A Generalized Exponential Formula for Forward and Differential Kinematics of Open-chain Multi-body Systems

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Abstract

This paper presents a generalized exponential formula for Forward and Differential Kinematics of open-chain multi-body systems with multi-degree-of-freedom, holonomic and nonholonomic joints. The notion of lower kinematic pair is revisited, and it is shown that the relative configuration manifolds of such joints are indeed Lie groups. Displacement subgroups, which correspond to different types of joints, are categorized accordingly, and it is proven that except for one class of displacement subgroups the exponential map is surjective. *Screw joint parameters* are defined to parameterize the relative configuration manifolds of displacement subgroups using the exponential map of Lie groups. For nonholonomic constraints the admissible screw joint speeds are introduced, and the Jacobian of the open-chain multi-body system is modified accordingly. Computational aspects of the developed formulation for Forward and Differential Kinematics of open-chain multi-body systems are explored by assigning coordinate frames to the initial configuration of the multi-body system, employing the matrix representation of $SE(3)$ and choosing a basis for $se(3)$. Finally, an example of a mobile manipulator mounted on a spacecraft, i.e., a six-degree-of-freedom moving base, elaborates the computational aspects.

Keywords: Exponential map, Lie groups, Open-chain multi-body system, Displacement subgroup, Holonomic/nonholonomic joint

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Operators.

L_r	Left composition/translation by r
R_r	Right composition/translation by r
K_r	Conjugation by r
Ad_r	Adjoint operator corresponding to r
ad_ξ	adjoint operator corresponding to ξ
$[\xi, \eta]$	Lie bracket or matrix commutator
$d_r f$	Differential of the map f at the point r
$T_r M$	Tangent space of the manifold M at the point r
TM	Tangent bundle of the manifold M
$\exp(\xi)$	Group/matrix exponential of ξ
$Lie(G)$	Lie algebra of the Lie group G
$diag(A_1, ..., A_n)$	Block diagonal matrix of the entries
$\ltimes$	Semi-direct product of groups
$\\|v\\|$	Euclidean norm of the vector v
$\tilde{v}$	Skew-symmetric matrix corresponding to the vector v
$R(\theta)$	2×2 rotation matrix for the angle θ
$R(\theta, v)$	3×3 rotation matrix of a rotation for the angle θ, about the vector v

1. Introduction

The product of exponentials formula for Forward Kinematics of serial-link multi-body systems with revolute and/or prismatic joints was first introduced by Brockett in 1984 [1]. This formulation was further developed and its roots in Lie group and screw theory were illustrated by Murray *et al.* in 1994 [2]. One of the most important contributions of this method of multi-body system modeling is the elimination of intermediate coordinate frames in the kinematic analysis of serial-link manipulators. Since then, a number of researchers have investigated the computational efficiency of this formulation [3], and have applied it to different robotic problems [4, 5, 6, 7, 8]. In 1995, Park *et al.* used this formulation to reformulate the dynamical equations of serial-link multi-body systems [9], and later in 2003 Müller *et al.* attempted to unify the kinematics and dynamics of open-chain multi-body systems with one degree-of-freedom (d.o.f.) joints [10].

The exponential map used in the product of exponentials formula is indeed the exponential map of Lie groups, which maps an element of corresponding Lie algebra to an element of the Lie group [11]. For a rigid body, this Lie group is $SE(3)$, which is called the configuration manifold, and the elements of its Lie algebra $se(3)$ are the screws associated with the possible motions of a rigid body in 3-dimensional space [2]. In [12] a family of approximation formulas is presented that allow reconstructing large rigid body motions from a given velocity field, up to a desired order. Screw theory, which was first introduced by Ball in 1900 [13] and also appeared in the work of Clifford [14, 15], has been extensively investigated as a powerful means for the kinematic modeling of mechanisms [16, 17, 18, 19, 20, 21] and robotic systems [5, 22, 23, 24], by defining the notion of screw systems [25]. Moreover, the relationship between screw theory, Lie groups and projective geometry in the study of rigid body motion was elaborated in a paper by Stramigioli in 2002 [26]. He subsequently defined the notions of relative configuration manifold and relative screw to study multi-body systems [27]. In 1999 Mladenova also applied Lie group theory to the modeling and control of multi-body systems [28]. As opposed to the geometric nature of most of the above-mentioned works, her approach was mainly algebraic.

Based on a well-known theorem in the theory of Lie groups, any element of a connected Lie group can be written as product of exponentials of some elements of its Lie algebra. Accordingly, Wei and Norman introduced a product of exponentials representation for the elements of a connected Lie group [29], which was adopted by Liu [30] and Leonard *et al.* [31] to reformulate Kane's equations for multi-body systems and solve nonholonomic control problems on Lie groups, respectively. However, this is not computationally the most efficient way of parameterizing Lie groups, since this parameterization does not use the minimum number of exponentials of the Lie algebra elements (in the product of exponentials). Therefore, in terms of computational efficiency, investigating the surjectivity of the exponential map for Lie groups is valuable. For $SE(3)$, surjectivity of the exponential map is a direct consequence of Chasles' Theorem [2], which implies that any element of $SE(3)$ can be written as the exponential of at least one element of $se(3)$. However, not much work has been done on the exponential parameterization of the Lie subgroups of $SE(3)$. Only for the one-parameter subgroups of $SE(3)$, which correspond to one-d.o.f. joints, the exponential map has been used to parameterize the relative configuration manifold that leads to the standard product of exponentials formula. In fact, it is going to be shown that the

Lie subgroups of $SE(3)$ correspond to the relative configuration manifolds of *displacement subgroups* [20, 32]. These joints are generally multi-d.o.f. holonomic joints. For generic multi-d.o.f. joints, Stramigioli in [27] briefly mentions that at each point the exponential map can be used as a local diffeomorphism between the relative configuration manifold and its tangent space. He later used this local diffeomorphism to introduce singularity-free dynamic equations of a generic open-chain multi-body system with holonomic and nonholonomic joints [33]. In the following sections, the necessary and sufficient conditions for surjectivity of the exponential map of the relative configuration manifolds of displacement subgroups are given, and under those conditions the corresponding manifolds are parameterized using the elements of their Lie algebras.

In this paper, as a natural extension of the product of exponentials formula, a generalized formulation for Forward and Differential Kinematics of open-chain multi-body systems with multi-d.o.f., holonomic and nonholonomic joints is formalized. Lie group theory and differential geometry are used in Section 2 to classify the multi-d.o.f. joints, and introduce screw joint parameters. In Section 3, exponential map of Lie groups is utilized for parameterization of the relative configuration manifolds of displacement subgroups, and the generalized exponential formula for Forward Kinematics of multi-body systems with displacement subgroups is formally derived. Using the differential of the Forward Kinematics map and an annihilator of the nonholonomic constraints matrix, a coordinate-independent formulation for the Differential Kinematics of an open-chain multi-body system with nonholonomic constraints is derived in Section 4. This formulation is indeed independent of the choice of coordinate chart and a basis for the Lie algebras. Section 5 introduces the computational tools for the utilization of the developed formulation in numerical modeling, and the paper is finalized by a case study in Section 6.

2. Holonomic and Nonholonomic Joints

A physical 3-dimensional (3D) space can be mathematically modeled as a 3D *affine space*, denoted by A, which is equipped with a vector space V, and a rigid body B is the closure of a bounded open subset of A. Considering a multi-body system $MS(N) = \{(A_i, B_i)|B_i \subset A_i, i = 0, ..., N\}$ and two of its bodies, namely B_i and B_j, the space of all relative poses (position and orientation) of B_i with respect to B_j forms a smooth manifold P_i^j. When $i =$

j this manifold, which is the space of all possible coordinate transformations of A_i, inherits Lie group structure isomorphic to $SE(3)$ with the identity element e_i and the Lie algebra denoted by $Lie(P_i^i)$. In the case of $i=j$, to simplify the notation only the lower index is used, e.g., $P_i := P_i^i$. A *relative motion* of B_i with respect to B_j is a smooth curve $r_i^j : [0,1] \to P_i^j$, and the *relative velocity* at time t is the vector $v_i^j(t) = (dr_i^j/dt)(t) \in T_{r_i^j(t)} P_i^j$, where $T_{r_i^j(t)} P_i^j$ is the tangent space of P_i^j at the element $r_i^j(t)$. At each instant t, one can show that this vector induces a vector field X_t on A_j corresponding to the relative motion of B_i with respect to B_j such that $\forall a \in A_j$,

$$X_t(a) = \lim_{\delta \to 0} \frac{\exp\left(\delta \left(d_{r_i^j(t)} R_{r_j^i(t)}\right) v_i^j(t)\right)(a) - (a)}{\delta}; \tag{1}$$

where $R_{r_j^i(t)} : P_i^j \to P_j$ denotes the *right composition* map by $r_j^i(t)$. For a relative motion, if this vector field is independent of time, the relative motion is called *relative screw motion.* In other words, a relative screw motion is the curve on P_i^j corresponding to the flow of a left-invariant vector field on P_j. An interpretation of the Chasles' Theorem indicates that from any initial relative pose, a deliberate relative pose of B_i with respect to B_j can be reached by a relative screw motion. Therefore, P_i^j can be parameterized using the exponential map of Lie groups [2].

Given two rigid bodies of a multi-body system, B_i and B_j, a *joint* is a mechanism that restricts the relative motion of B_i with respect to B_j, and specifies a subset D_i^j of TP_i^j. A joint may be time dependant, called *rheonomic* joint, or time independent, which is called *scleronomic* joint. A special type of scleronomic joints, which is mostly considered in the literature, is when we have $D_i^j \subseteq TP_i^j$ being a distribution on P_i^j that corresponds to admissible directions of the relative velocity of B_i with respect to B_j. We only consider this category of joints in this paper. We also assume that the distribution D_i^j is non-singular in this paper. If D_i^j is involutive, i.e. closed under the Lie bracket of vector fields, the joint is called *holonomic*, otherwise, it is a *nonholonomic* joint. For any non-involutive distribution D_i^j, let $\bar{D}_i^j$ be the *involutive closure* of D_i^j. The involutive closure of a distribution D_i^j is the smallest vector sub-bundle of TP_i^j containing D_i^j that is closed under the Lie bracket of vector fields. Based on the global Frobenius Theorem [34], either D_i^j or $\bar{D}_i^j$ (for a holonomic or nonholonomic joint) identifies a foliation of submanifolds of P_i^j. The leaf $Q_i^j \subseteq P_i^j$ that contains the initial relative pose of B_i with respect to B_j, $r_i^j(0)$, is called the *relative configuration manifold.*

The manifold Q_i^j is the space of all admissible relative poses considering the joint constraints. The dimension of this manifold, k, is called the *number of d.o.f.* of a joint, which is greater than or equal to the dimension of the joint distribution for a nonholonomic or holonomic joint, respectively.

