Nonholonomic Dynamical Reduction of Open-chain Multi-body Systems: A Geometric Approach

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Abstract

This paper studies the geometry behind nonholonomic Hamilton's equation to present a two-stage reduction procedure for the dynamical equations of nonholonomic open-chain multi-body systems with multi-degree-of-freedom joints. In this process, we use the Chaplygin reduction and an almost symplectic reduction theorem. We first restate the Chaplygin reduction theorem on cotangent bundle for nonholonomic Hamiltonian mechanical systems with symmetry. Then, under some conditions we extend this theorem to include a second reduction stage using an extended version of the symplectic reduction theorem for almost symplectic manifolds. We briefly introduce the displacement subgroups and accordingly open-chain multi-body systems consisting of such joints. For a holonomic open-chain multi-body system, the relative configuration manifold corresponding to the first joint is a symmetry group. Hence, we focus on a class of nonholonomic distributions on the configuration manifold of an open-chain multi-body system that is invariant under the action of this group. As the first stage of reduction procedure, we perform the Chaplygin reduction for such systems. We then introduce a number of sufficient conditions for a reduced system to admit more symmetry due to the action of the relative configuration manifolds of other joints. Under these conditions, we present the second stage of the reduction process for nonholonomic open-chain multi-body systems with multi-degree-of-freedom joints. Finally, we explicitly derive the reduced dynamical equations in the local coordinates for an example of a two degree-of-freedom crane mounted

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on a four-wheel car to illustrate the results of this paper.

Keywords: Nonholonomic open-chain multi-body system, Chaplygin reduction, almost symplectic manifold, nonholonomic Hamiltonian mechanical system

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Operators.

L_r	Left composition/translation by r
R_r	Right composition/translation by r
Ad_r	Adjoint operator corresponding to r
ad_ξ	adjoint operator corresponding to ξ
$[\xi, \eta]$	Lie bracket or matrix commutator
$T_m f$	Tangent map corresponding to the map f at the element m
$T_m^* f$	Cotangent map corresponding to the map f at the element m
$T_m M$	Tangent space of the manifold M at the element m
TM	Tangent bundle of the manifold M
$T_m^* M$	Cotangent space of the manifold M at the element m
$T^* M$	Cotangent bundle of the manifold M
$\exp(\xi)$	Group/matrix exponential of ξ
$Lie(G)$	Lie algebra of the Lie group G
$Lie^*(G)$	Dual of the Lie algebra of the Lie group G
G_μ	Coadjoint isotropy group for $\mu \in Lie^*(G)$
$\ltimes$	Semi-direct product of groups
$\ll \cdot, \cdot \gg$	Euclidean metric
$\|v\|_h$	Norm of the vector v with respect to the metric h
$\langle \cdot, \cdot \rangle$	Canonical pairing of the elements of tangent and cotangent space
$\mathcal{L}_X$	Lie derivative with respect to the vector field X
ξ_M	Vector field on the manifold M induced by the infinitesimal action of $\xi \in Lie(G)$
$\iota_X \Omega$	Interior product of the differential form Ω by the vector field X
$\mathfrak{X}(M)$	Space of all vector fields on the manifold M
$\Omega^2(M)$	Space of all differential 2-forms on the manifold M
$d\Omega$	Exterior derivative of the differential form Ω
dH	Exterior derivative of the function H
M/G	Quotient manifold corresponding to a free and proper action of the Lie group G

1. Introduction

Eliminating the nonholonomic constraints in the dynamical equations of nonholonomic Hamiltonian and Lagrangian systems is helpful to better understand the inherent behaviour of such systems, design controllers, and develop accurate simulation packages. A historic example is the work of Chaplygin in 1911, where he eliminated the Lagrange multipliers for nonholonomic Lagrangian systems with cyclic parameters [1]. This result was extended to the Lagrangian mechanical systems with non-abelian symmetry by Koiller in 1992 [2]. On the Hamiltonian side, van der Schaft and Maschke [3] eliminated the Lagrange multipliers by projecting Hamilton's equation onto a submanifold of the cotangent bundle corresponding to the nonholonomic distribution. They worked with the Poisson structure of the cotangent bundles.

In this paper, by *reducing the dynamical equations* we mean expressing the differential equations representing a (Lagrangian or Hamiltonian) system on a manifold whose dimension is less than the original phase space of the system, by restricting to a submanifold of the phase space or quotienting a group action. In the following, we first review the existing reduction theories for nonholonomic Hamiltonian and Lagrangian mechanical systems. Then, we state the contributions of this paper.

1.1. Background

From a geometric point of view, a Hamiltonian system is a vector field on a symplectic manifold (i.e., the phase space) that satisfies Hamilton's equation. If this system is invariant under a group action [4], and there is a conserved quantity (*momentum*) for the system, then we can perform the symplectic reduction [4, 5]. For a *mechanical system*, the phase space is the cotangent bundle of the configuration manifold $T^*\mathcal{Q}$ that admits a canonical symplectic 2-form, which is $\Omega_{can} := -dp \wedge dq$ in coordinates. The Hamiltonian in this case is the summation of the kinetic and potential energy. Let G be a Lie group. The cotangent lift of its proper action on $\mathcal{Q}$ is symplectic. If the Hamiltonian of the system is also invariant under this action, the group G is called the *symmetry group* of the mechanical system, and the system is called a *Hamiltonian mechanical system with symmetry* [6, 4]. Cotangent bundle also admits a canonical Poisson bracket $\{\cdot, \cdot\}$. For a mechanical system with symmetry, the Poisson bracket is also invariant under the cotangent lifted action. As a result, the Poisson bracket on $T^*\mathcal{Q}$ descends to a Poisson bracket on the quotient manifold $(T^*\mathcal{Q})/G$. This process is called

Poisson reduction [4, 7]. Momentum map is the main difference of the Poisson and symplectic reductions. This approach unifies the Euler-Poincaré and Lagrange-Poincaré equations for mechanical systems with symmetry [4]. Both of the above-mentioned reduction theories were developed and extended to Lagrangian systems in the 1990s [8, 9, 10]. Since during a reduction process the trivial behaviour of a mechanical system due to symmetry is eliminated, the system behaviour in the reduced phase space is more explicit. These theories are also helpful for extracting coordinate-independent control laws for mechanical systems with symmetry [7, 11].

A *nonholonomic* mechanical system with symmetry is a mechanical system with symmetry together with a G-invariant distribution $\mathcal{D}$. This distribution is a linear sub-bundle of $T\mathcal{Q}$, where the velocity of the physical trajectories of the system should lie. Generally, this distribution is non-involutive, and it is the result of kinematic nonholonomic constraints such as rolling without slipping. If $\mathcal{D}$ is involutive, we say that the constraints are *holonomic*. The distinguishing characteristics of nonholonomic systems (comparing to holonomic systems) are that (i) they satisfy the Lagrange-d'Alembert principle instead of the Hamilton principle [12], and (ii) the momentum is not generally conserved for them.

A Chaplygin system is a nonholonomic mechanical system with symmetry such that the sub-bundle corresponding to the infinitesimal G-action is complementary to the distribution $\mathcal{D}$ at each point $q \in \mathcal{Q}$. On the Lagrangian side, Chaplygin in [1] reduces such systems for abelian symmetries. His result was generalized to non-abelian symmetry groups, by Koiller [2]. A more general approach resulting in Lagrange-d'Alembert-Poincaré equation [7, 13] is reported in [12]. This method is centred at defining a nonholonomic connection as the summation of an Ehresmann connection and the mechanical connection, and introducing a nonholonomic momentum map. The analogue of this approach in Poisson formalism is also explained in [7], which is originated in a paper by van der Schaft and Maschke [3].

On the Hamiltonian side, Bates and Śniatycki first show that the vector field that is the solution of Hamilton's equation for a nonholonomic system is a section of the distribution $T\left(\mathbb{F}L(\mathcal{D})\right) \cap \{v \in T(T^*\mathcal{Q}) |\, T\pi_{\mathcal{Q}} v \in \mathcal{D}\} \subseteq T(T^*\mathcal{Q})$. Here, the fibre-wise linear map $\mathbb{F}L\colon T\mathcal{Q} \to T^*\mathcal{Q}$ is the Legendre transformation. Then under the symmetry hypotheses, after restricting Hamilton's equation to this distribution, they show that the flow of this vector field descends to the quotient manifold $\mathbb{F}L(\mathcal{D})/G$ [14, 15, 16, 17]. Later on, based on this method of reduction, which is called distributional Hamiltonian approach [17], the Noether theorem is extended to nonholonomic systems and accordingly a two-stage reduction procedure is introduced. In the first stage, the symplectic reduction theorem is applied to reduce Hamilton's equation by a normal subgroup $G_0 \subseteq G$, whose momentum is conserved, and yields another distributional Hamiltonian system. For the second stage, the method in [14] is used to reduce the equations by G/G_0 [18]. This method is further extended to singular reduction of nonholonomic systems, and it is reformulated for almost Poisson manifolds in [19]. An almost Poisson manifold is a manifold equipped with a bracket that satisfies the properties of the Poisson bracket except the Jacobi identity.

Multi-body systems are examples of interconnected mechanical systems, consisting of rigid bodies and joints. In order to unify the Hamiltonian and Lagrangian formalisms for these systems with nonholonomic constraints, Dirac structures can be employed. Van der Schaft and Maschke introduce the notion of implicit Hamiltonian system [20], while its analogue, implicit Lagrangian system, is formulated by Yoshimura and Marsden [21, 22]. Both of these formalisms have been used to model systems with nonholonomic constraints, and subsequently their reduction theories have been developed [21, 23, 24].

An extension of the reduction of Chaplygin systems is also reported in the concept of nonholonomic Hamilton-Jacobi theory [25, 26], which uses an extended version of the symplectic reduction theorem in the presence of further symmetries of the system to reduce a Chaplygin system in two stages. The first stage is performing the Chaplygin reduction [2], which results in a non-degenerate 2-form for describing the reduced Hamilton's equation. In the second stage, under some assumptions an almost symplectic reduction [27] is performed. Note that for Hamiltonian systems, researchers have also introduced the notion of reduction by stages [28], which consists of several stages of symplectic reduction of a Hamiltonian system. As opposed to the two-stage reduction of nonholonomic systems, in reduction by stages the resulting system is always symplectic in each stage of reduction. On the Lagrangian side, the notion of Lagrangian reduction by stages has also been introduced [8]. In particular, Routh reduction by stages can be regarded as the Lagrangian analogue of the symplectic reduction by stages on the Hamiltonian side [29].

1.2. Structure of the Paper and Statement of Contributions

In the field of robotics, researchers have been studying kinematics, dynamics and control of multi-body systems, specifically nonholonomic systems, from a geometric point of view [30]. However, the dynamical reduction of nonholonomic multi-body systems has been mostly focused on the restriction of dynamical equations to a submanifold of the phase space [31, 32]. And, the existing symmetries in such systems has not been explored. In this paper we systematically develop a two-stage reduction process, based on the Chaplygin reduction and an almost symplectic reduction theorem, for dynamical equations of nonholonomic open-chain multi-body systems. We try to exploit the proposed reduction procedure by finding the symmetry groups in the form of a Cartesian product of subgroups of $SE(3)$, and trivializing the principal bundles appearing in this process.

The following section gives a brief review of the Chaplygin reduction theorem on cotangent bundles, and its extension to a two-stage reduction process based on an almost symplectic reduction theorem. In Section 3, we introduce nonholonomic open-chain multi-body systems with multi-d.o.f. joints, and derive their corresponding Lagrangian and Hamiltonian. The symmetries of the kinetic energy metric of multi-body systems is investigated in Section 4. The main results of this paper are presented in Section 5, where we introduce the notion of nonholonomic open-chain multi-body systems with symmetry, and derive the reduced coordinate-independent dynamical equations of generic nonholonomic open-chain multi-body systems with symmetry in the cotangent bundle of a quotient manifold. Finally in Section 6, as an example, we reduce the dynamical equations of a two d.o.f. crane mounted on a four-wheel car, and we conclude the paper in Section 7.

2. Two-stage Dynamical Reduction of Nonholonomic Hamiltonian Mechanical Systems with Symmetry

In this section, we revisit Koiller's result [2], known as Chaplygin reduction theorem, for nonholonomic mechanical systems with symmetry in the Hamiltonian framework. An example of such systems is a two-wheel cart that was studied in [2]. Then, we extend his result to a two-stage reduction procedure, using an almost symplectic reduction theorem under some hypothesis. We study, for instance, the two-stage dynamical reduction of a four-wheel crane in Section 6 that admits a big symmetry group.

