{1}{2}\left(\gamma^2\|\boldsymbol{w}\|^2 - \|\boldsymbol{z}\|^2\right). \tag{23}$$

For any scalar $0 < \varpi < 1$, (23) can be rewritten as

$$\dot{V}(\boldsymbol{x}_1, \boldsymbol{\xi}) + \varpi\tilde{k}V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu + (1-\varpi)\tilde{k}V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu - \beta \le \frac{1}{2}\left(\gamma^2\|\boldsymbol{w}\|^2 - \|\boldsymbol{z}\|^2\right). \tag{24}$$

It is observed that if $V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu > \frac{\beta}{(1-\varpi)\tilde{k}}$, then (24) is rewritten as

$$\dot{V}(\boldsymbol{x}_1, \boldsymbol{\xi}) + \varpi\tilde{k}V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu \le \frac{1}{2}\left(\gamma^2\|\boldsymbol{w}\|^2 - \|\boldsymbol{z}\|^2\right). \tag{25}$$

Based on Lemma 1, the trajectories are driven to $V(\boldsymbol{x}_1, \boldsymbol{\xi})^\mu \le \frac{\beta}{(1-\varpi)\tilde{k}}$ in finite-time and the $\mathcal{L}_2$ gain is not greater than γ. This ends the proof. ■

Remark 4: Unlike the complicated constrained controls in the literature [2], [20]–[24], the proposed control framework employs a novel time-varying gain in the virtual control to constrain the tracking errors. The prescribed performance for the tracking error $x_{1i}(t)$ is achieved by incorporating the adaptive gain $\sigma_i(t) = \frac{1}{\rho_i(t)-|x_{1i}(t)|}$ in the virtual control $\varphi_i(x_{1i})$. The performance specifications are determined by the performance function $\rho_i(t)$. When the error trajectory approaches the boundary of the constraint region, i.e. $x_{1i} \to \rho_i$, the adaptive gain σ_i increases resulting in an increase in the virtual control φ_i. This, in turn, prevents the error trajectory from contacting the boundary and violating the constraint. The developed constrained control approach removes the need for a transformation error, simplifies the design procedure, and gives rise to a simple-structure controller. □

IV. Simulation Results

To evaluate effectiveness of the new control law, it is employed for trajectory tracking of a two-link robot manipulator whose dynamics is described by [28]:

$$\begin{bmatrix} h_{11}(q_2) & h_{12}(q_2) \\ h_{12}(q_2) & h_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -c_{12}(q_2)\dot{q}_1^2 - 2c_{12}(q_2)\dot{q}_1\dot{q}_2 \\ c_{12}(q_2)\dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} g_1(q_1, q_2)g \\ g_2(q_1, q_2)g \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \tag{26}$$

TABLE I

The Actual and Nominal Values of the Parameters

Parameter	r_1	r_2	J_1	J_2	m_1	m_2	$\hat{m}_1$	$\hat{m}_2$
Value	1 m	0.8 m	5 kgm	5 kgm	0.5 kg	1.5 kg	0.4 kg	1.2 kg

where

$$h_{11}(q_2) = (m_1 + m_2)r_1^2 + m_2r_2^2 + 2m_2r_1r_2\cos(q_2) + J_1$$
$$h_{12}(q_2) = m_2r_2^2 + m_2r_1r_2\cos(q_2), h_{22} = m_2r_2^2 + J_2$$
$$c_{12}(q_2) = m_2r_1r_2\sin(q_2), g_2(q_1, q_2) = m_2r_2\cos(q_1 + q_2)$$
$$g_1(q_1, q_2) = (m_1 + m_2)r_1\cos(q_2) + m_2r_2\cos(q_1 + q_2).$$