One can define the submanifold $Q_i \subseteq P_i$ as the left composition of Q_i^j by $r_j^i(0)$, i.e., $Q_i = L_{r_j^i(0)}(Q_i^j)$, where $r_i^j(0) \circ r_j^i(0) = e_j$ and $r_j^i(0) \circ r_i^j(0) = e_i$. This submanifold consists of the identity element of P_i that corresponds to the $r_i^j(0) \in Q_i^j$. A local coordinate chart for a neighbourhood $W \subset Q_i$ of e_i is a diffeomorphism $\varphi : U \subset \mathbb{R}^k \to W$ such that $\varphi([0, ..., 0]^T) = e_i$. Therefore, any element $r_i^j \in L_{r_i^j(0)}(W) \subseteq Q_i^j$ can be parameterized by a $q \in U$, which is called the *classic joint parameter*, through the diffeomorphism $L_{r_i^j(0)} \circ \varphi$. A velocity vector $v_i^j \in T_{r_i^j} Q_i^j$ can also be identified by a k-dimensional vector $\dot{q} \in T_q U \cong \mathbb{R}^k$ by the linear isomorphism $(d_{\varphi(q)} L_{r_i^j(0)})(d_q \varphi)$. Note that the coordinate chart φ induces a basis $\{(\frac{\partial}{\partial q_b})|_q | b = 1, ..., k\}$ for $T_{\varphi(q)} W$, where q_b is the b^{th} element of q, and in this basis $d_q \varphi$ is the identity matrix, id_k.

2.1. displacement subgroups

In this subsection, *displacement subgroups* are defined as a class of holonomic joints, and it is shown that their relative configuration manifolds are indeed connected Lie groups. In Proposition 2.2, the necessary and sufficient conditions for the surjectivity of the exponential map of these relative configuration manifolds are given. Based on this identification of displacement subgroups, a set of new joint parameters, called *screw joint parameters*, is introduced that can be physically interpreted as the initial classic joint speeds for a screw motion on the corresponding relative configuration manifold. Finally, the relationship between the screw joint parameters and the classic joint parameters is formalized in Theorem 2.3.

For a holonomic joint, define the distribution $D_j := T_{r_i^j} R_{r_j^i(0)}(D_i^j) \subseteq TP_j$. Based on the definition of a holonomic joint, D_j is involutive, i.e., its space of sections is closed under the Lie bracket of vector fields on P_j. This bracket coincides with the definition of the Lie bracket [35] on $Lie(P_j)$ if D_j is left-invariant, i.e., $D_j(r_j) = T_{e_j} L_{r_j}(D_j(e_j)), \forall r_j \in P_j$. We denote the integral manifold of D_j containing e_j by $Q_j \subseteq P_j$. Particularly, $D_j(e_j)$, which is a linear subspace of $Lie(P_j)$, is closed under the Lie bracket of $Lie(P_j)$; hence $T_{e_j} Q_j = D_j(e_j)$ is a Lie sub-algebra of $Lie(P_j)$.

Proposition 2.1. *For a holonomic joint, if D_j (defined above) is left-invariant, its integral manifold containing e_j, i.e., $Q_j \subseteq P_j$, is a unique k-dimensional connected Lie subgroup of P_j with the Lie algebra $Lie(Q_j) = D_j(e_j)$.*

Note that, conversely, for any Lie subgroup $Q'_j \subseteq P_j$, there exists a unique involutive distribution corresponding to a holonomic joint, by left translating $Lie(Q'_j)$ over P_j and right composing it with $r_i^j(0)$.

Definition 1. A holonomic joint is called *displacement subgroup* if the corresponding distribution D_j (defined above) on P_j is left-invariant.

Therefore, based on Proposition 2.1 and since $P_j \cong SE(3)$, different types of displacement subgroups are identified by the connected Lie subgroups of $SE(3)$, up to conjugation, which are tabulated in Table 1 [20, 25]. From this table, one can observe that the displacement subgroups consist of the six *lower kinematic pairs*, i.e., revolute, prismatic, helical, cylindrical, planar and spherical joints, and combinations of them. Therefore, in this joint categorization, the relative configuration manifolds of lower kinematic pairs are indeed subgroups of $SE(3)$. There also exist other types of holonomic joints, e.g., universal joint and higher kinematic pairs, which are not included in the category of displacement subgroups. However, the relative configuration manifolds of these joints are not subgroups of $SE(3)$. To parameterize the relative configuration manifolds of these joints one needs a product of exponentials of some elements of a basis for the tangent space of the relative configuration manifold at the identity element.

Proposition 2.2. *The group exponential map* $\exp : Lie(Q_j) \to Q_j$ *is surjective for all categories of displacement subgroups, except for a three-d.o.f. joint where a helical joint is combined with a two-d.o.f. prismatic joint such that the helical joint axis is perpendicular to the plane of the prismatic joint. This case is considered as two separate joints in the paper.*

Since this proposition is proved by coordinate chart assignment, its proof is presented in Section 5.

Definition 2. Let φ be a coordinate chart for a neighbourhood of e_i, by Proposition 2.2 any relative configuration manifold Q_i^j of a displacement subgroup can be parameterized by vectors $s \in \mathbb{R}^k$, called *screw joint parameters*, such that every $r_i^j \in Q_i^j \subseteq P_i^j$ can be expressed as

$$r_i^j = \exp(\tau_i^j s) \circ r_i^j(0) := \exp\Big((Ad_{r_i^j(0)})(d_{e_i}\iota)(d_0\varphi)s\Big) \circ r_i^j(0), \tag{2}$$

Table 1: Categories of displacement subgroups

Dim.	Subgroups of $SE(3)$/displacement subgroups			
6	$SE(3) = SO(3) \ltimes \mathbb{R}^3$ free[a]			
4	$SE(2) \times \mathbb{R}$ planar+prismatic[b]			
3	$SE(2) = SO(2) \ltimes \mathbb{R}^2$ planar	$SO(3)$ ball (spherical)	$\mathbb{R}^3$ 3-d.o.f. prismatic	$H_p \ltimes \mathbb{R}^2$ 2-d.o.f. prismatic + helical[c]
2	$SO(2) \times \mathbb{R}$ cylindrical[d]	$\mathbb{R}^2$ 2-d.o.f. prismatic		
1	$SO(2)$ revolute	$\mathbb{R}$ prismatic	H_p helical	
0	$\{e\}$ fixed[a]			

[a] These two subgroups are the trivial subgroups of $SE(3)$.
[b] The axis of the prismatic joint is always perpendicular to the plane of the planar joint.
[c] The axis of the helical joint is always perpendicular to the plane of the 2-d.o.f. prismatic joint.
[d] The axis of the revolute and prismatic joints are always aligned.

where $\iota : Q_i \to P_i$ is the inclusion map.

Therefore, for a relative motion $r_i^j : [0,1] \to Q_i^j$ the relationship between $(s, \dot{s})$ and $(q, \dot{q})$, the classic joint parameters and their speed, can be summarized in the following theorem.

Theorem 2.3. *For a displacement subgroup, consider a coordinate chart for Q_i, $\varphi : U \subset \mathbb{R}^k \to W$ such that $\varphi([0, ..., 0]^T) = e_i$, and a relative motion $r_i^j : [0,1] \to Q_i^j$ in the neighbourhood $W' := L_{r_i^j(0)}(W) \subseteq Q_i^j$ of $r_i^j(0)$. Then, $r_i^j(t) = \exp(\tau_i^j s(t)) \circ r_i^j(0)$ such that $s(0) = 0$, and*

$$\begin{aligned} q(s) &= \varphi^{-1} \circ \exp(d_0\varphi\ s), && \text{(3a)} \\ \dot{q}(s, \dot{s}) &= Z(s)\dot{s} \\ &:= (d_{q(s)}\varphi)^{-1} d_{e_j} L_{\exp(d_0\varphi s)} \left(\int_0^1 \exp(-x\ ad_{d_0\varphi s})\ dx \right) d_0\varphi\ \dot{s}, && \text{(3b)} \end{aligned}$$

where $\forall \eta \in Lie(Q_j)$ $ad_\eta : Lie(Q_j) \to Lie(Q_j)$ is an endomorphism of $Lie(Q_j)$ such that $\forall \xi \in Lie(Q_j)$ $ad_\eta(\xi) := [\eta, \xi]$ [35]. The linear map $Z(s)$ is an isomorphism between $T_0\mathbb{R}^k$ and $T_q\mathbb{R}^k$ if and only if $ad_{d_0\varphi s}$ has no eigenvalue in $\left\{2\pi i\mathbb{Z} | i = \sqrt{-1}\right\}$.

Proof. For the relative motion $r_i^j \subset W'$, let $r_i = L_{r_j^i(0)} \circ r_i^j \subset W$ be the corresponding curve on Q_i. This curve on P_i is $\iota \circ \varphi(q) = L_{r_j^i(0)} \circ R_{r_i^j(0)} \circ \exp(\tau_i^j s) = K_{r_j^i(0)} \circ \exp(\tau_i^j s)$. Based on (2) and the fact that exponential map is compatible with the Lie group homomorphisms [35], in this case conjugation and inclusion map, $\iota \circ \varphi(q) = K_{r_j^i(0)} \circ K_{r_i^j(0)} \circ \iota \circ \exp(d_0\varphi s) = \iota \circ \exp(d_0\varphi s)$. Therefore, (3a) is true since the inclusion map ι is an embedding, and φ is a diffeomorphism.