2.1. Chaplygin Reduction on Cotangent Bundles

For a mechanical system, the Lagrangian $L\colon T\mathcal{Q} \to \mathbb{R}$ is defined by $L(v_q) := \frac{1}{2}K_q(v_q, v_q) - V(q)$, where $\forall q \in \mathcal{Q}$ we have $v_q \in T_q\mathcal{Q}$, and $K_q\colon T_q\mathcal{Q} \times T_q\mathcal{Q} \to \mathbb{R}$ is a Riemannian metric, called the *kinetic energy metric*, and where $V\colon \mathcal{Q} \to \mathbb{R}$ is a smooth function, called the *potential energy function*. This Lagrangian is hyper-regular, and its corresponding Legendre transformation $\mathbb{F}L_q\colon T_q\mathcal{Q} \to T_q^*\mathcal{Q}$ is equal to the fibre-wise linear isomorphism that is induced by the metric K:

$$\langle \mathbb{F}L_q(v_q), w_q \rangle := K_q(v_q, w_q). \qquad \forall v_q, w_q \in T_q\mathcal{Q} \tag{2.1}$$

As a result, $\forall p_q \in T^*\mathcal{Q}$ the Hamiltonian $H\colon T^*\mathcal{Q} \to \mathbb{R}$ of the system is

$$H(p_q) := \frac{1}{2}K_q(\mathbb{F}L_q^{-1}(p_q), \mathbb{F}L_q^{-1}(p_q)) + V(q), \tag{2.2}$$

which is the total energy of the mechanical system. In this paper, we consider mechanical systems with (linear) nonholonomic constraints, which are called *nonholonomic Hamiltonian mechanical system*. Such a system can be identified by a five-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D})$, where $\Omega_{can} \in \Omega^2(T^*\mathcal{Q})$ is the canonical 2-form on the cotangent bundle $T^*\mathcal{Q}$, the distribution $\mathcal{D} \subset T\mathcal{Q}$ is a regular, non-involutive, linear distribution that is bracket generating, and H and K are defined as above. Suppose that $\mathcal{D}$ is specified by a set of (constraint) 1-forms $\{\omega_s \subset T^*\mathcal{Q} |\, s = 1, \cdots, f\}$ on $\mathcal{Q}$ such that

$$\mathcal{D}(q) = \{v_q \in T_q\mathcal{Q} |\, \omega_s(q)(v_q) = 0, s = 1, \cdots, f\}, \tag{2.3}$$

where f is the number of nonholonomic constraints. A nonholonomic Hamiltonian mechanical system satisfies Hamilton-d'Alembert equation (along with f constraint equations)

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & id \\ -id & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} - \begin{bmatrix} 0 \\ \omega^T(q)\kappa \end{bmatrix}, \qquad \omega(q)\dot{q} = 0, \tag{2.4}$$

where $\kappa = [\kappa_1 \quad \cdots \quad \kappa_f]^T$ is a set of Lagrange multipliers, (q, p) is a set of local coordinates for $T^*\mathcal{Q}$, and we define

$$\omega(q) := \begin{bmatrix} \omega_1(q) \\ \vdots \\ \omega_f(q) \end{bmatrix}.$$

Here, with an abuse of notation, we use $\omega_s(q)$ as the s^{th} row of the matrix whose kernel is the nonholonomic distribution. In this equation we have $2\dim(\mathcal{Q}) + f$ variables and the same number of equations consisting of $2\dim(\mathcal{Q})$ dynamical equations and f constraint equations.

Let $\mathcal{G}$ be a Lie group with the Lie algebra $Lie(\mathcal{G})$. Consider an action of $\mathcal{G}$ on $\mathcal{Q}$, and denote the action by $\Phi_{\mathfrak{g}} \colon \mathcal{Q} \to \mathcal{Q}$, $\forall \mathfrak{g} \in \mathcal{G}$. Note that wherever we consider the action of $\mathcal{G}$ on the tangent and cotangent bundle of $\mathcal{Q}$, we implicitly use the naturally induced action maps on those spaces. Consider the infinitesimal action of $Lie(\mathcal{G})$ on $\mathcal{Q}$. For any $\xi \in Lie(\mathcal{G})$, this action induces a vector field $\xi_{\mathcal{Q}} \in \mathfrak{X}(\mathcal{Q})$ such that $\forall q \in \mathcal{Q}$,

$$\xi_{\mathcal{Q}}(q) = \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} \left(\Phi_{\exp(\epsilon\xi)}(q)\right) =: \phi_q(\xi). \tag{2.5}$$

Here, we denote the fibre-wise linear map corresponding to the infinitesimal action of $Lie(\mathcal{G})$ by $\phi_q \colon Lie(\mathcal{G}) \to T_q\mathcal{Q}$. Now, consider the fibre-wise linear map $\mathbf{M} \colon T^*\mathcal{Q} \to Lie^*(\mathcal{G})$

$$\langle \mathbf{M}_q(p_q), \xi \rangle := \langle p_q, \xi_{\mathcal{Q}}(q) \rangle = \left\langle \phi_q^*(p_q), \xi \right\rangle, \tag{2.6}$$

called *momentum map*, which is an Ad^*-equivariant map with respect to the $\mathcal{G}$-action.

Definition 2.1. A nonholonomic Hamiltonian mechanical system $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D})$ is called a *nonholonomic Hamiltonian mechanical system with symmetry*, if H, K and $\mathcal{D}$ are invariant under the action of $\mathcal{G}$. We denote such a system by a six-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G})$, as defined above.

For instance, the nonholonomic distribution $\mathcal{D}$ is invariant under the coordinate change for the nonholonomic multi-bodies with fixed directions of constraint forces in the body coordinate frame.

Definition 2.2. (Chaplygin System) A nonholonomic Hamiltonian mechanical system with symmetry $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G})$ is called a *Chaplygin system* if $\forall q \in \mathcal{Q}$ we also have the dimensional assumption

$$T_q\mathcal{Q} = \mathcal{D}(q) \oplus T_q\mathcal{O}_q(\mathcal{G}), \tag{2.7}$$

where $\mathcal{O}_q(\mathcal{G}) := \{\, \Phi_{\mathfrak{g}}(q) |\, \mathfrak{g} \in \mathcal{G}\}$ is the orbit of the $\mathcal{G}$-action through q.

In this paper, we restrict our attention to nonholonomic Hamiltonian mechanical systems with symmetry whose symmetry group $\mathcal{G}$ possesses a Lie subgroup $G \subseteq \mathcal{G}$ that satisfies the definition of a Chaplygin system. That is, $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, G)$ is a Chaplygin system. Under this assumption, we can perform the *Chaplygin reduction* that was presented by Koiller in [2]. Here, we briefly review Koiller's result on cotangent bundles. For a Chaplygin system we may define a principal connection

$$\widehat{\mathcal{A}} := \sum_{s=1}^{f} \omega_s \varepsilon_s, \tag{2.8}$$

corresponding to the G-principal bundle $\mathcal{Q} \to \mathcal{Q}/G$. In this equation, $\{\varepsilon_s |\, s = 1, \cdots, f\}$ is a basis for $Lie(G)$. The horizontal vector bundle of this connection is the distribution $\mathcal{D}$ and its vertical vector bundle is the tangent to the G-orbits $\{\eta_{\mathcal{Q}}(q) |\, \eta \in Lie(G)\}$. Accordingly, we denote the horizontal lift corresponding to the G-principal bundle by $\widehat{\mathrm{hl}}_q$, which maps tangent vectors on the quotient manifold $\widehat{\mathcal{Q}} := \mathcal{Q}/G$ to the horizontal vector bundle. We also denote the momentum map corresponding to the G action by $\widehat{\mathbf{M}} \colon T^*\mathcal{Q} \to Lie^*(G)$. Let $\widehat{K}$ be the metric on $\widehat{\mathcal{Q}}$ induced by the kinetic energy metric K. That is, $\forall \widehat{u}_{\widehat{q}}, \widehat{w}_{\widehat{q}} \in T_{\widehat{q}}\widehat{\mathcal{Q}}$ we have

$$\widehat{K}_{\widehat{q}}(\widehat{u}_{\widehat{q}}, \widehat{w}_{\widehat{q}}) = K_q(\widehat{\mathrm{hl}}_q(\widehat{u}_{\widehat{q}}), \widehat{\mathrm{hl}}_q(\widehat{w}_{\widehat{q}})).$$

Then, we can define Legendre transformation on $\widehat{\mathcal{Q}}$ by

$$\left\langle \mathbb{F}\widehat{L}_{\widehat{q}}(\widehat{u}_{\widehat{q}}), \widehat{w}_{\widehat{q}} \right\rangle := \widehat{K}_{\widehat{q}}(\widehat{u}_{\widehat{q}}, \widehat{w}_{\widehat{q}}),$$

where $\widehat{q} \in \widehat{\mathcal{Q}}$ and $\widehat{u}_{\widehat{q}}, \widehat{w}_{\widehat{q}} \in T_{\widehat{q}}\widehat{\mathcal{Q}}$. Let $\mathcal{M} := \mathbb{F}L(\mathcal{D})$ be the vector subbundle of $T^*\mathcal{Q}$ corresponding to the nonholonomic distribution, which is invariant under the G-action. We may also define the horizontal lift map

$\widehat{\mathrm{hl}}_q^{\mathcal{M}}\colon T^*_{\widehat{q}}\widehat{\mathcal{Q}} \to \mathcal{M}(q)$ to $\mathcal{M}$ by

$$\widehat{\mathrm{hl}}_q^{\mathcal{M}} := \mathbb{F}L_q \circ \widehat{\mathrm{hl}}_q \circ \mathbb{F}\widehat{L}_{\widehat{q}}^{-1}. \tag{2.9}$$

Accordingly, we can define the reduced Hamiltonian $\widehat{H}\colon T^*\widehat{\mathcal{Q}} \to \mathbb{R}$ on the cotangent bundle of the quotient manifold by

$$\widehat{H}(\widehat{p}_{\widehat{q}}) = H \circ \widehat{\mathrm{hl}}_q^{\mathcal{M}}(\widehat{p}_{\widehat{q}}). \tag{2.10}$$

Theorem 2.3 (Chaplygin Reduction [2])**.** *A Chaplygin system $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, G)$, whose solution curves satisfy the nonholonomic Hamilton's equation (2.4), can be reduced to a system $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K})$, where $\widehat{\Omega}$ is a non-degenerate 2-form on the cotangent bundle of the quotient manifold $\widehat{\mathcal{Q}}$, $\widehat{H}$ is the reduced Hamiltonian defined by (2.10), $\widehat{K}$ is the induced metric on $\widehat{\mathcal{Q}}$. The reduced system satisfies Hamilton's equation for the reduced Hamiltonian $\widehat{H}$ with the non-degenerate 2-form $\widehat{\Omega}$. That is*

$$[\widehat{\Omega}](\widehat{p}_{\widehat{q}}) \begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} =: \begin{bmatrix} [\widehat{\Upsilon}](\widehat{p}_{\widehat{q}}) & -id \\ id & 0 \end{bmatrix} \begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix}, \tag{2.11}$$

where $[\widehat{\Omega}]\colon T(T^*\widehat{\mathcal{Q}}) \to T^*(T^*\widehat{\mathcal{Q}})$ is the naturally induced vector bundle map corresponding to the 2-form $\widehat{\Omega}$. The elements of the anti-symmetric matrix $[\widehat{\Upsilon}](\widehat{p}_{\widehat{q}})$ are specified in the following:

$$\widehat{\Upsilon}_{ij}(\widehat{p}_{\widehat{q}}) := \sum_{a=1}^{f} \widehat{\mathcal{F}}_a \left(\left(\frac{\partial \widehat{A}^a_j}{\partial \widehat{q}_i} - \frac{\partial \widehat{A}^a_i}{\partial \widehat{q}_j} \right) - \sum_{l<k} \widehat{\mathcal{E}}^a_{lk} (\widehat{A}^l_i \widehat{A}^k_j - \widehat{A}^l_j \widehat{A}^k_i) \right) \tag{2.12}$$

such that $\forall \mathfrak{g} \in \mathcal{G}$ we have the following identities:

$$\begin{aligned} \widehat{\mathcal{A}} &=: Ad_{\mathfrak{g}} \begin{bmatrix} T_{\mathfrak{g}} L_{\mathfrak{g}^{-1}} & \widehat{A}_{\widehat{q}} \end{bmatrix}, \\ \widehat{\mathcal{F}} &: = \widehat{\boldsymbol{M}} \circ \widehat{\mathrm{hl}}^{\mathcal{M}}(\widehat{p}_{\widehat{q}}), \\ [\varepsilon_l, \varepsilon_k] &=: \sum_{a=1}^{f} \widehat{\mathcal{E}}^a_{lk} \varepsilon_a, \end{aligned}$$

for $l, k \in \{1, \cdots, f\}$, and $i, j \in \{1, \cdots, \dim(\mathcal{Q}) - f\}$.

2.2. Second Stage Reduction of Nonholonomic Hamiltonian Mechanical Systems with Symmetry

In this section, we use a modified version of Noether's theorem to further reduce the dynamical equations of the reduced Chaplygin system $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K})$. Then using an extended version of the symplectic reduction theorem for almost symplectic manifolds [27] and embedding version of the cotangent bundle reduction [28], we identify the reduced space with a vector sub-bundle of the cotangent bundle of a quotient manifold. This process is an extension of the two-stage reduction introduced by Ohsawa *et al.* in the concept of Hamilton-Jacobi theory of nonholonomic systems [25]. An example of a nonholonomic system that admits a conserved quantity in the reduced phase space is a two-wheel cart with a dumbbell-shaped body rotating perpendicular to the plane of motion of the cart. The rotational momentum of the dumbbell is constant.

Proposition 1 (Noether's theorem for nonholonomic systems)**.** *For a reduced Chaplygin system $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K})$, a function $\widehat{h}\colon T^*\widehat{\mathcal{Q}} \to \mathbb{R}$ is constant of motion if and only if its Hamiltonian vector field $\widehat{X}_{\widehat{h}} \in \mathfrak{X}(T^*\widehat{\mathcal{Q}})$ corresponding to the non-degenerate 2-form $\widehat{\Omega}$ preserves the Hamiltonian $\widehat{H}$.*

Proof. Suppose that there exists a function $\widehat{h}\colon T^*\widehat{\mathcal{Q}} \to \mathbb{R}$ that is constant of motion. Its Hamiltonian vector field $\widehat{X}_{\widehat{h}}$ corresponding to $\widehat{\Omega}$ is defined by

$$\widehat{X}_{\widehat{h}} := [\widehat{\Omega}]^{-1} \begin{bmatrix} \frac{\partial \widehat{h}}{\partial \widehat{q}} \\ \frac{\partial \widehat{h}}{\partial \widehat{p}} \end{bmatrix}.$$

Then we have

$$0 = \mathcal{L}_{\widehat{X}}\widehat{h} = \left\langle d\widehat{h}, \widehat{X} \right\rangle = \left\langle [\widehat{\Omega}]\widehat{X}_{\widehat{h}}, \widehat{X} \right\rangle = -\left\langle [\widehat{\Omega}]\widehat{X}, \widehat{X}_{\widehat{h}} \right\rangle = -\left\langle d\widehat{H}, \widehat{X}_{\widehat{h}} \right\rangle = -\mathcal{L}_{\widehat{X}_{\widehat{h}}}\widehat{H},$$

where $\widehat{X} = [\dot{\widehat{q}}^T \;\; \dot{\widehat{p}}^T]^T$, and $\mathcal{L}_{\widehat{X}}\widehat{h}$ is the Lie derivative of $\widehat{h}$ along the trajectories of the reduced system. Conversely, based on the above calculation, if there exists a Hamiltonian vector field $\widehat{X}_{\widehat{h}}$ for the function $\widehat{h}$ that preserves the Hamiltonian $\widehat{H}$, then $\widehat{h}$ is a constant of motion. □

Note that if $\widehat{h}\colon T^*\widehat{\mathcal{Q}} \to \mathbb{R}$ is a constant of motion, then there exists a G-invariant function $h\colon T^*\mathcal{Q} \to \mathbb{R}$ defined by the equality $\widehat{h} = h \circ \widehat{\mathrm{hl}}^{\mathcal{M}}$ that is constant on the trajectories of the vector field $X = [\dot{q}^T \;\; \dot{p}^T]^T \in \mathfrak{X}(T^*\mathcal{Q})$, i.e.,

$\mathcal{L}_X h = 0$. Now assume that a reduced Chaplygin system $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K})$ is invariant under a group action in the following sense. Let $\mathcal{N} \subset \mathcal{G}$ be a Lie group with a free and proper action, such that

(i) The Hamiltonian $\widehat{H}$ is invariant under the $\mathcal{N}$-action.