The actual and nominal values of the parameters are given in Table I. The initial conditions of the system states are taken as: $q_1(0) = 1$, $q_2(0) = 0.5$, $\dot{q}_1(0) = 0.5$ and $\dot{q}_2(0) = 1$. The reference signals to be tracked are $q_{d_1} = \sin(1.5t)$ and $q_{d_2} = 1.5\cos(t)$. The parameters of the control law in (15) and the FTPPF (10) are selected as: $\alpha = 0.95$, $w_1 = w_2 = 1$, $\gamma = 0.8$, $\beta = 0.001$, $k_0 = 3$, $k_1 = k_2 = 0.7$, $\rho_0 = 1.5$, $\rho_T = 0.001$, $\kappa = 2$ and $T_f = 1$. It is assumed that the external disturbance $d(t) = [2\sin(t), \cos(2t)]^T + 0.5\sin(100\pi t)[1, 1]^T$ acts on the system and the actuators experience the partial loss of effectiveness and additive faults as given below

$$L_i = \begin{cases} 1, & t < 0.8s \\ 0.6 + 0.2\sin(0.1it), & t \ge 0.8s \end{cases}, \tau_{u_i} = \begin{cases} 0, & t < 1s \\ i, & t \ge 1s \end{cases}.$$

To assess the efficacy of the proposed constrained finite-time nonlinear $\mathcal{H}_\infty$ controller in (15), the finite-time $\mathcal{H}_\infty$ control (FTHC) [28] and the arctan nonsingular terminal sliding mode (ATNTSM) control [34] are simulated under the identical situation. As it is observed in Figs. 1 and 2, the joints 1 and 2 successfully track the reference trajectories within a finite time; however, the proposed controller leads to a faster convergence rate. The joint position tracking errors along with their partial zooms are depicted in Figs. 3–6. As expected, the prescribed performance for the tracking error is provided by the proposed controller by satisfying the constraint $-\rho_i(t) < x_{1i}(t) < \rho_i(t)$. Therefore, the desired convergence time and steady-state tracking error can be specified a priori. The behavior of the control inputs are depicted in Figs. 7 and 8. Although the novel control leads to better convergence behavior, it does not require large control effort. Moreover, the norm of the performance vector on logarithmic scale has been illustrated in Fig. 9. It is obvious the less the parameter γ, the less $\|\boldsymbol{z}\|$ and the better performance. However, it should be noted that since γ appears in the denominator of the control input Eq. (15), a small value of γ gives rise to a higher control effort. The ratio of energy of the performance vector to the disturbance is shown in Fig. 10. As expected the effect of the lumped disturbance on the performance vector is attenuated by the level γ. From the simulation results, it is concluded that the proposed control framework possesses substantial superiority over the FTHC and ATNTSM in terms of tracking accuracy, convergence rate and robustness.

V. Conclusion

This letter deals with the difficult issue of constrained finite-time nonlinear $\mathcal{H}_\infty$ control for uncertain robot manipulators in the presence of the actuators fault, system uncertainty and external disturbance. It was analytically proved that the closed-loop robot manipulator system is finite-time stable and the settling time is regardless of initial conditions. Unlike the existing complicated