Differentiating (3a) with respect to the curve parameter results in

$$\dot{q} = \left(d_{\exp(d_0\varphi s)}\varphi^{-1}\right)\left(d_{d_0\varphi s}\exp\right) d_0\varphi\dot{s} = \left(d_q\varphi\right)^{-1}\left(d_{d_0\varphi s}\exp\right) d_0\varphi\dot{s}.$$

For a Lie group G, it can be shown that the differential of the exponential map at $\xi \in Lie(G)$ is [36]

$$d_\xi \exp = d_e L_{\exp(\xi)} \int_0^1 \exp(-x\; ad_\xi)dx. \tag{4}$$

Hence, substituting (4) and (3a) in the above equation completes the proof for (3b).

In (3b), $Z(s)$ is defined as the composition of several linear operators, and it is invertible if and only if all of the linear operators are invertible. Since left translation is a global diffeomorphism and φ is a coordinate chart, it suffices to check the conditions under which $\Theta := \int_0^1 \exp(-x\; ad_{d_0\varphi s})\, dx$ is invertible. For $z \in \mathbb{C}$, consider the solution of $\int_0^1 \exp(-x\; z)\; dx$ that is equal to the entire holomorphic function $f(z) = \frac{1-\exp(-z)}{z}$ such that $f(0) = 1$. Thus, the eigenvalues of Θ are equal to $\frac{1-\exp(-\lambda_i)}{\lambda_i}$, where λ_i's are the eigenvalues of $ad_{d_0\varphi s}$. The Lie algebra endomorphism Θ is invertible if and only if it has no eigenvalues equal to zero, i.e., $\lambda_i \neq 2\pi i\mathbb{Z}$ where $i = \sqrt{-1}$. $\square$

Last part of Theorem 2.3 also gives a condition for the size of the image of the coordinate chart associated with the screw joint parameterization. On $P_j \cong SE(3)$ this condition dictates that the coordinate chart cannot include elements of P_j corresponding to 2π radian rotation about an axis in A_j. Also, note that the integral term in (4) is equal to the identity map for abelian Lie groups, and in general this term corresponds to the non-commutativity of $\xi, \dot{\xi} \in Lie(Q_j)$ with respect to the Lie bracket.

2.2. Nonholonomic displacement subgroups

A nonholonomic displacement subgroup is a displacement subgroup together with $\bar{k}$ linearly independent constraints in the space of the speeds of the classic joint parameters that are not integrable, i.e., $C(q)\dot{q} = 0$, where $C(q) \in \mathbb{R}^{\bar{k}\times k}$, and $C(q)$ is assumed to be a differentiable linear operator on Q_i. In other words, for the neighbourhood W of the initial relative pose $r_i^j(0)$, $\forall q \in U \subset \mathbb{R}^k$ $\dot{q} \in T_q\mathbb{R}^k$ should lie in the $\ker(C(q)) \cong \mathbb{R}^{k-\bar{k}}$ that can be considered as the range of another linear operator $\bar{C}(q)$, i.e., $C(q)\bar{C}(q) = 0$. The $\bar{C}(q) \in \mathbb{R}^{k\times(k-\bar{k})}$ is a differentiable linear operator on Q_i of constant rank $k-\bar{k}$. This linear operator identifies a smooth non-involutive distribution on Q_i^j corresponding to the space of all admissible instantaneous relative velocities of the joint. Therefore, an admissible joint speed has the form $\dot{q} = \bar{C}(q)\dot{\bar{q}}$ $\forall\dot{\bar{q}} \in \mathbb{R}^{k-\bar{k}}$. Note that the representation of $\bar{C}(q)$ in the local coordinates is not unique, and it could be chosen such that the admissible classic joint speeds are collocated with the joint control forces and torque to simplify the dynamic analysis. Based on (3b) in Theorem 2.3 and considering the screw joint parameters, the space of all admissible screw joint speeds at s can be identified by

$$\dot{s} = \Sigma(s)\dot{\bar{s}} := Z^{-1}(s)\bar{C}(q(s))\dot{\bar{s}}. \quad \forall\dot{\bar{s}} \in \mathbb{R}^{k-\bar{k}} \tag{5}$$

3. Forward Kinematics

Definition 3. An *open-chain multi-body system* is a multi-body system $MS(N)$ together with $N-1$ joints between the bodies, such that there exists a unique path between any two bodies of the multi-body system. In an open-chain multi-body system, bodies with only one neighbouring body are called *extremities.*

In robotics, the relative pose and velocity of the extremities with respect to a *base body*, labeled as B_0 in $MS(N)$, is usually of interest. The base body is possibly an inertial observer.

Definition 4. A *branch* of an open-chain multi-body system is a chain of $m+1 \le N$ bodies together with m joints that connects B_0 to an extremity.

In this paper, an open-chain multi-body system is assumed to have n branches with both holonomic and nonholonomic multi-d.o.f. joints. In the branch i, joint j connects body $B_{\mathsf{j}-1}$ to B_{j}. The *branch configuration* r_{i}

is defined as the collection of the relative poses of rigid bodies, i.e., $r_{\mathfrak{i}} := \left(r_1^0, ..., r_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}\right) \in Q_1^0 \times ... \times Q_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}$.

Index the $\mathfrak{j}^{th}$ body of the branch $\mathfrak{i}$ by $\mathfrak{j}_{\mathfrak{i}}$. Let $k_{\mathfrak{j}_{\mathfrak{i}}}$ be the number of d.o.f. of the joint $\mathfrak{j}$ in the $\mathfrak{i}^{th}$ branch, for an initial branch configuration, the set of all screw joint parameters of the branch is denoted by $G_{\mathfrak{i}} := \left\{ {}^{\mathfrak{i}}s = \left[{}^{\mathfrak{i}}s_1^T, ..., {}^{\mathfrak{i}}s_{m_{\mathfrak{i}}}^T\right]^T | {}^{\mathfrak{i}}s_{\mathfrak{j}} \in \mathbb{R}^{k_{\mathfrak{j}_{\mathfrak{i}}}}, \mathfrak{j} = 1, ..., m_{\mathfrak{i}} \right\}$. *Forward Kinematics* of the $\mathfrak{i}^{th}$ branch of an open-chain multi-body system is a smooth map $FK_{\mathfrak{i}}$ from the set of screw joint parameters of the branch to $P_{m_{\mathfrak{i}}}^0$ for an initial branch configuration that indicates the relative pose of the body $B_{m_{\mathfrak{i}}}$ with respect to B_0, i.e., $FK_{\mathfrak{i}} : G_{\mathfrak{i}} \to P_{m_{\mathfrak{i}}}^0$ such that $FK_{\mathfrak{i}}({}^{\mathfrak{i}}s) := r_1^0 \circ ... \circ r_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}$.

Theorem 3.1. *For an open-chain multi-body system $MS(N)$ along with N holonomic and nonholonomic displacement subgroups, the generalized exponential formula for the Forward Kinematics map corresponding to the $\mathfrak{i}^{th}$ branch can be formulated as*

$$FK_{\mathfrak{i}}({}^{\mathfrak{i}}s) = \exp\left({}^0\tau_1^0\,{}^{\mathfrak{i}}s_1\right) \circ ... \circ \exp\left({}^0\tau_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}\,{}^{\mathfrak{i}}s_{m_{\mathfrak{i}}}\right) \circ r_{m_{\mathfrak{i}}}^0, \tag{6}$$

where ${}^0\tau_{\mathfrak{j}}^{\mathfrak{j}-1} = (Ad_{r_{\mathfrak{j}}^0(0)})(d_{e_{\mathfrak{j}}}\iota_{\mathfrak{j}})(d_0\varphi_{\mathfrak{j}})$, $\iota_{\mathfrak{j}} : Q_{\mathfrak{j}} \to P_{\mathfrak{j}}$ is the inclusion map, and $\varphi_{\mathfrak{j}}$ is a coordinate chart for a neighbourhood of $e_{\mathfrak{j}} \in P_{\mathfrak{j}}$ $\forall \mathfrak{j} = 1, ..., m_{\mathfrak{i}}$.

PROOF. Using the screw joint parameters and the definition of the Forward Kinematics map,

$$FK_{\mathfrak{i}}({}^{\mathfrak{i}}s) = \left(\exp(\tau_1^0\,{}^{\mathfrak{i}}s_1) \circ r_1^0(0)\right) \circ ... \circ \left(\exp(\tau_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}\,{}^{\mathfrak{i}}s_{m_{\mathfrak{i}}}) \circ r_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}(0)\right).$$

Due to the fact that $r_{\mathfrak{j}}^{\mathfrak{j}-1}(0) = r_0^{\mathfrak{j}-1}(0) \circ r_{\mathfrak{j}}^0(0)$, associativity of the composition operator, and compatibility of the exponential map with the conjugation map,

$$\begin{aligned} FK_{\mathfrak{i}}({}^{\mathfrak{i}}s) &= \exp(\tau_1^0\,{}^{\mathfrak{i}}s_1) \circ \left(r_1^0(0) \circ \exp(\tau_2^1\,{}^{\mathfrak{i}}s_2) \circ r_0^1(0)\right) \circ ... \\ &\quad \circ \left(r_{m_{\mathfrak{i}}-1}^0(0) \exp(\tau_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}\,{}^{\mathfrak{i}}s_{m_{\mathfrak{i}}}) \circ r_0^{m_{\mathfrak{i}}-1}(0)\right) \circ r_{m_{\mathfrak{i}}}^0(0) \\ &= \exp(\tau_1^0\,{}^{\mathfrak{i}}s_1) \circ \exp(Ad_{r_1^0(0)}(\tau_2^1\,{}^{\mathfrak{i}}s_2)) \circ ... \circ \exp(Ad_{r_{m_{\mathfrak{i}}-1}^0(0)}(\tau_{m_{\mathfrak{i}}}^{m_{\mathfrak{i}}-1}\,{}^{\mathfrak{i}}s_{m_{\mathfrak{i}}})) \circ r_{m_{\mathfrak{i}}}^0(0). \end{aligned}$$

Substituting the definition of $\tau_{\mathfrak{j}}^{\mathfrak{j}-1}$, $\forall \mathfrak{j} = 1, ..., m_{\mathfrak{i}}$, from (2) completes the proof. □

Note that since Forward Kinematics is only a function of the relative poses, nonholonomic constraints do not appear in (6). Forward Kinematics of an open chain multi-body system, FK, is defined as the collection of the relative poses of the extremities with respect to the base body B_0, i.e., $FK : G_1 \times ... \times G_n \to P^0_{m_1} \times ... \times P^0_{m_n}$ such that

$$FK(s) := \begin{bmatrix} FK_1({}^1s) \\ \vdots \\ FK_n({}^ns) \end{bmatrix},$$

where $s = [{}^1s^T, ..., {}^ns^T]^T$.