(ii) Along the trajectories of the vector field $\widehat{X}$, this action has a conserved momentum map $\breve{\mathbf{M}}\colon T^*\widehat{\mathcal{Q}} \to Lie^*(\mathcal{N})$, defined by (2.6). That is for every $\zeta \in Lie(\mathcal{N})$,

$$\mathcal{L}_{\widehat{X}} \left(\left\langle \breve{\mathbf{M}}, \zeta \right\rangle \right) = 0.$$

Let $\vartheta \in Lie^*(\mathcal{N})$ be a regular value for the momentum $\breve{\mathbf{M}}$. Since $\breve{\mathbf{M}}$ is also a momentum map with respect to the non-degenerate 2-form $\widehat{\Omega}$, we can perform the almost symplectic reduction presented in [27] at ϑ. And consequently, drop the dynamics to $\breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta$, where $\mathcal{N}_\vartheta = \{\mathfrak{n} \in \mathcal{N} | \operatorname{Ad}^*_{\mathfrak{n}}\vartheta = \vartheta\}$ is the coadjoint isotropy group of $\vartheta \in Lie^*(\mathcal{N})$.

Proposition 2. *Under the assumptions (i) and (ii) stated above, we have*

(i) There exists a non-degenerate 2-form $\widehat{\Omega}_\vartheta \in \Omega^2(\breve{\boldsymbol{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta)$ *that is uniquely characterized by*

$$T^*\pi_\vartheta \widehat{\Omega}_\vartheta = T^* i_\vartheta \widehat{\Omega},$$

where $i_\vartheta\colon \breve{\boldsymbol{M}}^{-1}(\vartheta) \hookrightarrow T^*\widehat{\mathcal{Q}}$ *and* $\pi_\vartheta\colon \breve{\boldsymbol{M}}^{-1}(\vartheta) \to \breve{\boldsymbol{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta$ *are the canonical inclusion and projection maps, respectively.*

(ii) The reduced Hamilton's equation (2.11) can be further reduced to

$$[\widehat{\Omega}_\vartheta]\widehat{X}_\vartheta = d\widehat{H}_\vartheta, \tag{2.13}$$

where $[\widehat{\Omega}_\vartheta]$ *and* $\widehat{X}_\vartheta \in \mathfrak{X}(\breve{\boldsymbol{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta)$ *are respectively the naturally induced vector bundle map corresponding to the 2-form* $\widehat{\Omega}_\vartheta$ *and the vector field that represent the reduced dynamics and* $\widehat{H}_\vartheta$ *is uniquely defined by*

$$\widehat{H}_\vartheta \circ \pi_\vartheta = \widehat{H} \circ i_\vartheta.$$

In the theory of cotangent bundle reduction, there exist two equivalent ways to identify the symplectic reduced space with cotangent bundles and coadjoint orbits [28], either of which can be used to identify $\breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta$ along with its almost symplectic structure $\widehat{\Omega}_\vartheta$:

(i) Embedding version: in which the reduced space $\breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta$ is identified with a vector sub-bundle of the cotangent bundle of $\breve{\mathcal{Q}} := \widehat{\mathcal{Q}}/\mathcal{N}_\vartheta$.

(ii) Bundle version: in which the reduced space $\breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta$ is identified by a (locally trivial) fibre bundle of the coadjoint orbit through ϑ over the cotangent bundle of $\widehat{\mathcal{Q}}/\mathcal{N}$.

In this paper, the embedding version of the cotangent bundle reduction is used to write the reduced Hamilton's equation (2.13) in $T^*\breve{\mathcal{Q}}$. The quotient manifold $\widehat{\mathcal{Q}}/\mathcal{N}$ gives rise to the principal bundle $\widehat{\mathcal{Q}} \to \widehat{\mathcal{Q}}/\mathcal{N}$. We work with the mechanical connection $\breve{\mathcal{A}}\colon T\widehat{\mathcal{Q}} \to Lie(\mathcal{N})$ as the principal connection:

$$\breve{\mathcal{A}}_{\widehat{q}} = \breve{\mathbb{I}}_{\widehat{q}}^{-1} \circ \breve{\mathbf{M}}_{\widehat{q}} \circ \mathbb{F}\widehat{L}_{\widehat{q}}, \tag{2.14}$$

where the locked inertia tensor $\breve{\mathbb{I}}\colon Lie(\mathcal{N}) \to Lie^*(\mathcal{N})$ is defined by

$$\breve{\mathbb{I}}_{\widehat{q}} = \breve{\phi}_{\widehat{q}}^* \circ \mathbb{F}\widehat{L}_{\widehat{q}} \circ \breve{\phi}_{\widehat{q}}.$$

Here, the map $\breve{\phi}_{\widehat{q}}\colon Lie(\mathcal{N}) \to T_{\widehat{q}}\widehat{\mathcal{Q}}$ corresponds to the infinitesimal action of $Lie(\mathcal{N})$ on $\widehat{\mathcal{Q}}$, defined by (2.5). For any $\vartheta \in Lie^*(\mathcal{N})$, let us consider the action of $\mathcal{N}$ restricted to the subgroup $\mathcal{N}_\vartheta = \{\mathfrak{n} \in \mathcal{N} | \operatorname{Ad}^*_{\mathfrak{n}}\vartheta = \vartheta\}$. Similarly, for this action we have a principal bundle $\breve{\pi}\colon \widehat{\mathcal{Q}} \to \breve{\mathcal{Q}} := \widehat{\mathcal{Q}}/\mathcal{N}_\vartheta$. The locked inertia tensor $\breve{\mathbb{I}}^\vartheta_{\widehat{q}}\colon Lie(\mathcal{N}_\vartheta) \to Lie^*(\mathcal{N}_\vartheta)$ and the (mechanical) connection $\breve{\mathcal{A}}^\vartheta_{\widehat{q}}\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to Lie(\mathcal{N}_\vartheta)$ ($\forall \widehat{q} \in \widehat{\mathcal{Q}}$) for the $\mathcal{N}_\vartheta$-action are defined by

$$\breve{\mathbb{I}}^\vartheta_{\widehat{q}} := (\breve{\phi}^\vartheta_{\widehat{q}})^* \circ \mathbb{F}\widehat{L}_{\widehat{q}} \circ \breve{\phi}^\vartheta_{\widehat{q}}, \tag{2.15}$$

and

$$\breve{\mathcal{A}}^\vartheta_{\widehat{q}} := (\breve{\mathbb{I}}^\vartheta_{\widehat{q}})^{-1} \circ \breve{\mathbf{M}}^\vartheta_{\widehat{q}} \circ \mathbb{F}\widehat{L}_{\widehat{q}}, \tag{2.16}$$

respectively. Here, the map $\breve{\phi}^\vartheta_{\widehat{q}}\colon Lie(\mathcal{N}_\vartheta) \to T_{\widehat{q}}\widehat{\mathcal{Q}}$ corresponds to the infinitesimal $\mathcal{N}_\vartheta$-action, and $\breve{\mathbf{M}}^\vartheta\colon T^*\widehat{\mathcal{Q}} \to Lie^*(\mathcal{N}_\vartheta)$ is the momentum map for the $\mathcal{N}_\vartheta$-action, which are defined based on (2.5) and (2.6). For the principal bundle $\breve{\pi}\colon \mathcal{Q} \to \breve{\mathcal{Q}}$ with the principal connection $\breve{\mathcal{A}}^\vartheta$, the horizontal lift map is denoted by $\breve{\mathrm{hl}}_{\widehat{q}}\colon T\breve{\mathcal{Q}} \to T\widehat{\mathcal{Q}}$, where $\breve{q} := \breve{\pi}(\widehat{q})$.

Now, let us consider the 1-form $\alpha_\vartheta := \breve{\mathcal{A}}^* \vartheta \in \Omega^1(\widehat{\mathcal{Q}})$.

Lemma 2.4. *The 1-form α_ϑ takes values in $\breve{\boldsymbol{M}}^{-1}(\vartheta)$, and it is invariant under $\mathcal{N}_\vartheta$-action.*

Proof. Using the definition of the momentum map and principal connection, we have $\forall \zeta \in Lie(\mathcal{N})$

$$\left\langle \breve{\mathbf{M}}(\alpha_\vartheta), \zeta \right\rangle = \langle \alpha_\vartheta, \zeta_{\widehat{\mathcal{Q}}} \rangle = \left\langle \breve{\mathcal{A}}^*_{\widehat{q}} \vartheta, \breve{\phi}_{\widehat{q}}(\zeta) \right\rangle = \left\langle \vartheta, (\breve{\mathcal{A}}_{\widehat{q}} \circ \breve{\phi}_{\widehat{q}})(\zeta) \right\rangle = \langle \vartheta, \zeta \rangle .$$

As a result, $\alpha_\vartheta \in \breve{\mathbf{M}}^{-1}(\vartheta)$.

Finally, consider an arbitrary element $\mathfrak{n} \in \mathcal{N}_\vartheta$, and denote its action and induced tangent action simply by $\mathfrak{n} \cdot \widehat{q}$ and $\mathfrak{n} \cdot \widehat{v}_{\widehat{q}}$, respectively. Based on the Ad^*-equivariance of $\breve{\mathcal{A}}$ and the definition of $\mathcal{N}_\vartheta$, one can show that α_ϑ is $\mathcal{N}_\vartheta$ invariant. For all $\widehat{v}_{\widehat{q}} \in T_{\widehat{q}}\widehat{\mathcal{Q}}$,

$$\begin{aligned}\langle \alpha_\vartheta(\mathfrak{n} \cdot \widehat{q}), \mathfrak{n} \cdot \widehat{v}_{\widehat{q}} \rangle &= \left\langle \breve{\mathcal{A}}^*_{\mathfrak{n}\cdot\widehat{q}} \vartheta, \mathfrak{n} \cdot \widehat{v}_{\widehat{q}} \right\rangle = \left\langle \vartheta, \breve{\mathcal{A}}_{\mathfrak{n}\cdot\widehat{q}}(\mathfrak{n} \cdot \widehat{v}_{\widehat{q}}) \right\rangle \\ &= \left\langle \vartheta, \mathrm{Ad}_{\mathfrak{n}^{-1}} \breve{\mathcal{A}}_{\widehat{q}}(\widehat{v}_{\widehat{q}}) \right\rangle = \left\langle \mathrm{Ad}^*_{\mathfrak{n}^{-1}} \vartheta, \breve{\mathcal{A}}_{\widehat{q}}(\widehat{v}_{\widehat{q}}) \right\rangle = \left\langle \vartheta, \breve{\mathcal{A}}_{\widehat{q}}(\widehat{v}_{\widehat{q}}) \right\rangle .\end{aligned}$$

□

According to the Cartan Structure Equation [33], $\forall \widehat{Z}, \widehat{Y} \in \mathfrak{X}(\widehat{\mathcal{Q}})$ the exterior derivative of α_ϑ evaluated on $\widehat{Y}$ and $\widehat{Z}$ is equal to

$$d\alpha_\vartheta(\widehat{Z}, \widehat{Y}) = \left\langle \vartheta, d\breve{\mathcal{A}}(\widehat{Z}, \widehat{Y}) \right\rangle = \left\langle \vartheta, \breve{\mathcal{B}}(\widehat{Z}, \widehat{Y}) + [\breve{\mathcal{A}}(\widehat{Z}), \breve{\mathcal{A}}(\widehat{Y})] \right\rangle , \qquad (2.17)$$

where $\breve{\mathcal{B}}_{\widehat{q}}(\widehat{Z}_{\widehat{q}}, \widehat{Y}_{\widehat{q}}) := (d\breve{\mathcal{A}})_{\widehat{q}}(\breve{\mathrm{hor}}_{\widehat{q}}(\widehat{Z}_{\widehat{q}}), \breve{\mathrm{hor}}_{\widehat{q}}(\widehat{Y}_{\widehat{q}})) = -\breve{\mathcal{A}}_{\widehat{q}}([\breve{\mathrm{hor}}(\widehat{Z}), \breve{\mathrm{hor}}(\widehat{Y})]_{\widehat{q}})$ is the curvature of the connection $\breve{\mathcal{A}}$, and $[\cdot, \cdot]$ in (2.17) corresponds to the Lie bracket in $Lie(\mathcal{N})$.