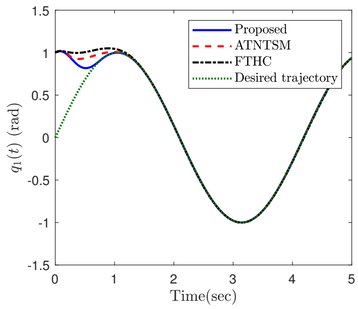

Fig. 1. Trajectories of q_1 and q_{1d}

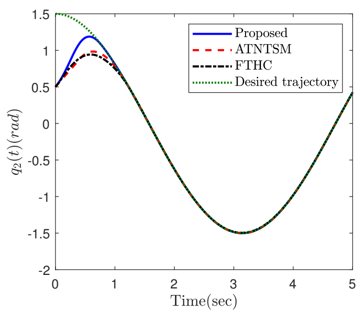

Fig. 2. Trajectories of q_2 and q_{2d}

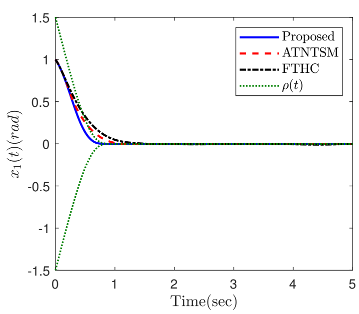

Fig. 3. Trajectory of tracking error x_1

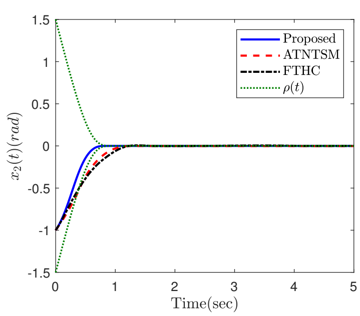

Fig. 4. Trajectory of tracking error x_2

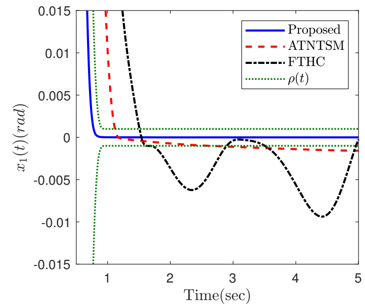

Fig. 5. Partial zoom of trajectory of tracking error x_1

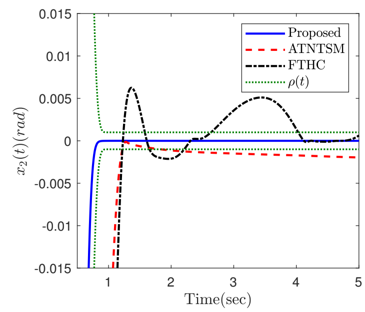

Fig. 6. Partial zoom of trajectory of tracking error x_2

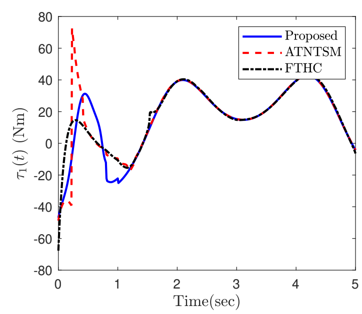

Fig. 7. Trajectory of control input τ_1

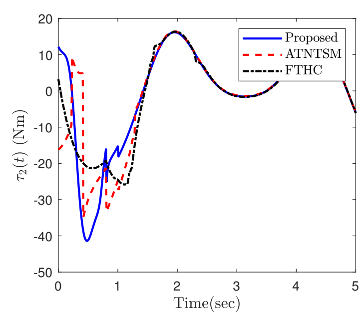

Fig. 8. Trajectory of control input τ_2

constrained controls in the literate, the proposed control approach achieves the desirable performance through adding a simple time-varying gain. Besides, to guarantee that the $\mathcal{L}_2$ gain of the robot manipulator system remains less than or equal to γ, a combination of backstepping control and nonlinear $\mathcal{H}_\infty$ was used. The simulation results illustrated the strong performance of the proposed control strategy.

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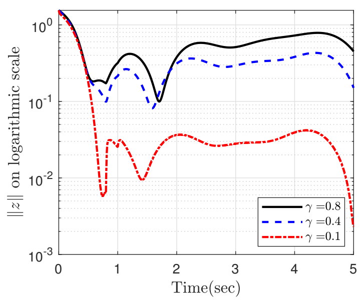

Fig. 9. $||z||$ on logarithmic scale

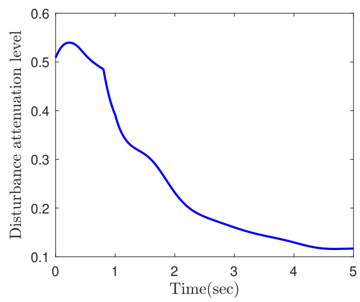

Fig. 10. Disturbance attenuation level

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