For a serial-link multi-body system $MS(N)$ with one-d.o.f. revolute and/or prismatic joints, $s_\mathrm{j}(t)\ \forall \mathrm{j} = 1, ..., N$ is a real number function, instead of a vector function. Based on the interpretation of the screw joint parameters given in the beginning of Subsection 2.1, $s_\mathrm{j}(t)$ is the constant speed of a classic joint parameter during a screw motion from 0 to $q_\mathrm{j}(t)$, in the interval of [0,1]. Therefore, its number is equal to the corresponding classic joint parameter. Moreover, since the joint has only one d.o.f., the linear operator ${}^0\tau^{\mathrm{j}-1}_\mathrm{j}$ reduces to the joint screw at the initial configuration, which corresponds to the axis of rotation for a revolute joint or the direction of translation for a prismatic joint [2, 25]. Consequently, it can be shown that in this special case the formulation for Forward Kinematics of an open-chain multi-body system is equivalent to the product of exponentials formula suggested by Brockett [1]. This relationship is further illustrated in the case study in Section 6.

4. Differential Kinematics

For the i^{th} branch of an open-chain multi-body system, *Differential Kinematics* is a linear map that relates the speed of the screw joint parameters of the branch to the instantaneous relative twist of B_{m_i} with respect to B_0 and observed in A_0, i.e., expressed in the vector space associated with A_0, V_0. The corresponding linear operator ${}^0J^0_{m_\mathrm{i}}({}^\mathrm{i}s)$, called the *Jacobian*, for an initial branch configuration is ${}^0J^0_{m_\mathrm{i}}({}^\mathrm{i}s) : T_{{}^\mathrm{i}s}G_\mathrm{i} \to Lie(P_0)$ such that ${}^0J^0_{m_\mathrm{i}}({}^\mathrm{i}s) := \left(d_{FK_\mathrm{i}({}^\mathrm{i}s)}R_{(FK_\mathrm{i}({}^\mathrm{i}s))^{-1}}\right) d_{{}^\mathrm{i}s}FK_\mathrm{i}$.

Theorem 4.1. *For an open-chain multi-body system $MS(N)$ along with N holonomic displacement subgroups, the generalized exponential formula for* *the Jacobian of the branch* $\mathfrak{i}$ *can be formulated as*

$$
\begin{aligned}
{}^0J^0_{m_\mathfrak{i}}({}^\mathfrak{i}s) = \quad & \Big[\left(\Delta_1\, {}^0\tau^0_1\right) \quad \left(\exp\left(ad_{{}^0\tau^0_1\,{}^\mathfrak{i}s_1}\right)\Delta_2\, {}^0\tau^1_2\right) \quad \cdots \\
& \left(\exp\left(ad_{{}^0\tau^0_1\,{}^\mathfrak{i}s_1}\right)...\exp\left(ad_{{}^0\tau^{m_\mathfrak{i}-2}_{m_\mathfrak{i}-1}\,{}^\mathfrak{i}s_{m_\mathfrak{i}-1}}\right)\Delta_{m_\mathfrak{i}}\, {}^0\tau^{m_\mathfrak{i}-1}_{m_\mathfrak{i}}\right)\Big], \quad (7)
\end{aligned}
$$

where $\Delta_\mathfrak{j} := \int_0^1 \exp(x\, ad_{{}^0\tau^{\mathfrak{j}-1}_\mathfrak{j}({}^\mathfrak{i}s_\mathfrak{j})})dx$ *is an endomorphism of* $Lie(P_0)$.

Proof. Consider a curve ${}^\mathfrak{i}s : [0,1] \to G_\mathfrak{i}$, such that $t \mapsto^\mathfrak{i} s(t)$, in the set of screw joint parameters of the branch $\mathfrak{i}$. Let $\gamma_\mathfrak{j}(t) := \exp({}^0\tau^{\mathfrak{j}-1}_\mathfrak{j}\,{}^\mathfrak{i}s_\mathfrak{j}(t))\ \forall \mathfrak{j} = 1, ..., m_\mathfrak{i}$. Using (6) and the product rule for Lie groups,

$$
\begin{aligned}
\frac{d}{dt}FK_\mathfrak{i}({}^\mathfrak{i}s(t)) \;&=\; d_{{}^\mathfrak{i}s(t)}FK_\mathfrak{i}\,{}^\mathfrak{i}\dot{s}(t) = \left(d_{\gamma_1}R_{\gamma_2\circ...\circ\gamma_{m_\mathfrak{i}}\circ r^0_{m_\mathfrak{i}}(0)}\right)\dot{\gamma}_1 \\
&+\; \left(d_{\gamma_2\circ...\circ\gamma_{m_\mathfrak{i}}\circ r^0_{m_\mathfrak{i}}(0)}L_{\gamma_1}\right)\left(d_{\gamma_2}R_{\gamma_3\circ...\circ\gamma_{m_\mathfrak{i}}\circ r^0_{m_\mathfrak{i}}(0)}\right)\dot{\gamma}_2 + ... \\
&+\; \left(d_{\gamma_{m_\mathfrak{i}}\circ r^0_{m_\mathfrak{i}}(0)}L_{\gamma_1\circ...\circ\gamma_{m_\mathfrak{i}-1}}\right)\left(d_{\gamma_{m_\mathfrak{i}}}R_{r^0_{m_\mathfrak{i}}(0)}\right)\dot{\gamma}_{m_\mathfrak{i}}.
\end{aligned}
$$

By the definition of the Differential Kinematics map and rearranging the differential of the right and left composition maps,

$$
\begin{aligned}
{}^0J^0_{m_\mathfrak{i}}({}^\mathfrak{i}s)\,{}^\mathfrak{i}\dot{s} \;&=\; \left(d_{\gamma_1}R_{\gamma_1^{-1}}\right)\dot{\gamma}_1 + \left(d_{\gamma_1\circ\gamma_2}R_{(\gamma_1\circ\gamma_2)^{-1}}\right)\left(d_{\gamma_2}L_{\gamma_1}\right)\dot{\gamma}_2 + ... \\
&+\; \left(d_{\gamma_1\circ...\circ\gamma_{m_\mathfrak{i}}}R_{(\gamma_1\circ...\circ\gamma_{m_\mathfrak{i}})^{-1}}\right)\left(d_{\gamma_{m_\mathfrak{i}}}L_{\gamma_1\circ...\circ\gamma_{m_\mathfrak{i}-1}}\right)\dot{\gamma}_{m_\mathfrak{i}}. \qquad (8)
\end{aligned}
$$

Now, use (4) for the exponential map $\exp : Lie(P_0) \to P_0$, and the equality of operators [35]

$$
Ad_{\exp(\xi)} = \exp(ad_\xi), \quad \forall \xi \in Lie(P_0) \qquad (9)
$$

to calculate $\dot{\gamma}_\mathfrak{j}(t) = (d_{e_0}L_{\gamma_\mathfrak{j}})\left(\int_0^1 Ad_{\exp(-x\,{}^0\tau^{\mathfrak{j}-1}_\mathfrak{j}\,{}^\mathfrak{i}s_\mathfrak{j})}dx\right){}^0\tau^{\mathfrak{j}-1}_\mathfrak{j}\,{}^\mathfrak{i}\dot{s}_\mathfrak{j}(t)$. Substitute $\dot{\gamma}_\mathfrak{j}$ and use the identity $Ad_r := d_rR_{r^{-1}}\,d_{e_0}L_r\ \forall r \in P_0$ in (8) to achieve

$$
\begin{aligned}
{}^0J^0_{m_\mathfrak{i}}({}^\mathfrak{i}s)\,{}^\mathfrak{i}\dot{s} \;&=\; Ad_{\gamma_1}\left(\int_0^1 Ad_{\exp(-x\,{}^0\tau^0_1\,{}^\mathfrak{i}s_1)}dx\right){}^0\tau^0_1\,{}^\mathfrak{i}\dot{s}_1 + ... \\
&+\; Ad_{\gamma_1\circ...\circ\gamma_{m_\mathfrak{i}}}\left(\int_0^1 Ad_{\exp(-x\,{}^0\tau^{m_\mathfrak{i}-1}_{m_\mathfrak{i}}\,{}^\mathfrak{i}s_{m_\mathfrak{i}})}dx\right){}^0\tau^{m_\mathfrak{i}-1}_{m_\mathfrak{i}}\,{}^\mathfrak{i}\dot{s}_{m_\mathfrak{i}}. \quad (10)
\end{aligned}
$$

Define $\Delta_\mathfrak{j}$ $\forall \mathfrak{j} = 1, ..., m_\mathfrak{i}$ as

$$\begin{aligned}\Delta_\mathfrak{j} := Ad_{\gamma_\mathfrak{j}} \left(\int_0^1 Ad_{\exp(-x\,{}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\,{}^\mathfrak{i}s_\mathfrak{j})} dx \right) &= \int_0^1 Ad_{\exp((1-x)\,{}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\,{}^\mathfrak{i}s_\mathfrak{j})} dx \\ &= \int_0^1 \exp\left(x\, ad_{{}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\,{}^\mathfrak{i}s_\mathfrak{j}} \right) dx,\end{aligned}$$

where the first equality holds since $[x\ {}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\,{}^\mathfrak{i}s_\mathfrak{j}, {}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\,{}^\mathfrak{i}s_\mathfrak{j}] = 0$, and the second equality is the consequence of a change of variable and using (9). Finally, by substituting $\Delta_\mathfrak{j}$ in (10) and employing the equality of operators in (9) one can show the desired expression for the Jacobian in (7) . $\square$

For a serial-link multi-body system with one-d.o.f. revolute and/or prismatic joints, since $s_\mathfrak{j}(t)$ is a real number function, ${}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}s_\mathfrak{j}(t) \in Lie(P_0)$ and ${}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\dot{s}_\mathfrak{j}(t) \in Lie(P_0)$ commute, i.e., $[{}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}s_\mathfrak{j}, {}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\dot{s}_\mathfrak{j}] = 0$, and hence $\Delta_\mathfrak{j}$ becomes the identity map. In this case, the developed formulation simplifies to the existing product of exponentials formula for Differential Kinematics [2, 25].