Lemma 2.5. *For all $\zeta \in Lie(\mathcal{N}_\vartheta)$, the interior product of the 2-form $d\alpha_\vartheta$ with $\zeta_{\widehat{\mathcal{Q}}}$ is zero, i.e., $[d\alpha_\vartheta]\zeta_{\widehat{\mathcal{Q}}} = 0$.*

By this lemma and Lemma 2.4 the 2-form $d\alpha_\vartheta$ is basic (it is invariant in the directions of the group action and it does not see those directions); hence, a closed 2-form $\beta_\vartheta \in \Omega^2(\breve{\mathcal{Q}})$ can be uniquely defined by the relation $T^*\breve{\pi}(\beta_\vartheta) = d\alpha_\vartheta$, and its pullback Ξ_ϑ by the cotangent bundle projection

$\pi_{\breve{\mathcal{Q}}}\colon T^*\breve{\mathcal{Q}} \to \breve{\mathcal{Q}}$ will be a closed 2-form on $T^*\breve{\mathcal{Q}}$,

$$\Xi_\vartheta := T^*\pi_{\breve{\mathcal{Q}}}(\beta_\vartheta).$$

There is an embedding $\varphi_\vartheta : \breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta \hookrightarrow T^*\breve{\mathcal{Q}}$ onto the vector sub-bundle $\breve{\mathcal{S}} = \left\{ \breve{p}_{\breve{q}} \in T^*\breve{\mathcal{Q}} \middle| \left\langle \breve{p}_{\breve{q}}, T\breve{\pi}\left(\zeta_{\breve{\mathcal{Q}}}(\breve{q})\right)\right\rangle = 0, \forall\zeta \in Lie(\mathcal{N}) \right\}$ of $T^*\breve{\mathcal{Q}}$ that is identified by

$$\langle \varphi_\vartheta([\gamma_{\widehat{q}}]_\vartheta), T_{\widehat{q}}\breve{\pi}(\widehat{v}_{\widehat{q}})\rangle = \langle \gamma_{\widehat{q}} - \alpha_\vartheta(\widehat{q}), \widehat{v}_{\widehat{q}}\rangle\,, \tag{2.18}$$

Its inverse exists only on $\breve{\mathcal{S}} \subset T^*\breve{\mathcal{Q}}$, and it is a diffeomorphism on this vector sub-bundle. Hence, the map $\varphi_\vartheta^{-1}\colon \breve{\mathcal{S}} \to \breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta$ is only defined on $\breve{\mathcal{S}}$. As the result, we can write the reduced Hamilton's equation (2.13) in $\breve{\mathcal{S}} \subset T^*\breve{\mathcal{Q}}$ as

$$\left([\breve{\Omega}_\vartheta](\breve{q}_1, \widetilde{q}, \widetilde{p}) - [\Xi_\vartheta](\breve{q})\right) \begin{bmatrix} \dot{\breve{q}}_1 \\ \dot{\widetilde{q}} \\ \dot{\widetilde{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \breve{H}_\vartheta}{\partial \breve{q}_1} \\ \frac{\partial \breve{H}_\vartheta}{\partial \widetilde{q}} \\ \frac{\partial \breve{H}_\vartheta}{\partial \widetilde{p}} \end{bmatrix}, \tag{2.19}$$

where $\breve{\Omega}_\vartheta := T^*\varphi_\vartheta^{-1}(\widehat{\Omega}_\vartheta)$, $(\breve{q}_1, \widetilde{q}, \widetilde{p})$ is a set of coordinates for the reduced phase space $\breve{\mathcal{S}}$ (such that $\breve{q} = (\breve{q}_1, \widetilde{q})$), and the Hamiltonian $\breve{H}_\vartheta$ is defined by the relation $\breve{H}_\vartheta := \widehat{H}_\vartheta \circ \varphi_\vartheta^{-1}$.

The 2-form $\Xi_\vartheta \in \Omega^2(\breve{\mathcal{S}})$ is defined by

$$\begin{aligned}
\Xi_\vartheta =& \sum_{i<j}\sum_{a=1}^{d_n} \breve{\mathcal{F}}_a \left(\left(\frac{\partial \breve{A}^a_j}{\partial \widetilde{q}_i} - \frac{\partial \breve{A}^a_i}{\partial \widetilde{q}_j} \right) - \sum_{l<k} \breve{\mathcal{E}}^a_{lk}(\breve{A}^l_i \breve{A}^k_j - \breve{A}^l_j \breve{A}^k_i) \right) (d\widetilde{q}_i \wedge d\widetilde{q}_j) \\
&+ \sum_{i'<j'}\sum_{l<k}\sum_{a=1}^{d_n} \left(\vartheta_a \breve{\mathcal{E}}^a_{lk}(\mathcal{H}^l_{i'}\mathcal{H}^k_{j'} - \mathcal{H}^l_{j'}\mathcal{H}^k_{i'}) \right) (d\breve{q}_{i'} \wedge d\breve{q}_{j'}) \\
=:& \sum_{i'<j'} \breve{\Upsilon}_{i'j'}(\breve{q}) d\breve{q}_{i'} \wedge d\breve{q}_{j'},
\end{aligned} \tag{2.20}$$

such that we have the following identities:

$$\begin{aligned}
\breve{\mathcal{A}} &=: \mathrm{Ad}_{\mathfrak{n}} \left[T_{\mathfrak{n}} L_{\mathfrak{n}^{-1}} \quad \breve{A}\right], \\
\breve{\mathcal{F}} &: = \vartheta^T \mathrm{Ad}_{(e_\vartheta, \breve{q}_1)}, \\
\mathcal{H} &: = -\breve{A}^\vartheta + \left[T_{(e_\vartheta,\breve{q}_1)} R_{(e_\vartheta,\breve{q}_1)} \quad \mathrm{Ad}_{(e_\vartheta,\breve{q}_1)} \breve{A}\right], \\
\breve{\mathcal{A}}^\vartheta &=: \mathrm{Ad}_{\mathfrak{n}_\vartheta} \left[T_{\mathfrak{n}_\vartheta} L_{\mathfrak{n}_\vartheta^{-1}} \quad \breve{A}^\vartheta\right], \\
[E_l, E_k] &=: \sum_{a=1}^{d_n} \breve{\mathcal{E}}_{lk}^a E_a,
\end{aligned}$$

for $l, k \in \{1, \cdots, d_n\}$, and $i, j \in \{1, \cdots \quad, \dim(\mathcal{Q}) - d_n\}$, and $i', j' \in \{1, \cdots, \dim(\mathcal{Q}) - \dim(\mathcal{N}_\vartheta)\}$, $d_n := \dim(\mathcal{N})$ and $\{E_1, \cdots, E_{d_n}\}$ being a basis for $Lie(\mathcal{N})$. The element $\mathfrak{n}_\vartheta$ is an arbitrary element of $\mathcal{N}_\vartheta$ and e_ϑ is the identity element of this Lie group. Note that for a local trivialization of $\mathcal{N}_\vartheta$ principal bundle, we have $\mathfrak{n} = (\mathfrak{n}_\vartheta, \breve{q}_1)$. In matrix form we have

$$[\Xi_\vartheta](\breve{q}) = \begin{bmatrix} [\breve{\Upsilon}](\breve{q}) & 0 \\ 0 & 0 \end{bmatrix},$$

where zero matrices have appropriate size. Now, we can finalize this section in the following theorem.

Theorem 2.6. *We say that a reduced Chaplygin system with symmetry* $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K}, \mathcal{N})$, *whose solution curves satisfy the reduced nonholonomic Hamilton's equation (2.11), can be further reduced to the system* $(\breve{\mathcal{S}} \subset T^*\breve{\mathcal{Q}}, \breve{\Omega}_\vartheta - \Xi_\vartheta, \breve{H}_\vartheta)$. *The 2-form* $\breve{\Omega}_\vartheta := T^*\varphi_\vartheta^{-1}(\widehat{\Omega}_\vartheta)$. *The Hamiltonian* $\breve{H}_\vartheta \colon \breve{\mathcal{S}} \to \mathbb{R}$ *is the further reduced Hamiltonian defined by* $\breve{H}_\vartheta := \widehat{H}_\vartheta \circ \varphi_\vartheta^{-1}$, *and* Ξ_ϑ *is a closed 2-form that is defined by (2.20). The reduced system satisfies Hamilton's equation (2.19) for the Hamiltonian* $\breve{H}_\vartheta$.

3. Nonholonomic Open-chain Multi-body Systems

Let $B_0, \cdots, B_N$ be $N+1$ rigid bodies and $J_1, \cdots, J_M$ be M multi-d.o.f. holonomic or nonholonomic joints. A *multi-body system* is the collection of these $N+1$ bodies and M joints $MBS(N, M) = \{A_i, B_i, J_j \,|\, B_i \subset A_i, i = 0, \cdots, N, j = 1, \cdots, M\}$ such that each joint restricts the relative motion of a body with respect to another.

Definition 3.1. A *nonholonomic open-chain multi-body system* is a multi-body system $MBS(N, N)$, which consists of $N + 1$ rigid bodies and N holonomic or nonholonomic multi-d.o.f. joints, such that there exists a unique path between any two bodies of the multi-body system. In an open-chain multi-body system, bodies with only one neighbouring body are called *extremities*.

Here, we set a convention to simplify the calculations. We label the bodies starting from the inertial coordinate frame (ground), B_0, outwards. That is, we label the bodies connected to B_0 by joints successively as $B_1, \cdots, B_{N_0}$ ($N_0 \leq N$), and we repeat the same procedure for all N_0 bodies starting from B_1. We number the joints using the bigger body label, e.g., we label the joint between B_i and B_j, where $i > j$, as J_i. In this paper we only consider a class of multi-d.o.f. joints whose relative configuration manifold is diffeomorphic to a connected Lie subgroup of $SE(3)$, which are called *displacement subgroup* [34]. Let J_i be a joint connecting B_i to B_j. its relative configuration manifold is denoted by Q_i^j. Considering $r_{i,0}^0 \in SE(3)$ and $r_{j,0}^0 \in SE(3)$ as the initial poses of B_i and B_j with respect to the inertial coordinate frame, respectively, $Q_i^j \cong L_{r_{i,0}^0} R_{(r_{j,0}^0)^{-1}}(Q_i^j) =: \mathcal{Q}_i \subseteq SE(3)$. Here, $L_\bullet \colon SE(3) \to SE(3)$ and $R_\bullet \colon SE(3) \to SE(3)$ are the standard left and right translation maps on $SE(3)$, respectively. We use $\mathcal{Q}_i$ as the relative configuration manifold of the joint J_i. Note that, every $\mathcal{Q}_i$ is a d_i dimensional Lie subgroup of $SE(3)$, where d_i is the number of degrees of freedom of J_i, and $D := \sum_{i=1}^{N} d_i$ is the total number of degrees of freedom of $MBS(N, N)$. Any state of the system can be realized by $q := (q_1, \cdots, q_N) \in \mathcal{Q} := \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_N$, and $\mathcal{Q}$ is the configuration manifold. The manifold $\mathcal{Q}$ along with the group structure induced by $\mathcal{Q}_i$'s is also a Lie group. Let $r_{cm,i} \in SE(3)$ be the initial pose of the centre of mass of B_i with respect to the inertial coordinate frame. We define the map $F \colon \mathcal{Q} \to SE(3) \times \cdots \times SE(3) =: \mathcal{P}$ by

$$F(q) := (q_1 r_{cm,1}, q_1 q_2 r_{cm,2}, \cdots, q_1 \cdots q_N r_{cm,N}), \tag{3.21}$$

where the i^{th} component of this map only contains the joint parameters that are in the path from B_0 to B_i. This map determines the pose of the coordinate frames attached to the centre of mass of all bodies with respect to the inertial coordinate frame. For any motion of $MBS(N, N)$, i.e., a curve $t \mapsto q(t) \in \mathcal{Q}$, the velocity of the centre of mass of the bodies with respect to the inertial coordinate frame (absolute velocity) is calculated by $\dot{\mathfrak{p}} :=$

$\frac{d}{dt}F(q(t)) = T_qF(\dot{q})$.

A *MBS*(N,N) is a mechanical system with the Lagrangian $L\colon T\mathcal{Q} \to \mathbb{R}$ that is determined by $L(v_q) = \frac{1}{2}K_q(v_q, v_q) - V(q)$. In the following, we describe how the Lagrangian L and subsequently the Hamiltonian H of a *MBS*(N,N) is calculated. Let h_i for $i = 1, \cdots, N$ be the left-invariant kinetic energy metric for the rigid body B_i in the *MBS*(N,N). They induce the metric $h := h_1 \oplus \cdots \oplus h_N$ on $\mathcal{P}$, which is left-invariant. The kinetic energy metric of an open-chain multi-body system is defined by $K := T^*F(h)$, where $T^*F(h)$ is the pull back of the metric h by the map F. That is, $\forall q \in \mathcal{Q}$ and $\forall v_q, w_q \in T_q\mathcal{Q}$ we have

$$\begin{aligned} K_q(v_q, w_q) &= h_{F(q)}\left(T_qF(v_q), T_qF(w_q)\right) \\ &= h_{\mathfrak{e}}\left(T_{F(q)}L_{F(q)^{-1}}(T_qF(v_q)), T_{F(q)}L_{F(q)^{-1}}(T_qF(w_q))\right), \end{aligned} \tag{3.22}$$

where $\mathfrak{e}$ is the identity element of the Lie group $\mathcal{P}$ and $L_{\mathfrak{p}}$ is the left translation map by an element $\mathfrak{p} \in \mathcal{P}$. Furthermore, we can simplify the above expression by calculating the following linear map for the *MBS*(N,N):

$$\begin{aligned} &T_{F(q)}L_{F(q)^{-1}}(T_qF) \\ &\quad= \left(\mathrm{Ad}_{r_{cm,1}^{-1}} \oplus \cdots \oplus \mathrm{Ad}_{r_{cm,N}^{-1}}\right) \mathcal{J}_q \left(T_{q_1}(L_{q_1^{-1}} \circ \iota_1) \oplus \cdots \oplus T_{q_N}(L_{q_N^{-1}} \circ \iota_N)\right) \\ &\quad= \begin{bmatrix} \mathrm{Ad}_{r_{cm,1}^{-1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{Ad}_{r_{cm,N}^{-1}} \end{bmatrix} \mathcal{J}_q \begin{bmatrix} T_{q_1}(L_{q_1^{-1}} \circ \iota_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_{q_N}(L_{q_N^{-1}} \circ \iota_N) \end{bmatrix}, \end{aligned}$$

where $\mathcal{J}_q\colon \mathit{Lie}(\mathcal{P}) \to \mathit{Lie}(\mathcal{P})$ is the linear map calculated in the following way. This map is a lower triangular linear map with the blocks of identity map on the diagonal and a combination of zero blocks and the blocks of the linear maps in the form of $\mathrm{Ad}_{(\prod_r q_r)^{-1}}$. Here, $\prod_r q_r$ is the product of elements of the relative configuration manifolds $\mathcal{Q}_i \subseteq SE(3)$, which appear in the unique path from B_0 to B_i. A zero blocks means that a certain joint does not appear in the path connecting B_0 and B_i in the *MBS*(N,N).

In this paper, we consider the potential energy function induced by a constant gravitational field g in the ambient space, i.e., the inertial coordinate frame. The potential energy function for the *MBS*(N,N) is defined as

$$V(q) := \sum_{i=1}^{N} \langle m_i g, F_i(q)(O_i) - O_0 \rangle \,, \tag{3.23}$$

where m_i is the mass of the rigid body B_i, and $F_i(q)$ is the i^{th} component of the map F that can be considered as an isometry between the inertial coordinate frame and the coordinate frame attached to B_i. The points O_0 and O_i are the base points of these coordinate frames, where O_i is located at the centre of mass.

Using the Legendre transformation induced by the metric K, we define the Hamiltonian $H\colon T^*\mathcal{Q} \to \mathbb{R}$ for the *MBS*(N,N) by

$$H(p_q) := \langle p_q, \mathbb{F}L_q^{-1}(p_q)\rangle - L(\mathbb{F}L_q^{-1}(p_q)). \tag{3.24}$$

Here, we remind the reader that $\mathbb{F}L\colon T\mathcal{Q} \to T^*\mathcal{Q}$ is the fibre-wise invertible Legendre transformation induced by the kinetic energy metric, i.e., $\forall v_q, w_q \in T_q\mathcal{Q}$, $\langle \mathbb{F}L_q(v_q), w_q\rangle = K_q(v_q, w_q)$.

Accordingly, the *MBS*(N,N) can be considered as a Hamiltonian mechanical system described by the four-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K)$. Here, the metric K and the Hamiltonian H are defined by (3.22) and (3.24), respectively.

4. Symmetries of the Kinetic Energy Metric of Open-chain Multibody Systems

In this section, in Theorem 4.1 we state that the relative configuration manifold of the first joint is always a symmetry group for the kinetic energy metric K of the *MBS*(N,N). Then, we give some sufficient conditions, under which K is also invariant under the action of a subgroup of the configuration manifold of the rest of the joints.

Theorem 4.1. *For a MBS*(N,N)*, the action of* $\mathcal{G}_1 = \mathcal{Q}_1$ *on* $\mathcal{Q}$ *by left translation on the first component leaves the kinetic energy metric* K *invariant. For any* $\mathfrak{g}_1 \in \mathcal{G}_1$ *we denote the action map by* $\Phi_{\mathfrak{g}_1}\colon \mathcal{Q} \to \mathcal{Q}$ *such that* $\forall q = (q_1, \cdots, q_N) \in \mathcal{Q}$ *we have* $\Phi_{\mathfrak{g}_1}(q) = (\mathfrak{g}_1 q_1, q_2, \cdots, q_N)$.

We also investigate two approaches to identify other groups in addition to the one presented in Theorem 4.1 whose actions leave the kinetic energy metric of a *MBS*(N,N) invariant.

AP1) Identifying symmetry groups due to left invariance of the kinetic energy metric h on $\mathcal{P} = SE(3) \times \cdots \times SE(3)$. See Section 3 for the definition of the metric h.

AP2) Identifying symmetry groups by studying the metric K on $\mathcal{Q}$.

4.1. Identifying Symmetry Groups using AP1

As for the approach AP1, we consider the embedding $F\colon \mathcal{Q} \to \mathcal{P}$, defined by (3.21). For any element $(a_1, \cdots, a_N) \in \mathcal{P}$ we define the group action $\Theta^{\mathcal{N}}_{(a_1,\cdots,a_N)}\colon \mathcal{P} \to \mathcal{P}$ by

$$\Theta^{\mathcal{N}}_{(a_1,\cdots,a_N)}(\mathfrak{p}) := (a_1\mathfrak{p}_1, (a_1a_2)\mathfrak{p}_2, \cdots, (a_1 \cdots a_N)\mathfrak{p}_N),$$

where $\mathfrak{p} = (\mathfrak{p}_1, \cdots, \mathfrak{p}_N) \in \mathcal{P}$. Since the metric h on $\mathcal{P}$ is left-invariant, it is also invariant under this action. That is, we have $T^*\Theta^{\mathcal{N}}_{(a_1,\cdots,a_N)}(h) = h$. This action induces an action on $\mathcal{Q}$, if and only if the image of the map F, i.e., $F(\mathcal{Q})$, is invariant under the action $\Theta^{\mathcal{N}}$ for a Lie subgroup of $\mathcal{P}$. We denote this Lie subgroup by $\mathcal{G}_1 \times \cdots \times \mathcal{G}_N$, where $\mathcal{G}_i \subseteq SE(3)$ $(i = 1, \cdots, N)$ is a Lie subgroup of $SE(3)$. Then the induced action on $\mathcal{Q}$, denoted by $\Phi^{\mathcal{N}}_{(a_1,\cdots,a_N)}\colon \mathcal{Q} \to \mathcal{Q}$, is defined by $\Phi^{\mathcal{N}}_{(a_1,\cdots,a_N)} := F^{-1} \circ \Theta^{\mathcal{N}}_{(a_1,\cdots,a_N)} \circ F$, where $(a_1, \cdots, a_N) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_N$. Here, $F^{-1}\colon F(\mathcal{Q}) \to \mathcal{Q}$ is only defined on the image of the map F. In order to identify the group $\mathcal{G}_1 \times \cdots \times \mathcal{G}_N$, we impose the condition that $F(\mathcal{Q})$ is invariant under the action of this group. By the definition of the map F and $\Theta^{\mathcal{N}}_{(a_1,\cdots,a_N)}$, we have

$$\Theta^{\mathcal{N}}_{(a_1,\cdots,a_N)} \circ F(q) = (a_1q_1r_{cm,1}, (a_1a_2)q_1q_2r_{cm,2}, \cdots, (a_1 \cdots a_N)q_1 \cdots q_Nr_{cm,N}).$$

The image of F is invariant under the group action if and only if we have the following conditions:

$$\begin{aligned}
&a_1 \in \mathcal{Q}_1\\
&q_1^{-1}a_2q_1 \in \mathcal{Q}_2 \qquad \forall q_1 \in \mathcal{Q}_1,\\
&\vdots\\
&(q_1 \cdots q_{N-1})^{-1}a_N(q_1 \cdots q_{N-1}) \in \mathcal{Q}_N \qquad \forall q_1 \in \mathcal{Q}_1 \text{ and } \cdots \text{ and } \forall q_{N-1} \in \mathcal{Q}_{N-1}.
\end{aligned}$$

Hence, the biggest symmetry group $\mathcal{G}_1 \times \cdots \mathcal{G}_N$ that leaves the kinetic energy metric K invariant under the induced action $\Phi^{\mathcal{N}}$ is equal to

$$\begin{aligned}\mathcal{G}_1 \times \cdots \times \mathcal{G}_N =& \{ (a_1, \cdots, a_N) | \, a_1 \in \mathcal{Q}_1, a_2 \in \bigcap_{q_1 \in \mathcal{Q}_1} (q_1 \mathcal{Q}_2 q_1^{-1}), \cdots \\ &, a_N \in \bigcap_{\substack{q_1 \in \mathcal{Q}_1 \\ \cdots \\ q_{N-1} \in \mathcal{Q}_{N-1}}} ((q_1 \cdots q_{N-1}) \mathcal{Q}_N (q_{N-1}^{-1} \cdots q_1^{-1})) \} \subseteq \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_N.\end{aligned}$$

Noteworthy examples of *MBS*(N, N) whose kinetic energy metric K is invariant under the action of this group include but not limited to the systems with identical multi-d.o.f. joints, and systems with commutative joints. In general, this symmetry group may be as small as $\mathcal{G}_1 = \mathcal{Q}_1$, specially when the joints are actuated, since the actuation force may break the symmetry.

4.2. Identifying Symmetry Groups using AP2

For any velocity vector $\dot{q} \in T_q\mathcal{Q}$, we denote the left translation of $\dot{q}$ to $Lie(\mathcal{Q})$ by

$$\tau = (\tau_1, \cdots, \tau_N) := q^{-1}\dot{q} = (q_1^{-1}\dot{q}_1, \cdots, q_N^{-1}\dot{q}_N) \in Lie(\mathcal{Q})$$

Now, let ${}^i\tau_i^j$ $(i, j = 1, \cdots, N)$ be the relative twist of the body B_i with respect to B_j and expressed in the coordinate frame attached to B_i. With the similar calculation we did in Section 3, we have

$${}^i\tau_i^0 = \mathrm{Ad}_{r_{cm,i}^{-1}} \left(\mathrm{Ad}_{(q_2 \cdots q_i)^{-1}}(\tau_1) + \cdots + \mathrm{Ad}_{q_i^{-1}}(\tau_{i-1}) + \tau_i \right)$$

for a sequence of joints connecting B_0 to B_i [34]. Then, the kinetic energy of a *MBS*(N, N) can alternatively be calculated by

$$\frac{1}{2} K_q(\dot{q}, \dot{q}) = \frac{1}{2} \sum_{i=1}^{N} \| \, {}^i\tau_i^0 \, \|_{h_i}^2, \tag{4.25}$$

where h_i denotes the left invariant metric corresponding to the body B_i on $se(3)$, and $\| \cdot \|_{h_i}$ refers to its induced norm on $se(3)$. To simplify the computations, we first consider the case of *MBS*$(2, 2)$ in the sequel, and then we generalize the result for the case of *MBS*$(2, 2)$.

Let $\mathcal{G}_1 = \mathcal{Q}_1$ and $\mathcal{G}_2 \subseteq \mathcal{Q}_2$ be a Lie subgroup of $\mathcal{Q}_2$, and consider the action of $\mathcal{G}_1 \times \mathcal{G}_2$ by left translation on the configuration manifold $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2$, i.e., $\forall (a_1, a_2) \in \mathcal{G}_1 \times \mathcal{G}_2$ we have $(q_1, q_2) \mapsto (a_1 q_1, a_2 q_2)$ for all $q = (q_1, q_2) \in \mathcal{Q}$. It is easy to show that under this action the kinetic energy of the system becomes

$$\frac{1}{2} K_{(a_1 q_1, a_2 q_2)}(a_1 \dot{q}_1, a_2 \dot{q}_2) = \frac{1}{2} \left(\| \operatorname{Ad}_{r_{cm,1}^{-1}} \tau_1 \|_{h_1}^2 + \| \operatorname{Ad}_{r_{cm,2}^{-1}} \left(\operatorname{Ad}_{(a_2 q_2)^{-1}} \tau_1 + \tau_2 \right) \|_{h_2}^2 \right),$$

where $(a_1 \dot{q}_1, a_2 \dot{q}_2)$ denotes the left translation of the velocity vector $(\dot{q}_1, \dot{q}_2)$ to $(a_1 q_1, a_2 q_2)$. As it was expected, the kinetic energy remains invariant under the $\mathcal{G}_1$-action. We define the metric $h'_2 := \operatorname{Ad}^*_{r_{cm,2}^{-1}} (h_2)_e$ on $se(3)$ corresponding to the body B_2. Note that, here $e \in SE(3)$ denotes the identity element of $SE(3)$. Kinetic energy is invariant under the action of $\mathcal{G}_1 \times \mathcal{G}_2$ if and only if it is invariant under the infinitesimal action of all elements $\varpi \in Lie(\mathcal{G}_2)$ at the identity element $e_2 \in \mathcal{G}_2$. As a result, we have the following necessary and sufficient condition for the metric K being invariant under the action of $\mathcal{G}_1 \times \mathcal{G}_2$ by left translation:

$$\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \left(\frac{1}{2} \| \operatorname{Ad}_{(\exp(-\epsilon \varpi) q_2)^{-1}} \tau_1 + \tau_2 \|_{h'_2}^2 \right) = h'_2(\operatorname{Ad}_{q_2^{-1}} \operatorname{ad}_{\varpi}(\tau_1), \operatorname{Ad}_{q_2^{-1}} \tau_1 + \tau_2) = 0. \tag{4.26}$$

$\forall q_2 \in \mathcal{Q}_2, \quad \forall \tau_1 \in Lie(\mathcal{Q}_1)$ and $\forall \tau_2 \in Lie(\mathcal{Q}_2)$

The largest Lie sub-algebra of $Lie(\mathcal{Q}_2)$ whose elements satisfy the above condition is the Lie algebra of $\mathcal{G}_2$, and $\mathcal{G}_2$ is identified by integrating this Lie sub-algebra on $\mathcal{Q}_2$. Noteworthy examples of the systems that admit such a symmetry group are any two commutative joints, a planar cart with a rotary joint orthogonal to it, and a planar cart moving on a rotating disc. With similar calculations, we can extend this result to the case of $MBS(N, N)$, and write the condition (4.26) as

$$\sum_{\mathfrak{i}=2}^{N} h'_{\mathfrak{i}}(\operatorname{Ad}_{(q_2 \cdots q_{\mathfrak{i}})^{-1}} \operatorname{ad}_{\varpi}(\tau_1), \operatorname{Ad}_{(q_2 \cdots q_{\mathfrak{i}})^{-1}}(\tau_1 + \cdots + \operatorname{Ad}_{(q_2 \cdots q_{\mathfrak{i}})} \tau_{\mathfrak{i}})) = 0. \tag{4.27}$$

$\forall q_{\mathfrak{i}} \in \mathcal{Q}_{\mathfrak{i}} \;\; (\mathfrak{i} = 2, \cdots, N)$ and $\forall \tau_i \in Lie(\mathcal{Q}_i) \;\; (i = 1, \cdots, N)$

where $h'_{\mathfrak{i}} := \operatorname{Ad}^*_{r_{cm,\mathfrak{i}}^{-1}} (h_{\mathfrak{i}})_e$. Note that, the expression in the parentheses in the second argument of $h'_{\mathfrak{i}}$ is the relative twist of $B_{\mathfrak{i}}$ with respect to B_0 and expressed in a coordinate frame attached to B_1. Based on this condition, we may derive a sufficient condition for the metric K being invariant under the action of $\mathcal{G}_1 \times \mathcal{G}_2$ by left translation.