Based on the definition of the Differential Kinematics map, ${}^0J^0_{m_\mathfrak{i}}({}^\mathfrak{i}s){}^\mathfrak{i}\dot{s}$ is the twist of $B_{m_\mathfrak{i}}$ with respect to B_0 and expressed in A_0. This twist can be viewed in the affine space attached to the body $\mathfrak{j}$ of the branch $\mathfrak{i}$, $A_{\mathfrak{j}_\mathfrak{i}}$, using the Adjoint operator, i.e.,

$$ {}^{\mathfrak{j}_\mathfrak{i}}J^0_{m_\mathfrak{i}}({}^\mathfrak{i}s) = Ad_{r_0^{\mathfrak{j}_\mathfrak{i}}({}^\mathfrak{i}s)}\ {}^0J^0_{m_\mathfrak{i}}({}^\mathfrak{i}s), \tag{11}$$

where according to (6) $r^0_{\mathfrak{j}_\mathfrak{i}}({}^\mathfrak{i}s) = \exp\left({}^0\tau_1^0\,{}^\mathfrak{i}s_1\right) \circ ... \circ \exp\left({}^0\tau_\mathfrak{j}^{\mathfrak{j}-1}\,{}^\mathfrak{i}s_\mathfrak{j}\right) \circ r^0_{\mathfrak{j}_\mathfrak{i}}(0)$. In addition, following the same calculations performed in the proof of Theorem 4.1, the Jacobian for the instantaneous relative twist of the body $B_\mathfrak{j}$ with respect to $B_\mathfrak{l}$ in the $\mathfrak{i}^{th}$ branch of $MS(N)$ and observed in A_0, i.e., ${}^0J^\mathfrak{l}_\mathfrak{j}({}^\mathfrak{i}s)$ $\mathfrak{j} > \mathfrak{l} > 0$, can be determined to be the truncated version of the Jacobian in (7):

$$\begin{aligned}{}^0J^\mathfrak{l}_\mathfrak{j}({}^\mathfrak{i}s) = \Big[&\exp\left(ad_{{}^0\tau_1^0\,{}^\mathfrak{i}s_1}\right) ... \exp\left(ad_{{}^0\tau_\mathfrak{l}^{\mathfrak{l}-1}}\,{}^\mathfrak{i}s_\mathfrak{l}\right) \Delta_{\mathfrak{l}+1}\,{}^0\tau^\mathfrak{l}_{\mathfrak{l}+1} \quad \cdots \\ &\exp\left(ad_{{}^0\tau_1^0\,{}^\mathfrak{i}s_1}\right) ... \exp\left(ad_{{}^0\tau_{\mathfrak{j}-1}^{\mathfrak{j}-2}\,{}^\mathfrak{i}s_{\mathfrak{j}-1}}\right) \Delta_\mathfrak{j}\,{}^0\tau_\mathfrak{j}^{\mathfrak{j}-1} \Big] \quad .\end{aligned} \tag{12}$$

In order to include the nonholonomic constraints in the Jacobian of the $\mathfrak{i}^{th}$ branch of $MS(N)$, one can define admissible screw joint speeds according to (5). Therefore, the Jacobian in (7) can be modified to introduce the *modified Jacobian* for the $\mathfrak{i}^{th}$ branch of a multi-body system consisting of both holonomic and nonholonomic joints.

$$^{0}\bar{J}^{0}_{m_{\mathfrak{i}}}(^{\mathfrak{i}}s) := {}^{0}J^{0}_{m_{\mathfrak{i}}}(^{\mathfrak{i}}s)diag\left(\Sigma_{1}(^{\mathfrak{i}}s_{1}), \cdots, \Sigma_{m_{\mathfrak{i}}}(^{\mathfrak{i}}s_{m_{\mathfrak{i}}})\right); \tag{13}$$

where $diag\left(\Sigma_{1}(^{\mathfrak{i}}s_{1}), \cdots, \Sigma_{m_{\mathfrak{i}}}(^{\mathfrak{i}}s_{m_{\mathfrak{i}}})\right)$ is the block diagonal matrix of its entries, and $\Sigma_{\mathfrak{j}} = id_{k_{\mathfrak{j}_{\mathfrak{i}}}}$ for a holonomic joint. The modified Jacobian is a linear operator from the space of all admissible screw joint speeds, i.e., $\bar{G}_{\mathfrak{i}} := \left\{ {}^{\mathfrak{i}}\dot{\bar{s}} = \left[{}^{\mathfrak{i}}\dot{\bar{s}}^{T}_{1}, ..., {}^{\mathfrak{i}}\dot{\bar{s}}^{T}_{m_{\mathfrak{i}}}\right]^{T} | {}^{\mathfrak{i}}\dot{\bar{s}}_{\mathfrak{j}} \in \mathbb{R}^{k_{\mathfrak{j}_{\mathfrak{i}}} - \bar{k}_{\mathfrak{j}_{\mathfrak{i}}}}, \mathfrak{j} = 1, ..., m_{\mathfrak{i}} \right\}$, to $Lie(P_0)$. For an open-chain multi-body system $MS(N)$, the modified Jacobian is defined as the collection of the modified Jacobians of the extremities with respect to the base body and observed in A_0, i.e., $\bar{J}(s)\dot{\bar{s}} := diag\left({}^{0}\bar{J}^{0}_{m_1}(^{1}s), ..., {}^{0}\bar{J}^{0}_{m_n}(^{n}s)\right)\dot{\bar{s}}$, where $\dot{\bar{s}} = \left[{}^{1}\dot{\bar{s}}^{T}, ..., {}^{n}\dot{\bar{s}}^{T}\right]^{T}$.

5. Coordinate Assignment

At the computational level, consider a base point O_i for the affine space A_i in a multi-body system $MS(N)$. Every point in this affine space can now be realized by a vector in $V_i \cong \mathbb{R}^3$ through the action of $(V_i, +)$ on A_i [37]. Therefore, any relative pose $r^j_i \in P^j_i$ can be represented by an orientation preserving isometry, $H^j_i : V_i \to V_j$ such that $H^j_i := \sigma_{Oj} \circ r^j_i \circ (\sigma_{O_i})^{-1} \in SE(3)$, where $\sigma_{O_l} : V_l \to A_l$ for $l = i, j$ is the map induced by the vector space action of V_l on A_l. A matrix representation of $SE(3)$ is the group of orientation preserving linear isometries of $\mathbb{R}^4$ that preserve the plane $x_4 = 1$ [37], i.e.,

$$SE(3) \cong \left\{ H^j_i = \begin{bmatrix} R^j_i & p^j_i \\ 0_{1\times 3} & 1 \end{bmatrix} | R^j_i \in SO(3), \, p^j_i \in \mathbb{R}^3 \right\},$$

where R^j_i is the rotation matrix whose columns are the elements of a basis for V_i expressed in terms of a basis for V_j and p^j_i is the position of the point $r^j_i(O_i)$ from O_j and expressed in V_j. In this representation, the Lie algebra of $SE(3)$ is denoted by

$$se(3) \cong \left\{ T^j_i = \begin{bmatrix} \tilde{\omega}^j_i & w^j_i \\ 0_{1\times 3} & 0 \end{bmatrix} | \tilde{\omega}^j_i \in so(3), w^j_i \in \mathbb{R}^3 \right\},$$

where w^j_i is the relative velocity of the point $r^i_j(O_j)$ with respect to O_j and expressed in V_j. The element $\tilde{\omega}^j_i \in so(3)$ can be identified by the column matrix ω_i^j that is the relative angular velocity vector of B_i with respect to B_j and expressed in V_j.

By choosing a basis for $se(3)$ as

$$\left\{ e_1 := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_3 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. e_4 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_5 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_6 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\},$$

and using the propositions presented in the sequel, one can perform the computations for Forward and Differential Kinematics in the matrix representation of $SE(3)$.

Proposition 5.1. *For any element $\xi = [w^T, \omega^T]^T \in se(3)$, where $\omega, w \in \mathbb{R}^3, \omega \neq 0$, expressed in the basis $\{e_1, ..., e_6\}$,*

$$\exp(\xi) = \begin{bmatrix} \exp(\tilde{\omega}) & (id_3 - \exp(\tilde{\omega}))\frac{\tilde{\omega}w}{\|\omega\|^2} + \frac{\omega\omega^T w}{\|\omega\|^2} \\ 0_{1\times 3} & 1 \end{bmatrix}, \tag{14}$$

where$\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^3$ and $\exp(\tilde{\omega})$ is evaluated using the Rodrigues' formula for the exponential of skew-symmetric matrices,

$$\exp(\tilde{\omega}) = id_3 + \frac{\tilde{\omega}}{\|\omega\|}\sin(\|\omega\|) + \frac{\tilde{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|)). \tag{15}$$

When $\omega = 0$, $\exp(\xi) = \begin{bmatrix} id_3 & w \\ 0_{1\times 3} & 1 \end{bmatrix}$.

Proof. See Appendix A in [2]. □

Now, using the matrix representation of $SE(3)$ and the above proposition, the proof for Proposition 2.2 is presented.

Proof. (*Proposition 2.2*) In the matrix representation, the exponential map for a connected Lie subgroup of $SE(3)$ coincides with the restriction of the matrix exponential to the Lie sub-algebra corresponding to the subgroup.

Up to conjugation, all of the connected Lie subgroups of $SE(3)$ are listed in Table 1. Hence, to prove this proposition, it suffices to check the surjectivity of the exponential map for the matrix representation of each connected Lie subgroup, individually. Consider the following two lemmas.