Proposition 3. *For an open-chain multi-body system MBS*(N,N)*, the metric* K *is invariant under the action of* $\mathcal{G}_1 \times \mathcal{G}_2$*, as defined above, by left translation, if* $\forall \varpi \in Lie(\mathcal{G}_2)$ *and* $\forall \tau_1 \in Lie(\mathcal{Q}_1)$ *we have*

$$ad_{\varpi}(\tau_1) = 0.$$

Similarly, we can derive sufficient conditions for invariance of the metric K under the action of a group in the form of $\mathcal{G}_1 \times \cdots \times \mathcal{G}_N$ by left translation. Here $\mathcal{G}_\mathfrak{i} \subseteq \mathcal{Q}_\mathfrak{i}$ is a Lie subgroup of $\mathcal{Q}_\mathfrak{i}$ for $\mathfrak{i} = 2, \cdots, N$. However, since it is unlikely that K is invariant under the action of such a big group, we do not go through the calculations for this most general case.

Finally, suppose that B_{i_0} is an extremity of $MBS(N,N)$. Consider the action of $\mathcal{G}_{i_0}$ as a Lie subgroup of $\mathcal{Q}_{i_0}$ by right translation. The kinetic energy of the system after the action of an element $a_{i_0} \in \mathcal{G}_{i_0}$ becomes

$$\frac{1}{2} K_{qa_{i_0}}(\dot{q}a_{i_0}, \dot{q}a_{i_0}) = \frac{1}{2} \sum_{\substack{i=1 \\ i \neq i_0}}^{N} \parallel {}^{i}\tau_i^0 \parallel_{h_i}^2 + \frac{1}{2} \parallel \mathrm{Ad}_{a_{i_0}^{-1}} \mathrm{Ad}_{r_{cm,i_0}} {}^{i_0}\tau_{i_0}^0 \parallel_{h'_{i_0}}^2 . \tag{4.28}$$

The kinetic energy metric is invariant under this action if and only if it is invariant under the infinitesimal action of any element $\varrho \in Lie(\mathcal{G}_{i_0})$ at the identity element.

$$\begin{aligned} \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} & \left(\frac{1}{2} \parallel \mathrm{Ad}_{(\exp(-\epsilon\varrho))^{-1}} (\mathrm{Ad}_{r_{cm,i_0}} {}^{i_0}\tau_{i_0}^0) \parallel_{h'_{i_0}}^2 \right) \\ &= h'_{i_0}(\mathrm{ad}_{\varrho}(\mathrm{Ad}_{r_{cm,i_0}} {}^{i_0}\tau_{i_0}^0), \mathrm{Ad}_{r_{cm,i_0}} {}^{i_0}\tau_{i_0}^0) = 0, \end{aligned} \tag{4.29}$$

for all ${}^{i_0}\tau_{i_0}^0$, i.e., all admissible relative twists of B_{i_0} with respect to the inertial coordinate frame and expressed in the same frame. The largest Lie sub-algebra of $Lie(\mathcal{Q}_{i_0})$ that satisfies the above condition is $Lie(\mathcal{G}_{i_0})$, and $\mathcal{G}_{i_0} \subseteq \mathcal{Q}_{i_0}$ is identified by integrating this Lie sub-algebra on $\mathcal{Q}_{i_0}$. Therefore, the kinetic energy K is invariant under the $\mathcal{G}_{i_0}$-action by right translation on $\mathcal{Q}_{i_0}$ if and only if we have the above condition.

5. Dynamical Reduction of Nonholonomic Open-chain Multi-body Systems

A nonholonomic open-chain multi-body system with displacement subgroups is a $MBS(N, N)$ with at least one nonholonomic joint whose dynamics can be identified by the five-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D})$. Here, $\mathcal{Q} = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_N$ is the configuration manifold, $H\colon T^*\mathcal{Q} \to \mathbb{R}$ is the Hamiltonian of the system, which is defined by (3.24), K is the kinetic energy metric defined by (3.22), and the distribution $\mathcal{D} \subset T\mathcal{Q}$ corresponds to the nonholonomic joints. This distribution may be identified by the constraint one-forms $\{\omega_s \subset T^*\mathcal{Q} | \, s = 1, \cdots, f\}$, where $f < D = \dim(\mathcal{Q})$ is the number of linear constraints on the relative velocities at nonholonomic joints.

Since in robotics nonholonomic joints usually appear in the form of wheeled mobile platforms, in this paper we restrict our attention to the case where ω_s's only depend on the elements of the configuration manifold of the mobile platform, labelled as $\mathcal{Q}_1'$. We also label the configuration manifold corresponding to the rest of the joints in $MBS(N, N)$ by $\overline{\mathcal{Q}}$; hence, we have $\mathcal{Q} = \mathcal{Q}_1' \times \overline{\mathcal{Q}}$. Let $\mathcal{G} := \mathcal{G}_1' \times \mathcal{N} \subseteq \mathcal{Q}$, such that $\mathcal{G}_1' \subseteq \mathcal{Q}_1'$ and $\mathcal{N} \subseteq \overline{\mathcal{Q}}$, be a Lie group whose action on $\mathcal{Q}$ leaves the kinetic energy metric K invariant. The action of $\mathcal{G}$ is defined in Section 4. We also assume that

NHR1) The potential energy function V is $\mathcal{G}$-invariant. As a result, the Hamiltonian H is also invariant under the cotangent lift of the $\mathcal{G}$-action.

NHR2) The distribution $\mathcal{D} \subset T\mathcal{Q}$ is invariant under the $\mathcal{G}$-action, i.e., $\forall q \in \mathcal{Q}$ and $\forall \mathfrak{g} \in \mathcal{G}$ we have $\mathcal{D} \circ \Phi_{\mathfrak{g}}(q) = T_q\Phi_{\mathfrak{g}}(\mathcal{D}(q))$.

NHR3) There exists a Lie subgroup of $\mathcal{G}_1'$, namely G, for which we have the Chaplygin assumption (2.7).

We call a $MBS(N, N)$ that satisfies the above assumptions, a nonholonomic open-chain multi-body system with symmetry and denote its dynamics by the six-tuple $(T^*\mathcal{Q}, \Omega_{can}, H, K, \mathcal{D}, \mathcal{G})$, as defined above. The nonholonomic Hamilton's equation for such a multi-body system is written on $T^*\mathcal{Q}$ by (2.4). Under the assumption NHR3, we have a G-principal bundle, and the corresponding connection $\widehat{\mathcal{A}}\colon T\mathcal{Q} \to Lie(G)$ may be defined by

$$\widehat{\mathcal{A}} := \sum_{s=1}^{f} \omega_s \varepsilon_s,$$

where ε_s for $s \in \{1, \cdots, f\}$ are elements of a basis for $Lie(G)$. We represent any element of $\mathcal{Q}/G$ by $\widehat{q} = (\widehat{q}_1, \overline{q}) \in \mathcal{Q}_1'/G \times \overline{\mathcal{Q}}$, where $\widehat{q}_1 \in \mathcal{Q}_1'/G$ is the equivalence class corresponding to $q_1' \in \mathcal{Q}_1'$ and $\overline{q} \in \overline{\mathcal{Q}}$. We consider the principal bundle $\widehat{\pi}_1 \colon \mathcal{Q}_1' \to \mathcal{Q}_1'/G$ to locally trivialize the Lie group $\mathcal{Q}_1'$. Let $U \subseteq \mathcal{Q}_1'/G$ be an open neighbourhood of $\widehat{e}_1$, where $\widehat{e}_1$ is the equivalence class corresponding to the identity element $e_1' \in \mathcal{Q}_1'$. We denote the map corresponding to a local trivialization of the principal bundle $\widehat{\pi}_1$ by $\widehat{\chi} \colon G \times U \to \mathcal{Q}_1'$. This map can be defined by embedding U in $\mathcal{Q}_1'$, for example by using the exponential map of Lie groups. We denote this embedding by $\chi \colon U \hookrightarrow \mathcal{Q}_1'$ such that $\forall \widehat{q}_1 \in \mathcal{Q}_1'/G$ we have $\chi(\widehat{q}_1) = \exp(\zeta)$ for some $\zeta \in \widehat{\mathcal{C}}$, where $\widehat{\mathcal{C}} \subset Lie(\mathcal{Q}_1')$ is a complementary subspace to $Lie(G) \subseteq Lie(\mathcal{G}_1') \subseteq Lie(\mathcal{Q}_1')$. Accordingly, $\forall \mathfrak{h} \in G$ we define the map $\widehat{\chi}$ by the equality $\widehat{\chi}((\mathfrak{h}, \widehat{q}_1)) := \mathfrak{h}\chi(\widehat{q}_1)$. The map $\widehat{\chi}$ is a diffeomorphism onto its image. Using this diffeomorphism, any element $q_1' \in \widehat{\pi}_1^{-1}(U) \subseteq \mathcal{Q}_1'$ can be uniquely identified by an element $(\mathfrak{h}, \widehat{q}_1) \in G \times U$. As a result, we have $q = (q_1', \overline{q}) = (\widehat{\chi}((\mathfrak{h}, \widehat{q}_1)), \overline{q})$. Note that, from now on, for brevity we write $q = (\mathfrak{h}, \widehat{q}_1, \overline{q})$.

The map corresponding to the infinitesimal action of G on $\mathcal{Q}$ is denoted by $\widehat{\phi}_q \colon Lie(G) \to T_q\mathcal{Q}$. Based on the above local trivialization, $\forall (\mathfrak{h}, \widehat{q}_1, \overline{q}) \in G \times U \times \overline{\mathcal{Q}}$ this map is calculated by

$$\widehat{\phi}_q = \begin{bmatrix} T_{e_1'} R_{\mathfrak{h}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where with an abuse of notation we show the identity element of G by e_1'. Accordingly, we calculate the momentum map $\widehat{\mathbf{M}} \colon T^*\mathcal{Q} \to Lie^*(G)$ by

$$\widehat{\mathbf{M}}_q = \widehat{\phi}_q^* = \begin{bmatrix} T_{e_1'}^* R_{\mathfrak{h}} & 0 & \dots & 0 \end{bmatrix}.$$

Then by defining the fibre wise linear map $\widehat{A}_{\widehat{q}_1} \colon T_{\widehat{q}_1} U \to Lie(G)$ according to the nonholonomic constraint 1-forms ω_s's and based on the properties of principal connections, for all $q = (\mathfrak{h}, \widehat{q}_1, \overline{q}) \in G \times U \times \overline{\mathcal{Q}}$ we can write $\widehat{\mathcal{A}}$ as

$$\widehat{\mathcal{A}}_q = \mathrm{Ad}_{\mathfrak{h}} \begin{bmatrix} T_{\mathfrak{h}} L_{\mathfrak{h}^{-1}} & \widehat{A}_{\widehat{q}_1} & 0 \end{bmatrix},$$

As a result, at each point $q = (\mathfrak{h}, \widehat{q}_1, \overline{q})$, we have the horizontal lift map

$\widehat{\mathrm{hl}}_q\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to T_q\mathcal{Q}$, which is determined by

$$\widehat{\mathrm{hl}}_q = \begin{bmatrix} \begin{bmatrix} -(T_{e_1'}L_{\mathfrak{h}})\widehat{A}_{\widehat{q}_1} & 0 \end{bmatrix} \\ id_{T_{\widehat{q}_1}U} \oplus id_{T_{\overline{q}}\overline{\mathcal{Q}}} \end{bmatrix},$$

where $id_{T_{\widehat{q}_1}U}$ and $id_{T_{\overline{q}}\overline{\mathcal{Q}}}$ are the identity maps on the tangent spaces $T_{\widehat{q}_1}U$ and $T_{\overline{q}}\overline{\mathcal{Q}}$, respectively.

We denote the block components of the kinetic energy tensor, which is equal to the Legendre transformation for Hamiltonian mechanical systems, by $K_{ij}(q)dq_i \otimes dq_j$ $(i,j=1,\cdots,N)$. We show $\mathbb{F}L_q$ in matrix form as

$$\mathbb{F}L_q = \begin{bmatrix} K_{11}'(q) & K_{12}'(q) \\ K_{21}'(q) & K_{22}'(q) \end{bmatrix} = \mathbb{F}L_q = \begin{bmatrix} K_{11}(q) & \cdots & K_{1N}(q) \\ \vdots & \ddots & \vdots \\ K_{N1}(q) & \cdots & K_{NN}(q) \end{bmatrix},$$

where $K_{11}'(q)$ and $K_{22}'(q)$ are the blocks of $\mathbb{F}L_q$ corresponding to the mobile platform and rest of the $MBS(N,N)$, respectively, and $K_{12}'(q) = K_{21}'^*(q)$ is the block representing the interconnection between them. Using the local trivialization we can rewrite this matrix as follows:

$$\mathbb{F}L_{(\mathfrak{h},\widehat{q})} = \begin{bmatrix} K_1^G((\mathfrak{h},\widehat{q})) & K_1^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) & K_{12}^G((\mathfrak{h},\widehat{q})) \\ K_2^G((\mathfrak{h},\widehat{q})) & K_2^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) & K_{12}^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) \\ K_{21}^G((\mathfrak{h},\widehat{q})) & K_{21}^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) & K_{22}'((\mathfrak{h},\widehat{q})) \end{bmatrix},$$

where $\widehat{q} = (\widehat{q}_1, \overline{q})$, $q_1' = \widehat{\chi}(\mathfrak{h}, \widehat{q}_1)$, and we have

$$\begin{bmatrix} K_1^G((\mathfrak{h},\widehat{q})) & K_1^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) \\ K_2^G((\mathfrak{h},\widehat{q})) & K_2^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) \end{bmatrix} = T^*_{(\mathfrak{h},\widehat{q}_1)}\widehat{\chi}\left(K_{11}'(\widehat{\chi}(\mathfrak{h},\widehat{q}))\right) T_{(\mathfrak{h},\widehat{q}_1)}\widehat{\chi},$$

$$\begin{bmatrix} K_{21}^G((\mathfrak{h},\widehat{q})) & K_{21}^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) \end{bmatrix}^* = \begin{bmatrix} K_{12}^G((\mathfrak{h},\widehat{q})) \\ K_{12}^{\mathcal{Q}_1'/G}((\mathfrak{h},\widehat{q})) \end{bmatrix} = T^*_{(\mathfrak{h},\widehat{q}_1)}\widehat{\chi}(K_{12}'(\widehat{\chi}(\mathfrak{h},\widehat{q}))),$$

$$K_{22}'((\mathfrak{h},\widehat{q})) = K_{22}'(\widehat{\chi}(\mathfrak{h},\widehat{q})).$$

Now, since K is invariant under the G-action, we have the following equality:

$$\mathbb{F}L_{(\mathfrak{h},\widehat{q})} = \begin{bmatrix} (T_{\mathfrak{h}}^* L_{\mathfrak{h}^{-1}})(\widehat{K}_1^G(\widehat{q}))(T_{\mathfrak{h}} L_{\mathfrak{h}^{-1}}) & (T_{\mathfrak{h}}^* L_{\mathfrak{h}^{-1}})(\widehat{K}_1^{\mathcal{Q}_1'/G}(\widehat{q})) & (T_{\mathfrak{h}}^* L_{\mathfrak{h}^{-1}})(\widehat{K}_{12}^G(\widehat{q})) \\ (\widehat{K}_2^G(\widehat{q}))(T_{\mathfrak{h}} L_{\mathfrak{h}^{-1}}) & \widehat{K}_2^{\mathcal{Q}_1'/G}(\widehat{q}) & \widehat{K}_{12}^{\mathcal{Q}_1'/G}(\widehat{q}) \\ (\widehat{K}_{21}^G(\widehat{q}))(T_{\mathfrak{h}} L_{\mathfrak{h}^{-1}}) & \widehat{K}_{21}^{\mathcal{Q}_1'/G}(\widehat{q}) & \widehat{K}_{22}'(\widehat{q}) \end{bmatrix},$$

where we introduce the new block components by

$$\begin{aligned} &\widehat{K}_1^G(\widehat{q}) = K_1^G((e_1', \widehat{q})), \qquad \widehat{K}_1^{\mathcal{Q}_1'/G}(\widehat{q}) = K_1^{\mathcal{Q}_1'/G}((e_1', \widehat{q})), \qquad \widehat{K}_2^G(\widehat{q}) = K_2^G((e_1', \widehat{q})), \\ &\widehat{K}_2^{\mathcal{Q}_1'/G}(\widehat{q}) = K_2^{\mathcal{Q}_1'/G}((e_1', \widehat{q})), \quad \widehat{K}_{12}^G(\widehat{q}) = K_{12}^G((e_1', \widehat{q})), \quad \widehat{K}_{12}^{\mathcal{Q}_1'/G}(\widehat{q}) = K_{12}^{\mathcal{Q}_1'/G}((e_1', \widehat{q})), \\ &\widehat{K}_{21}^G(\widehat{q}) = K_{21}^G((e_1', \widehat{q})), \qquad \widehat{K}_{21}^{\mathcal{Q}_1'/G}(\widehat{q}) = K_{21}^{\mathcal{Q}_1'/G}((e_1', \widehat{q})), \qquad \widehat{K}_{22}'(\widehat{q}) = K_{22}'((e_1', \widehat{q})). \end{aligned}$$

Since K is G-invariant, it induces a metric on $\widehat{\mathcal{Q}}$, namely $\widehat{K}$, which defines the Legendre transformation on $\widehat{\mathcal{Q}}$ by

$$\begin{aligned} \langle \mathbb{F}\widehat{L}_{\widehat{q}}(\widehat{u}_{\widehat{q}}), \widehat{w}_{\widehat{q}} \rangle :&= \widehat{K}_{\widehat{q}}(\widehat{u}_{\widehat{q}}, \widehat{v}_{\widehat{q}}) = K_q(\widehat{\mathrm{hl}}_q(\widehat{u}_{\widehat{q}}), \widehat{\mathrm{hl}}_q(\widehat{w}_{\widehat{q}})) \\ &= \langle \mathbb{F}L_q \circ \widehat{\mathrm{hl}}_q(\widehat{u}_{\widehat{q}}), \widehat{\mathrm{hl}}_q(\widehat{w}_{\widehat{q}}) \rangle = \langle \widehat{\mathrm{hl}}_q^* \circ \mathbb{F}L_q \circ \widehat{\mathrm{hl}}_q(\widehat{u}_{\widehat{q}}), \widehat{w}_{\widehat{q}} \rangle, \end{aligned}$$

where $\widehat{q} = (\widehat{q}_1, \overline{q})$ and $\forall \widehat{u}_{\widehat{q}}, \widehat{w}_{\widehat{q}} \in T_{\widehat{q}}\widehat{\mathcal{Q}}$. Therefore,

$$\mathbb{F}\widehat{L}_{\widehat{q}} = \begin{bmatrix} \widehat{K}_{11}'(\widehat{q}) & \widehat{K}_{12}'(\widehat{q}) \\ \widehat{K}_{21}'(\widehat{q}) & \widehat{K}_{22}'(\widehat{q}) \end{bmatrix},$$

with the following equalities:

$$\begin{aligned} &\widehat{K}_{11}'(\widehat{q}) = (\widehat{A}_{\widehat{q}_1}^*)(\widehat{K}_1^G(\widehat{q}))(\widehat{A}_{\widehat{q}_1}) - (\widehat{A}_{\widehat{q}_1}^*)(\widehat{K}_1^{\mathcal{Q}_1'/G}(\widehat{q})) - (\widehat{K}_2^G(\widehat{q}))(\widehat{A}_{\widehat{q}_1}) + \widehat{K}_2^{\mathcal{Q}_1'/G}(\widehat{q}), \\ &\widehat{K}_{12}'(\widehat{q}) = -(\widehat{A}_{\widehat{q}_1}^*)(\widehat{K}_{12}^G(\widehat{q})) + \widehat{K}_{12}^{\mathcal{Q}_1'/G}(\widehat{q}), \\ &\widehat{K}_{21}'(\widehat{q}) = -(\widehat{K}_{21}^G(\widehat{q}))(\widehat{A}_{\widehat{q}_1}) + \widehat{K}_{21}^{\mathcal{Q}_1'/G}(\widehat{q}). \end{aligned}$$

Let $\mathcal{M} = \mathbb{F}L(\mathcal{D})$ be the vector sub-bundle of $T^*\mathcal{Q}$ corresponding to the nonholonomic distribution. We then define the horizontal lift map $\widehat{\mathrm{hl}}_{(\mathfrak{h},\widehat{q})}^{\mathcal{M}} \colon T_{\widehat{q}}^*\widehat{\mathcal{Q}} \to$

$\mathcal{M}((\mathfrak{h},\widehat{q}))$ on the cotangent bundle of the reduced space by

$$\begin{aligned}\widehat{\mathrm{hl}}^{\mathcal{M}}_{(\mathfrak{h},\widehat{q})} &:= \mathbb{F}L_{(\mathfrak{h},\widehat{q})}\circ \widehat{\mathrm{hl}}_{(\mathfrak{h},\widehat{q})}\circ \mathbb{F}\widehat{L}^{-1}_{\widehat{q}}\\ &= \begin{bmatrix} T^*_{\mathfrak{h}}L_{\mathfrak{h}^{-1}} & 0\\ 0 & id_{T_{\widehat{q}}\widehat{\mathcal{Q}}}\end{bmatrix}\begin{bmatrix} -(\widehat{K}^G_1(\widehat{q}))(\widehat{A}_{\widehat{q}_1})+\widehat{K}^{\mathcal{Q}_1'/G}_1(\widehat{q}) & \widehat{K}^G_{12}(\widehat{q})\\ -(\widehat{K}^G_2(\widehat{q}))(\widehat{A}_{\widehat{q}_1})+\widehat{K}^{\mathcal{Q}_1'/G}_2(\widehat{q}) & \widehat{K}^{\mathcal{Q}_1'/G}_{12}(\widehat{q})\\ -(\widehat{K}^G_{21}(\widehat{q}))(\widehat{A}_{\widehat{q}_1})+\widehat{K}^{\mathcal{Q}_1'/G}_{21}(\widehat{q}) & \widehat{K}'_{22}(\widehat{q})\end{bmatrix}\mathbb{F}\widehat{L}^{-1}_{\widehat{q}},\end{aligned}$$

where $id_{T_{\widehat{q}}\widehat{\mathcal{Q}}}$ is the identity map on $T_{\widehat{q}}\widehat{\mathcal{Q}}$. Based on the definition of $\widehat{H}(\widehat{p}_{\widehat{q}}) := H\circ \widehat{\mathrm{hl}}^{\mathcal{M}}_q(\widehat{p}_{\widehat{q}})$, where $\widehat{p}_{\widehat{q}}\in T^*\widehat{\mathcal{Q}}$ and $\widehat{q}=\widehat{\pi}(q)$, we calculate $\widehat{H}$ on $T^*\widehat{\mathcal{Q}}$ using the local trivialization and the definition of the map $\widehat{\mathrm{hl}}^{\mathcal{M}}$:

$$\widehat{H}(\widehat{p}_{\widehat{q}}) = \frac{1}{2}\left\langle \widehat{\mathrm{hl}}^{\mathcal{M}}_{(\mathfrak{h},\widehat{q})}(\widehat{p}_{\widehat{q}}), \mathbb{F}L^{-1}_{(\mathfrak{h},\widehat{q})}\circ \widehat{\mathrm{hl}}^{\mathcal{M}}_{(\mathfrak{h},\widehat{q})}(\widehat{p}_{\widehat{q}})\right\rangle + V(e_1,\widehat{q}) = \frac{1}{2}\left\langle \widehat{p}_{\widehat{q}}, \mathbb{F}\widehat{L}^{-1}_{\widehat{q}}(\widehat{p}_{\widehat{q}})\right\rangle + \widehat{V}(\widehat{q}), \tag{5.30}$$

where the function $\widehat{V}(\widehat{q}) := V(e_1',\widehat{q})$.

Performing the Chaplygin reduction in Theorem 2.3 we can write the reduced dynamical equations for nonholonomic multi-body systems on $T^*\widehat{\mathcal{Q}}$.

Theorem 5.1. *A nonholonomic open-chain multi-body system MBS(N,N) with symmetry whose dynamics is represented by $(T^*\mathcal{Q},\Omega_{can},H,K,\mathcal{D},G)$ is reduced to a system $(T^*\widehat{\mathcal{Q}},\widehat{\Omega},\widehat{H},\widehat{K})$, where $\widehat{H}$ is defined by (5.30) and $\widehat{K}$ is the induced metric on $\widehat{\mathcal{Q}}$. Here, in the local coordinates $\widehat{\Omega}$ is calculated by (2.11) and (2.12), where*

$$\widehat{\mathcal{F}} := \begin{bmatrix} -(\widehat{K}^G_1(\widehat{q}))(\widehat{A}_{\widehat{q}_1})+\widehat{K}^{\mathcal{Q}_1'/G}_1(\widehat{q}) & \widehat{K}^G_{12}(\widehat{q})\end{bmatrix}\mathbb{F}\widehat{L}^{-1}_{\widehat{q}}(\widehat{p}_{\widehat{q}}).$$

And, Hamilton's equation in the reduced phase space reads (2.11).

5.1. Second Stage Reduction of Nonholonomic Open-chain Multi-body Systems

Let $\mathcal{N}$ be a Lie subgroup of $\overline{\mathcal{Q}}$. We define the action of $\mathcal{N}$ on $\widehat{\mathcal{Q}}$, i.e., $\breve{\Phi}_{\mathfrak{n}}\colon \widehat{\mathcal{Q}}\to\widehat{\mathcal{Q}}$, by left translation on $\overline{\mathcal{Q}}$. For any element $\mathfrak{n}\in\mathcal{N}$ we have

$$\breve{\Phi}_{\mathfrak{n}}(\widehat{q}_1,\overline{q}) = (\widehat{q}_1,\mathfrak{n}\overline{q}).$$

Hence, the tangent and cotangent lift of the $\mathcal{N}$-action are

$$
\begin{aligned}
T_{\widehat{q}}\breve{\Phi}_{\mathfrak{n}}(\widehat{v}_{\widehat{q}}) &= \begin{bmatrix} id_{T_{\widehat{q}_1}\widehat{\mathcal{Q}}_1} & 0 \\ 0 & T_{\overline{q}}L_{\mathfrak{n}} \end{bmatrix} \begin{bmatrix} \widehat{v}_1 \\ \overline{v} \end{bmatrix} \\
T^*_{\breve{\Phi}_{\mathfrak{n}}(\widehat{q})}\breve{\Phi}_{\mathfrak{n}^{-1}}(\widehat{p}_{\widehat{q}}) &= \begin{bmatrix} id_{T_{\widehat{q}_1}\widehat{\mathcal{Q}}_1} & 0 \\ 0 & T^*_{\mathfrak{n}\overline{q}}L_{\mathfrak{n}^{-1}} \end{bmatrix} \begin{bmatrix} \widehat{p}_1 \\ \overline{p} \end{bmatrix}.
\end{aligned}
$$

Let us assume that the Hamiltonian $\widehat{H}$ and the metric $\widehat{K}$ of the reduced nonholonomic open-chain multi-body system $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K})$ are invariant under the $\mathcal{N}$-action. We locally trivialize $\widehat{\mathcal{Q}}$ such that we have $\widehat{q} = (\widehat{q}_1, \mathfrak{n}, \widetilde{q}) \in U \times \mathcal{N} \times \widetilde{U}$, where $\widetilde{U} \subseteq \widetilde{\mathcal{Q}} := \overline{\mathcal{Q}}/\mathcal{N}$ is an open subset of $\widetilde{\mathcal{Q}}$. In this trivialization, the map corresponding to the infinitesimal $\mathcal{N}$-action $\breve{\phi}_{\widehat{q}}\colon Lie(\mathcal{N}) \subset Lie(\overline{\mathcal{Q}}) \to T\widehat{\mathcal{Q}}$ is calculated by

$$
\breve{\phi}_{\widehat{q}} = \begin{bmatrix} 0 \\ T_{\overline{e}}R_{\mathfrak{n}} \\ 0 \end{bmatrix},
$$

where $\overline{e} \in \mathcal{N} \subseteq \overline{\mathcal{Q}}$ is the identity element. Since the $\mathcal{N}$-action leaves $\widehat{p}_1$ invariant, it satisfies the condition (i) and (ii) on Page 11. We also define the momentum $\breve{\mathbf{M}}_{\widehat{q}}\colon T^*_{\widehat{q}}\widehat{\mathcal{Q}} \to Lie^*(\mathcal{N})$ of the $\mathcal{N}$-action by

$$
\breve{\mathbf{M}}_{\widehat{q}} = \breve{\phi}^*_{\widehat{q}} = \begin{bmatrix} 0 & T^*_{\overline{e}}R_{\mathfrak{n}} & 0 \end{bmatrix}.
$$