Lemma 5.2. *The exponential map of a compact, connected Lie group is surjective [36].*

Lemma 5.3. *For a vector space $\mathcal{V}$, $Lie(\mathcal{V}) = \mathcal{V}$ with zero Lie bracket, and the exponential map is the identity map, i.e., $\exp(v) = v$, $\forall v \in \mathcal{V}$.*

Based on these lemmas and Chasles' Theorem [2], immediately the exponential maps corresponding to the subgroups $SO(2)$, $SO(3)$, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ and $SE(3)$ are surjective. In addition, since $SO(2) \times \mathbb{R}$ is the direct product of two subgroups with surjective exponential maps, its own exponential map is also surjective. The subgroup H_p with $p \neq 0$ is a one dimensional subgroup of $SE(3)$ that can be represented as

$$H_p \cong \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} | \theta \in \mathbb{R} \right\}. \tag{16}$$

It is easy to check that the Lie algebra of H_p is

$$Lie(H_p) = T_{id}H_p = span_{\mathbb{R}} \left\{ e_p := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}. \tag{17}$$

Therefore, based on (14),

$$\forall H = \begin{bmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 \\ 0 & 0 & 1 & h_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in H_p$$

there exists $\theta = h_{34}/p$ such that $\exp(\theta e_p) = H$. For

$$SE(2) = SO(2) \ltimes \mathbb{R}^2 \cong \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x \\ \sin(\theta) & \cos(\theta) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} | \theta \in \mathbb{S}^1,\ x, y \in \mathbb{R} \right\}, \tag{18}$$

the corresponding Lie algebra is $span_{\mathbb{R}}\{e_1, e_2, e_6\}$. Based on Lemma 5.3,

$$\forall H = \begin{bmatrix} 1 & 0 & 0 & h_{14} \\ 0 & 1 & 0 & h_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(2),$$

$\exp(h_{14}e_1 + h_{24}e_2) = H$, and otherwise for a general element of $SE(2)$,

$$H = \begin{bmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{21} & h_{22} & 0 & h_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(2),$$

there exists $\theta = \text{atan2}(h_{21}, h_{11})$, where, based on (14), one has

$$\begin{aligned} &\exp\left(\theta e_6 + \left(\frac{\theta h_{24}}{2} + \frac{\theta h_{14}}{2}\cot(\frac{\theta}{2})\right)e_1 + \left(\frac{\theta h_{24}}{2}\cot(\frac{\theta}{2}) - \frac{\theta h_{14}}{2}\right)e_2\right) \\ &= \begin{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} &= \begin{bmatrix} 1-\cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & 1-\cos(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\theta} & 0 \\ \frac{1}{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\theta h_{24}}{2} + \frac{\theta h_{14}}{2}\cot(\frac{\theta}{2}) \\ \frac{\theta h_{24}}{2}\cot(\frac{\theta}{2}) - \frac{\theta h_{14}}{2} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} h_{14} \\ h_{24} \\ 0' \end{bmatrix}. \end{aligned} \tag{20}$$

Hence, the exponential map of $SE(2)$ is surjective, and since $SE(2) \times \mathbb{R}$ is the direct product of two subgroups with surjective exponential maps, its own exponential map is also surjective.

In the case of

$$H_p \ltimes \mathbb{R}^2 \cong \left\{ \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x \\ \sin(\theta) & \cos(\theta) & 0 & y \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} | \theta, x, y \in \mathbb{R} \right\}, \tag{21}$$

the Lie algebra is equal to $span_{\mathbb{R}}\{e_p, e_1, e_2\}$. If $\theta \in \{2\pi\mathbb{Z}\} \setminus \{0\}$, then

$$H = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & p\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \in H_p \ltimes \mathbb{R}^2,$$

and there does not exist any $\tau \in span_{\mathbb{R}}\{e_p, e_1, e_2\}$ such that $\exp(\tau) = H$. Therefore, for $H_p \ltimes \mathbb{R}^2$ the exponential map is not surjective. $\square$

The following proposition presents closed form formulae for $\exp(ad_{xi})$, for any $\xi \in se(3)$, and its integral that are used in the Differential Kinematics of open-chain multi-body systems with displacement subgroups.

Proposition 5.4. *For any element $\xi = [w^T, \omega^T]^T \in se(3)$, where $\omega, w \in \mathbb{R}^3$ and $\omega \neq 0$, expressed in the basis $\{e_1, ..., e_6\}$,*

$$ad_\xi = \begin{bmatrix} \tilde{\omega} & \tilde{w} \\ 0_{3\times3} & \tilde{\omega} \end{bmatrix},$$

$$\exp(ad_\xi) = \begin{bmatrix} \exp(\tilde{\omega}) & \frac{1}{\|\omega\|^2}\left[[\tilde{\omega}, \tilde{w}], \exp(\tilde{\omega})\right] + \frac{\partial}{\partial\mu}|_{\mu=1} \exp\left(\frac{\tilde{\omega}\omega^T w}{\|\omega\|^2}\mu\right) \\ 0_{3\times3} & \exp(\tilde{\omega}) \end{bmatrix}, \tag{22}$$

where $[\cdot, \cdot]$ is the matrix commutator, $\exp(\tilde{\omega})$ is evaluated using (15) and,

$$\int_0^1 \exp(x\, ad_\xi)\, dx = \begin{bmatrix} M_1 & M_2 \\ 0_{3\times3} & M_1 \end{bmatrix}, \tag{23}$$

where,
$M_1 = id_3 + \frac{\tilde{\omega}}{\|\omega\|^2}(1 - \cos(\|\omega\|)) + \frac{\tilde{\omega}^2}{\|\omega\|^2}\left(1 - \frac{1}{\|\omega\|}\sin(\|\omega\|)\right)$*, and*
$M_2 = \frac{1}{\|\omega\|^2}\left[[\tilde{\omega}, \tilde{w}], M_1\right] - \frac{\tilde{\omega}}{\omega^T w} + \left(\frac{\tilde{\omega}}{\omega^T w} - \frac{\tilde{\omega}^2}{\|\omega\|^2}\right)\cos\left(\frac{\omega^T w}{\|\omega\|}\right) + \left(\frac{\tilde{\omega}}{\|\omega\|} + \frac{\tilde{\omega}^2}{\|\omega\|\omega^T w}\right)\sin\left(\frac{\omega^T w}{\|\omega\|}\right).$
For the case $\omega = 0$,

$$\exp(ad_\xi) = \begin{bmatrix} id_3 & \tilde{w} \\ 0_{3\times3} & id_3 \end{bmatrix},$$

and

$$\int_0^1 \exp(x\, ad_\xi)\, dx = \begin{bmatrix} id_3 & \tilde{w}/2 \\ 0_{3\times3} & id_3 \end{bmatrix}.$$

PROOF. *Case 1)* When $\omega = 0$,

$$ad_\xi = \begin{bmatrix} 0_{3\times 3} & \tilde{w} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}.$$

Using the Taylor expansion of the matrix exponential, $\exp(ad_\xi) = \sum_{i=0}^{\infty} \left(ad_\xi^i / i!\right)$, and the fact that ad_ξ is nilpotent of degree two, i.e., $ad_\xi^i = 0$ for $i \geq 2$, it is easy to show the result.

Case 2) To prove the result for $\omega \neq 0$, the following lemma is required.

Lemma 5.5. *$\forall \omega, w \in \mathbb{R}^3$ and $\tilde{\omega} \in so(3)$,*

(i) $\tilde{\omega}^2 = \omega\omega^T - \|\omega\|^2 id_3$ *[2],*

(ii) $\tilde{\omega}^3 = -\|\omega\|^2 \tilde{\omega}$ *[2],*

(iii) $\tilde{\omega} w = -\tilde{w}\omega = \omega \times w$,

(iv) $\widetilde{\tilde{\omega} w} = [\tilde{\omega}, \tilde{w}]$.

The proof for the above lemma is a straight forward computation. Now, consider the Adjoint operator corresponding to the element H, Ad_H, for

$$H = \begin{bmatrix} id_3 & \frac{-\tilde{\omega} w}{\|\omega\|^2} \\ 0_{1\times 3} & 1 \end{bmatrix} \in SE(3),$$

and its action on $\xi \in se(3)$. Based on Lemma 5.5,

$$\begin{aligned} \xi' : &= Ad_H \xi = \begin{bmatrix} id_3 & -\frac{[\tilde{\omega}, \tilde{w}]}{\|\omega\|^2} \\ 0_{3\times 3} & id_3 \end{bmatrix} \begin{bmatrix} w \\ \omega \end{bmatrix} = \begin{bmatrix} w - \tilde{\omega}\tilde{w}\frac{\omega}{\|\omega\|^2} + \tilde{w}\tilde{\omega}\frac{\omega}{\|\omega\|^2} \\ \omega \end{bmatrix} \\ &= \begin{bmatrix} w + \left(\omega\omega^T - \|\omega\|^2 id_3\right) \frac{w}{\|\omega\|^2} \\ \omega \end{bmatrix} \begin{bmatrix} \frac{(\omega^T w)\omega}{\|\omega\|^2} \\ \omega \end{bmatrix} =: \begin{bmatrix} h\omega \\ \omega \end{bmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \exp(ad_{\xi'}) = \sum_{i=0}^{\infty} \frac{ad_{\xi'}^i}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{bmatrix} \tilde{\omega}^i & i(h\tilde{\omega})^i \\ 0_{3\times 3} & \tilde{\omega}^i \end{bmatrix} &= \begin{bmatrix} \exp(\tilde{\omega}) & \sum_{i=1}^{\infty} \frac{(h\tilde{\omega})^i}{(i-1)!} \\ 0_{3\times 3} & \exp(\tilde{\omega}) \end{bmatrix} \\ &= \begin{bmatrix} \exp(\tilde{\omega}) & \frac{\partial}{\partial \mu}|_{\mu=1} \exp(h\tilde{\omega}\mu) \\ 0_{3\times 3} & \exp(\tilde{\omega}) \end{bmatrix}. \end{aligned}$$

According to the definition of the adjoint operator, one has the equality of operators $ad_\xi = Ad_{H^{-1}} ad_{\xi'} Ad_H$. Based on this equality and the facts that $Ad_{H^{-1}} = (Ad_H)^{-1}$ and $\exp(ABA^{-1}) = A\exp(B)A^{-1}$, $\forall A, B \in \mathbb{R}^{n\times n}$ and A invertible, $\exp(ad_\xi) = Ad_{H^{-1}} \exp(ad_{\xi'}) Ad_H$. A straightforward calculation proves the first part of the proposition. For the second part of the proposition,

$$\begin{aligned}&\int_0^1 \exp(x\, ad_\xi)\, dx \\ &= \int_0^1 \begin{bmatrix} \exp(x\tilde{\omega}) & \frac{1}{x^2\|\omega\|^2}\left[[x\tilde{\omega}, x\tilde{w}], \exp(x\tilde{\omega})\right] + \frac{\partial}{\partial\mu}|_{\mu=1} \exp(xh\tilde{\omega}\mu) \\ 0_{3\times 3} & \exp(x\tilde{\omega}) \end{bmatrix} dx.\end{aligned}$$

Since the matrix commutator is a bilinear operator, and the integral operator and partial derivative can commute,

$$\begin{aligned}&\int_0^1 \exp(x\, ad_\xi)\, dx \\ &= \begin{bmatrix} \int_0^1 \exp(x\tilde{\omega})\, dx & \frac{1}{\|\omega\|^2}\left[[\tilde{\omega}, \tilde{w}], \int_0^1 \exp(s\tilde{\omega})\, dx\right] + \frac{\partial}{\partial\mu}|_{\mu=1} \int_0^1 \exp(xh\tilde{\omega}\mu)\, dx \\ 0_{3\times 3} & \int_0^1 \exp(x\tilde{\omega})\, dx \end{bmatrix}.\end{aligned}$$

Using (15) and substituting $h = \frac{\omega^T w}{\|\omega\|^2}$, one can show the second part of the proposition. □

6. Case Study

In this section, the kinematic analysis of a mobile manipulator moving on a spacecraft is performed to elaborate the computational aspects of the proposed formulation for Forward and Differential Kinematics of open-chain multi-body systems. The spacecraft can be considered as a six-d.o.f. moving base for the mobile manipulator that is shown in Figure 1. The multi-body system $MS(6) = \{(B_i, A_i) | i = 0, ..., 6,\ B_i \subset A_i\}$ consists of two branches and six joints. The first branch consists of B_0 to B_5. The second branch contains B_6 and joint six is its last joint. Joint one is a free joint, the second joint is a nonholonomic three-d.o.f. planar joint, the next joint is a three-d.o.f. spherical joint and the rest of the joints are one-d.o.f. revolute joints. The coordinate frames assigned to $A_0, ..., A_6$ at the initial configuration are shown in Figure 2. In the sequel, the joint parameters are specified, and Forward and Differential Kinematics maps of $MS(6)$ are determined. Note that, in the following, a basis for V_{j} at the initial configuration is denoted by $\{\hat{X}_{\mathrm{j}}, \hat{Y}_{\mathrm{j}}, \hat{Z}_{\mathrm{j}}\}$,

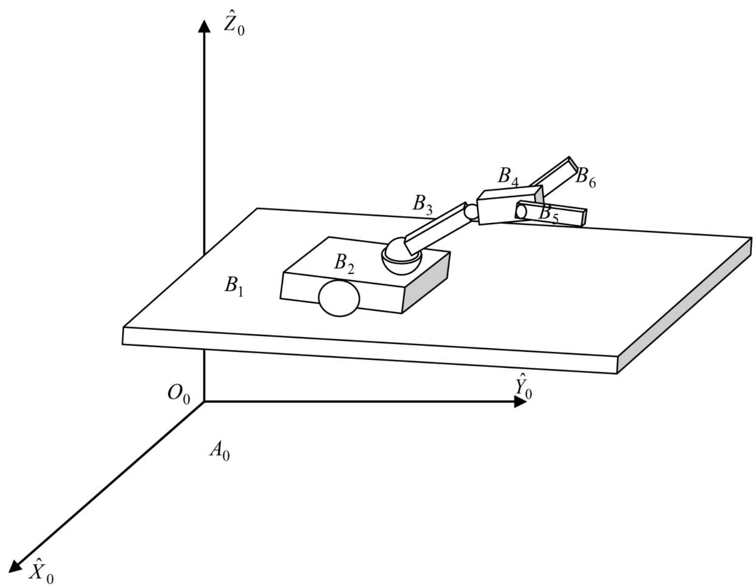

Figure 1: A mobile manipulator on a six d.o.f. moving base

and the linear operator ${}^0\tau_{\mathrm{j}}^{\mathrm{j}-1}$ in the chosen coordinates is represented by the matrix ${}^0T_{\mathrm{j}}^{\mathrm{j}-1}$.

6.1. Forward Kinematics

The first joint is a six-d.o.f. holonomic joint between B_0 and B_1. The classic joint parameters are $q_1 = [x_1, y_1, z_1, \theta_{1,x}, \theta_{1,y}, \theta_{1,z}]^T$, where $[x_1, y_1, z_1]^T$ is the position of $H_1^0(t)(O_1)$ with respect to $H_1^0(0)(O_1)$ and expressed in V_0, and $[\theta_{1,x}, \theta_{1,y}, \theta_{1,z}]^T$ is the rotation angles of V_1 with respect to the axes of V_1 at the initial configuration. Therefore, the local coordinate chart φ_1 for Q_1 is

$$\varphi_1(q_1) = \begin{bmatrix} R(\theta_{1,x}, \hat{X}_1)R(\theta_{1,y}, \hat{Y}_1)R(\theta_{1,z}, \hat{Z}_1) & [x_1, y_1, z_1]^T \\ 0_{1\times 3} & 1 \end{bmatrix},$$

where $R(\theta, \hat{W})$ is the 3×3 rotation matrix corresponding to θ radian rotation about the vector $\hat{W}$. For this coordinate chart, any element of $Lie(P_0)$ corresponding to the relative pose of B_1 with respect to B_0 is parameterized with the screw joint parameters $s_1 = [s_{1,1}, ..., s_{1,6}]^T$, such that

$${}^0T_1^0 s_1 = \left(Ad_{H_1^0(0)}\right)(d_{id_6}\iota_1)(d_0\varphi_1)\, s_1.$$

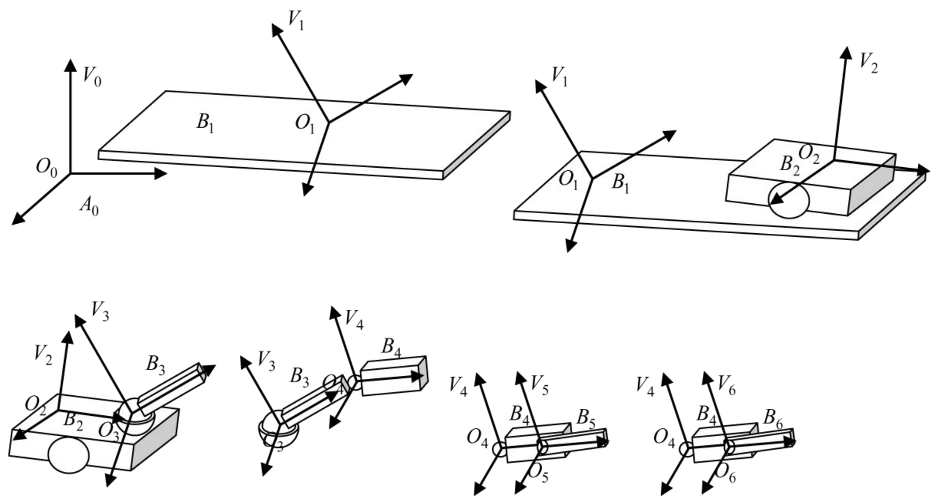

Figure 2: Coordinate frames assigned to $A_0, ..., A_6$ at the initial configuration

With some basic calculations one can show that

$$\frac{\partial \varphi_1}{\partial x_1}|_0 = e_1, \frac{\partial \varphi_1}{\partial y_1}|_0 = e_2, \frac{\partial \varphi_1}{\partial z_1}|_0 = e_3, \frac{\partial \varphi_1}{\partial \theta_{1,x}}|_0 = e_4, \frac{\partial \varphi_1}{\partial \theta_{1,y}}|_0 = e_5, \text{and} \frac{\partial \varphi_1}{\partial \theta_{1,z}}|_0 = e_6,$$

which coincides with the basis selected for $se(3) \cong Lie(P_1)$. For this joint since $Q_1 = P_1$, $d_{id_6}\iota_1$ and $d_0\varphi_1$ are equal to the identity matrix. In the basis $\{e_1, ..., e_6\}$,

$$\forall H_i^j(0) = \begin{bmatrix} R_i^j(0) & p_i^j(0) \\ 0_{1\times 3} & 1 \end{bmatrix}$$

the Adjoint operator can be represented by the matrix [27]

$$Ad_{H_i^j(0)} = \begin{bmatrix} R_i^j(0) & \tilde{p}_i^j(0) R_i^j(0) \\ 0_{3\times 3} & R_i^j(0) \end{bmatrix}.$$

Therefore, ${}^0T_1^0 s_1 = Ad_{H_1^0(0)} s_1$.

Joint number two is a three-d.o.f. nonholonomic joint between B_1 and B_2. The classic joint parameters can be chosen as $q_2 = [x_2, y_2, \theta_{2,z}]^T$, where $[x_2, y_2, 0]^T$ is the position of $H_2^1(t)(O_2)$ with respect to $H_2^1(0)(O_2)$ and expressed in V_2, and $\theta_{2,z}$ is the rotation angle of V_2 about $\hat{Z}_2$. Hence, the local coordinate chart φ_2 for Q_2 is

$$\varphi_2(q_2) = \begin{bmatrix} R(\theta_{2,z}) & R(\theta_{2,z})[x_2, y_2]^T \\ 0_{1\times 2} & 1 \end{bmatrix},$$

where $R(\theta_{2,z})$ is the 2×2 rotation matrix for $\theta_{2,z}$. For this coordinate chart, any element of $Lie(P_0)$ corresponding to the relative pose of B_2 with respect to B_1 is parameterized by the screw joint parameters $s_2 = [s_{2,1}, s_{2,2}, s_{2,3}]^T$, such that

$$^0T_2^1 s_2 = \left(Ad_{H_2^1(0)}\right)(d_{id_3}\iota_2)(d_0\varphi_2)\, s_2,$$

where

$$d_{id_3}\iota_2 \frac{\partial \varphi_2}{\partial x_2}|_0 = e_1, d_{id_3}\iota_2 \frac{\partial \varphi_2}{\partial y_2}|_0 = e_2, \text{ and } d_{id_3}\iota_2 \frac{\partial \varphi_2}{\partial \theta_{2,z}}|_0 = e_6.$$

Thus,

$$^0T_2^1 s_2 = Ad_{H_2^0(0)} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}^T s_2.$$

The third joint is a three-d.o.f. holonomic joint between B_2 and B_3. The classic joint parameters are $q_3 = [\theta_{3,x}, \theta_{3,y}, \theta_{3,z}]^T$, and the local coordinate chart for Q_3 is $\varphi_3(q_3) = R(\theta_{3,x}, \hat{X}_3)R(\theta_{3,y}, \hat{Y}_3)R(\theta_{3,z}, \hat{Z}_3)$. The elements of $Lie(P_0)$ corresponding to the relative poses of B_3 with respect to B_2 are parameterized by the screw joint parameters $s_3 = [s_{3,1}, s_{3,2}, s_{3,3}]^T$, such that

$$^0T_3^2 s_3 = Ad_{H_3^0(0)} \begin{bmatrix} 0_{3\times 3} \\ id_3 \end{bmatrix} s_3.$$

Joint 4 is a one-d.o.f. revolute joint, its classic joint parameter is $q_4 = \theta_{4,z}$, and the local coordinate chart for Q_4 is $\varphi_4(q_4) = R(\theta_{4,z})$.The line in $Lie(P_0)$ corresponding to the relative pose of B_4 with respect to B_5 is parameterized by the screw joint parameter s_4, such that

$$^0T_4^3 s_4 = Ad_{H_4^0(0)}[0, ..., 1]^T s_4.$$

By a simple calculation

$$^0T_4^3 = \begin{bmatrix} p_4^0(0) \times^0 \hat{Z}_4 \\ ^0\hat{Z}_4 \end{bmatrix},$$

where $^0\hat{Z}_4$ is the joint screw axis expressed in V_0. Hence, $^0T_4^3 s_4$ coincides with the argument of the exponential map in the existing product of exponentials formula for a revolute joint [1, 2, 25]. Similarly, for the fifth and sixth joints

$$^0T_5^4 s_5 = Ad_{H_5^0(0)}[0, ..., 1]^T s_5,$$
$$^0T_6^4 s_6 = Ad_{H_6^0(0)}[0, ..., 1]^T s_6,$$

respectively.

Therefore, based on (6), the Forward Kinematics map corresponding to $MS(6)$ is

$$FK(s) = \begin{bmatrix} \exp({}^0T_1^0 s_1) ... \exp({}^0T_5^4 s_5) H_5^0(0) \\ \exp({}^0T_1^0 s_1) ... \exp({}^0T_6^4 s_6) H_6^0(0) \end{bmatrix},$$

where exp is the matrix exponential for $SE(3)$ that can be evaluated by (14) and $s = [s_1^T, ..., v_6^T]^T$.

According to the calculation performed in the case of joint four, for a serial-link multi-body system with revolute and/or prismatic joints, where the multi-body system consists of one branch, the above formulation for FK reduces to the existing product of exponentials formula.

6.2. Differential Kinematics

Based on Proposition 5.1 and 5.4, the Jacobian maps of B_5 and B_6 with respect to B_0 and expressed in V_0, i.e., ${}^0J_5^0(s)$ and ${}^0J_6^0(s)$, can be determined as 6×14 matrices. The nonholonomic constraints at the second joint can be expressed in terms of the classical joint parameters as

$$C_2(q_2)\dot{q}_2 = [0, 1, 0]\dot{q}_2 = 0,$$

which indicates that the mobile base cannot drift side way. The annihilator of C_2 can be selected to be

$$\bar{C}_2(q_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T,$$

and therefore using (3b) and (5)

$$\Sigma_2(s_2) = \begin{bmatrix} \frac{\sin(s_{2,3})}{s_{2,3}} & s_{2,2}\frac{(\cos(s_{2,3})+s_{2,3}\sin(s_{2,3})-1)}{s_{2,3}^2} + s_{2,1}\frac{(\cos(s_{2,3})+\sin(s_{2,3})/s_{2,3})}{s_{2,3}} \\ \frac{(\cos(s_{2,3})-1)}{s_{2,3}} & s_{2,1}\frac{(1-\cos(s_{2,3})-s_{2,3}\sin(s_{2,3}))}{s_{2,3}^2} + s_{2,2}\frac{(\cos(s_{2,3})-\sin(s_{2,3})/s_{2,3})}{s_{2,3}} \\ 0 & 1 \end{bmatrix}.$$

Note that when $s_{2,3} = 0$,

$$\Sigma_2(s_2) = \begin{bmatrix} 1 & 0 & 0 \\ s_{2,2}/2 & -s_{2,1}/2 & 1 \end{bmatrix}^T.$$

Finally, according to (13) the modified Jacobian of the multi-body system $MS(6)$ becomes

$$\bar{J}(s) = \begin{bmatrix} {}^0\bar{J}_5^0(s) & 0_{6\times13} \\ 0_{6\times13} & {}^0\bar{J}_6^0(s) \end{bmatrix},$$

which can be calculated as a 12×26 matrix using Proposition 5.1 and 5.4.

7. Conclusion

An extension of the product of exponentials formula for Forward and Differential Kinematics of generic open-chain multi-body systems with multi-d.o.f., holonomic and nonholonomic joints was formalized using Lie group theory and differential geometry. Towards this goal, multi-d.o.f. joints were classified and the notion of displacement subgroup was generalized. It was shown that the relative configuration manifolds of such joints were Lie groups, and the exponential map was surjective for all types of displacement subgroups except for one type. The screw joint parameters were defined, and their relationship with the classic joint parameters was formalized. The nonholonomic constraints in the Pfaffian form were considered on displacement subgroups, and by introducing admissible screw joint speeds the Jacobian of an open-chain multi-body system was modified, accordingly. The proposed generalized exponential formulation for Forward and Differential Kinematics is independent of the intermediate coordinate assignment to the bodies and the choice of the joint parameterization and a basis for the Lie algebra of the relative configuration manifold. The computational aspects of the developed formulation were explored through an example where Forward and Differential Kinematics of a mobile manipulator mounted on a spacecraft were calculated.

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Figure 1: A mobile manipulator on a six d.o.f. moving base

Figure 2: Coordinate frames assigned to $A_0, ..., A_6$ at the initial configuration

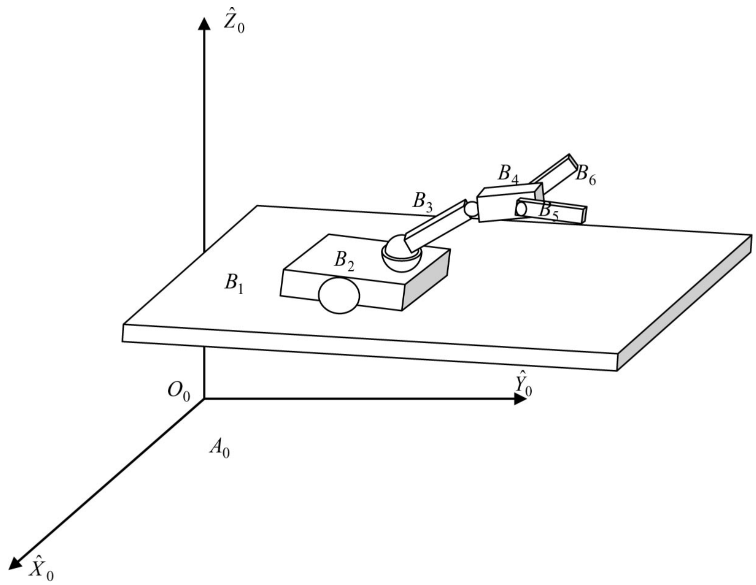

Figure 1: A mobile manipulator on a six d.o.f. moving base

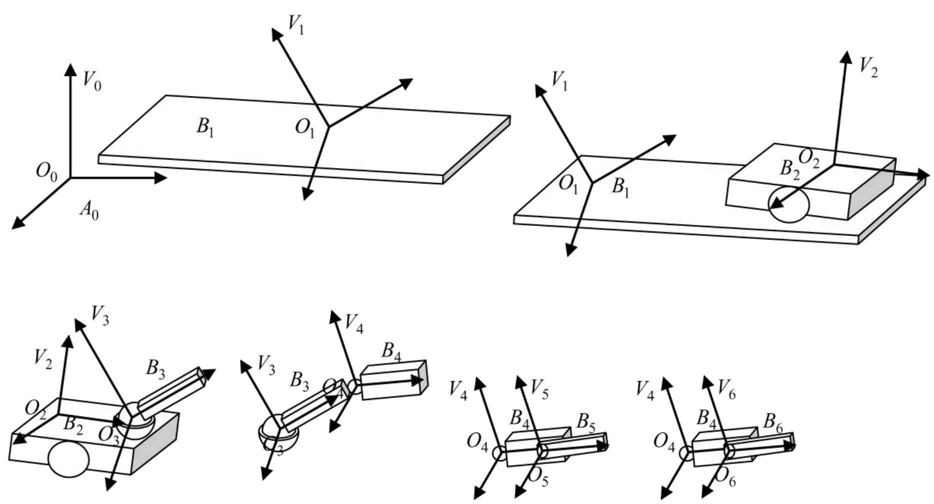

Figure 2: Coordinate frames assigned to $A_0, ..., A_6$ at the initial configuration

Table 1: Categories of displacement subgroups

Dim.	Subgroups of $SE(3)$/displacement subgroups			
6	$SE(3) = SO(3) \ltimes \mathbb{R}^3$ free[a]			
4	$SE(2) \times \mathbb{R}$ planar+prismatic[b]			
3	$SE(2) = SO(2) \ltimes \mathbb{R}^2$ planar	$SO(3)$ ball (spherical)	$\mathbb{R}^3$ 3-d.o.f. prismatic	$H_p \ltimes \mathbb{R}^2$ 2-d.o.f. prismatic + helical[c]
2	$SO(2) \times \mathbb{R}$ cylindrical[d]	$\mathbb{R}^2$ 2-d.o.f. prismatic		
1	$SO(2)$ revolute	$\mathbb{R}$ prismatic	H_p helical	
0	$\{e\}$ fixed[a]			

[a] These two subgroups are the trivial subgroups of $SE(3)$.
[b] The axis of the prismatic joint is always perpendicular to the plane of the planar joint.
[c] The axis of the helical joint is always perpendicular to the plane of the 2-d.o.f. prismatic joint.
[d] The axis of the revolute and prismatic joints are always aligned.