Accordingly, the locked inertia tensor $\breve{\mathbb{I}}_{\widehat{q}}\colon Lie(\mathcal{N}) \to Lie^*(\mathcal{N})$ and the principal connection $\breve{\mathcal{A}}_{\widehat{q}}\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to Lie(\mathcal{N})$ for the $\mathcal{N}$-action are calculated as

$$
\begin{aligned}
\breve{\mathbb{I}}_{\widehat{q}} &= \breve{\phi}^*_{\widehat{q}} \circ \mathbb{F}\widehat{L}_{\widehat{q}} \circ \breve{\phi}_{\widehat{q}} = \mathrm{Ad}^*_{\mathfrak{n}^{-1}}(K^{\mathcal{N}}_1(\widehat{q}_1, \widetilde{q}))\mathrm{Ad}_{\mathfrak{n}^{-1}} \\
\breve{\mathcal{A}}_{\widehat{q}} &= \breve{\mathbb{I}}^{-1}_{\widehat{q}} \circ \breve{\mathbf{M}}_{\widehat{q}} \circ \mathbb{F}\widehat{L}_{\widehat{q}} \\
&= \mathrm{Ad}_{\mathfrak{n}} \begin{bmatrix} (K^{\mathcal{N}}_1(\widehat{q}_1, \widetilde{q}))^{-1}K^{\mathcal{N}}_{12}(\widehat{q}_1, \widetilde{q}) & T_{\mathfrak{n}}L_{\mathfrak{n}^{-1}} & (K^{\mathcal{N}}_1(\widehat{q}_1, \widetilde{q}))^{-1}K^{\overline{\mathcal{Q}}/\mathcal{N}}_1(\widehat{q}_1, \widetilde{q}) \end{bmatrix} \\
&=: \begin{bmatrix} \breve{A}_{(\widehat{q}_1, \widetilde{q})} & T_{\mathfrak{n}}L_{\mathfrak{n}^{-1}} & \breve{B}_{(\widehat{q}_1, \widetilde{q})} \end{bmatrix}, \qquad (5.31)
\end{aligned}
$$

where we define the linear maps $\breve{A}_{\breve{q}}\colon TU \to Lie(\mathcal{N})$ and $\breve{B}_{\breve{q}}\colon T\widetilde{U} \to Lie(\mathcal{N}_\vartheta)$

by the last equality, and we have

$$\mathbb{F}\widehat{L}_{\widehat{q}} = \begin{bmatrix} \widehat{K}'_{11}(\widehat{q}) & \widehat{K}'_{12}(\widehat{q}) \\ \widehat{K}'_{21}(\widehat{q}) & \widehat{K}'_{22}(\widehat{q}) \end{bmatrix}$$
$$=: \begin{bmatrix} \widehat{K}'_{11}(\widehat{q}_1, \overline{e}, \widetilde{q}) & (K^{\mathcal{N}}_{12}(\widehat{q}_1, \widetilde{q}))T_{\mathfrak{n}}L_{\mathfrak{n}^{-1}} & K^{\overline{\mathcal{Q}}/\mathcal{N}}_{12}(\widehat{q}_1, \widetilde{q}) \\ T^*_{\mathfrak{n}}L_{\mathfrak{n}^{-1}}(K^{\mathcal{N}}_{21}(\widehat{q}_1, \widetilde{q})) & T^*_{\mathfrak{n}}L_{\mathfrak{n}^{-1}}(K^{\mathcal{N}}_{1}(\widehat{q}_1, \widetilde{q}))T_{\mathfrak{n}}L_{\mathfrak{n}^{-1}} & T^*_{\mathfrak{n}}L_{\mathfrak{n}^{-1}}(K^{\overline{\mathcal{Q}}/\mathcal{N}}_{1}(\widehat{q}_1, \widetilde{q})) \\ K^{\overline{\mathcal{Q}}/\mathcal{N}}_{21}(\widehat{q}_1, \widetilde{q}) & (K^{\mathcal{N}}_{2}(\widehat{q}_1, \widetilde{q}))T_{\mathfrak{n}}L_{\mathfrak{n}^{-1}} & K^{\overline{\mathcal{Q}}/\mathcal{N}}_{2}(\widehat{q}_1, \widetilde{q}) \end{bmatrix}.$$

We also locally trivialize the principal bundle $\mathcal{N} \to \mathcal{N}/\mathcal{N}_\vartheta$, and similarly we calculate the (mechanical) principal connection $\breve{\mathcal{A}}^\vartheta$ corresponding to the principal bundle $\widehat{\mathcal{Q}} \to \breve{\mathcal{Q}} = \widehat{\mathcal{Q}}/\mathcal{N}_\vartheta$. We use this connection to calculate the horizontal lift map $\breve{\text{hl}}$. Let us assume that the principal connection $\breve{\mathcal{A}}^\vartheta$ in the local trivialization is written as:

$$\breve{\mathcal{A}}^\vartheta_{\widehat{q}} := \begin{bmatrix} \breve{A}^\vartheta_{\breve{q}} & T_{\mathfrak{k}}L_{\mathfrak{k}^{-1}} & \breve{B}^\vartheta_{\breve{q}} \end{bmatrix},$$

for all $\widehat{q} \in U \times \mathcal{N}_\vartheta \times U_\vartheta \times \widetilde{U}$, where $U_\vartheta \subseteq \mathcal{N}/\mathcal{N}_\vartheta$ is an open subset of $\mathcal{N}/\mathcal{N}_\vartheta$, $\mathfrak{k} \in \mathcal{N}_\vartheta$ and $\breve{q} \in U \times U_\vartheta \times \widetilde{U} \subseteq \breve{\mathcal{Q}} = \widehat{\mathcal{Q}}/\mathcal{N}_\vartheta$. Here, the linear maps $\breve{A}^\vartheta_{\breve{q}} \colon TU \to Lie(\mathcal{N}_\vartheta)$ and $\breve{B}^\vartheta_{\breve{q}} \colon T(U_\vartheta \times \widetilde{U}) \to Lie(\mathcal{N}_\vartheta)$ are defined based on the Legendre transformation $\mathbb{F}\widehat{L}_{\widehat{q}}$ in the local trivialization of the principal bundle $\mathcal{N} \to \mathcal{N}/\mathcal{N}_\vartheta$. Consequently, the horizontal lift map $\breve{\text{hl}}_{\widehat{q}} \colon T_{\breve{q}}(U \times U_\vartheta \times \widetilde{U}) \to T_{\widehat{q}}(U \times \mathcal{N}_\vartheta \times U_\vartheta \times \widetilde{U})$ is calculated by

$$\breve{\text{hl}}_{\widehat{q}} = \begin{bmatrix} -T_{\overline{e}}L_{\mathfrak{k}} \begin{bmatrix} \breve{A}^\vartheta_{\breve{q}} & \breve{B}^\vartheta_{\breve{q}} \end{bmatrix} \\ id_{T_{\breve{q}}(U \times U_\vartheta \times \widetilde{U})} \end{bmatrix}, \tag{5.32}$$

where $id_{T_{\breve{q}}(U \times U_\vartheta \times \widetilde{U})}$ is the identity map on the tangent space $T_{\breve{q}}(U \times U_\vartheta \times \widetilde{U})$. Now, we use (5.31) and (5.32) to calculate the 2-form Ξ_ϑ in (2.20) for a reduced $MBS(N, N)$. Furthermore, for a reduced $MBS(N, N)$ we have the inverse of the map $\varphi_\vartheta \colon \breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}_\vartheta \to T^*\breve{\mathcal{Q}}$ is defined on $\breve{\mathcal{S}}$ and in the local trivialization $\forall \breve{p}_{\breve{q}} = (\widehat{p}_1, p_\vartheta, \widetilde{p}) \in T^*_{\breve{q}}(U \times U_\vartheta \times \widetilde{U})$,

$$\varphi_\vartheta^{-1}(\widehat{p}_1, p_\vartheta, \widetilde{p}) = \begin{bmatrix} \widehat{p}_1 + \breve{A}^*_{(\widehat{q}_1, \widetilde{q})}(\text{Ad}^*_{(\overline{e}, \widehat{\mathfrak{n}})}\vartheta) \\ T^*_{(\mathfrak{k}, \widehat{\mathfrak{n}})}R_{(\mathfrak{k}, \widehat{\mathfrak{n}})^{-1}}(\vartheta) \\ \widetilde{p} + \breve{B}^*_{(\widehat{q}_1, \widetilde{q})}(\text{Ad}^*_{(\overline{e}, \widehat{\mathfrak{n}})}\vartheta) \end{bmatrix}_\vartheta, \tag{5.33}$$

where in local trivialization we have $\mathfrak{n} = (\mathfrak{k}, \widetilde{\mathfrak{n}}) \in \mathcal{N}$. It is easy to show that on the vector sub-bundle $\breve{\mathcal{S}}$, $p_\vartheta = 0$. Hence, we can determine the reduced Hamiltonian $\breve{H}_\vartheta \colon \breve{\mathcal{S}} \to \mathbb{R}$ by

$$\breve{H}_\vartheta(\breve{q}, \widehat{p}_1, \widetilde{p}) := \widehat{H}_\vartheta(\varphi_\vartheta^{-1}(\widehat{p}_1, 0, \widetilde{p})). \tag{5.34}$$

Theorem 5.2. *We say that a reduced MBS(N, N) with symmetry whose dynamics is represented by $(T^*\widehat{\mathcal{Q}}, \widehat{\Omega}, \widehat{H}, \widehat{K}, \mathcal{N})$, and whose solution curves satisfy the reduced nonholonomic Hamilton's equation (2.11), can be further reduced to the system $(\breve{\mathcal{S}} \subset T^*\breve{\mathcal{Q}}, \breve{\Omega}_\vartheta - \Xi_\vartheta, \breve{H}_\vartheta)$. The 2-form $\breve{\Omega}_\vartheta := T^*\varphi_\vartheta^{-1}(\widehat{\Omega}_\vartheta)$ is calculated based on (5.33). The Hamiltonian $\breve{H}_\vartheta \colon \breve{\mathcal{S}} \to \mathbb{R}$ is the further reduced Hamiltonian in (5.34). Also the closed 2-form Ξ_ϑ is defined by (2.20), using (5.31) and (5.32). The further reduced system satisfies Hamilton's equation (2.19) for the Hamiltonian $\breve{H}_\vartheta$.*

6. Case Study (Four-wheel Crane)

In this section we study the dynamics of an example of nonholonomic open-chain multi-body systems. In this example, we study the two-stage reduction of a two-d.o.f. crane on a four-wheel car whose top and side view in the initial configuration is shown in Figure 1. Using the indexing introduced in the previous section and starting with the car without the rear wheels and the crane as B_1, we first number the bodies and joints. The following graph shows the structure of the nonholonomic open-chain multi-body system.

$$\begin{array}{ccccccc} B_4 & \xrightarrow{J_4} & B_2 & \xrightarrow{J_5} & B_5 & & \\ & & \Big| J_2 & & & & \\ B_0 & \xrightarrow{J_1} & B_1 & \xrightarrow{J_3} & B_3 & \xrightarrow{J_6} & B_6 \end{array}$$

We first identify the relative configuration manifolds corresponding to the joints of the robotic system. The relative configuration manifold corresponding to the first joint, which is a three-d.o.f. planar joint, is

$$Q_1^0 = \left\{ r_1^2 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x \\ \sin(\theta) & \cos(\theta) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \,\middle|\, x, y \in \mathbb{R}, \theta \in \mathbb{S}^1 \right\}.$$

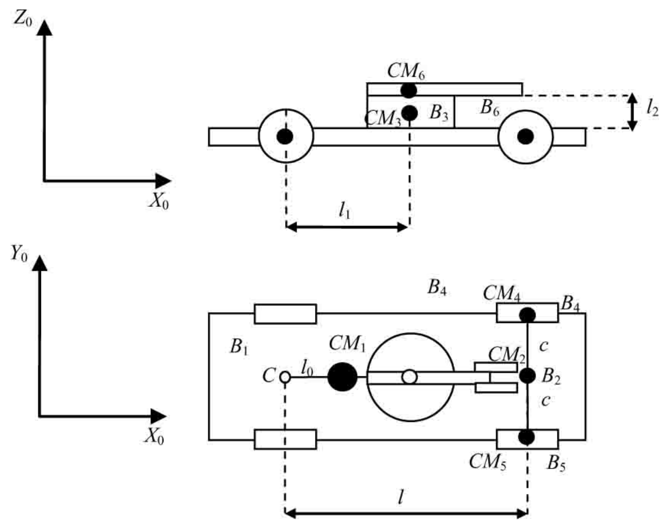

Figure 1: An example of a crane

Here, (x, y) is the position of C with respect to the inertial coordinate frame and θ is the angle between the X_1-axis and X_0-axis (see Figure 2). The second joint is a one-d.o.f. revolute joint between B_2 and B_1, and its corresponding relative configuration manifold is given by

$$Q_2^1 = \left\{ r_2^1 = \begin{bmatrix} \cos(\psi_1) & -\sin(\psi_1) & 0 & l \\ \sin(\psi_1) & \cos(\psi_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_1 \in \mathbb{S}^1 \right\},$$

where l is the distance between the front and rear wheels. Similarly, for the third joint we have

$$Q_3^1 = \left\{ r_3^1 = \begin{bmatrix} \cos(\varphi_1) & -\sin(\varphi_1) & 0 & l_1 \\ \sin(\varphi_1) & \cos(\varphi_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_1 \in \mathbb{S}^1 \right\}.$$

The forth and fifth joints are one-d.o.f. revolute joints whose axes of revolution are the Y_i-axis ($i = 4, 5$). The relative configuration manifolds of these

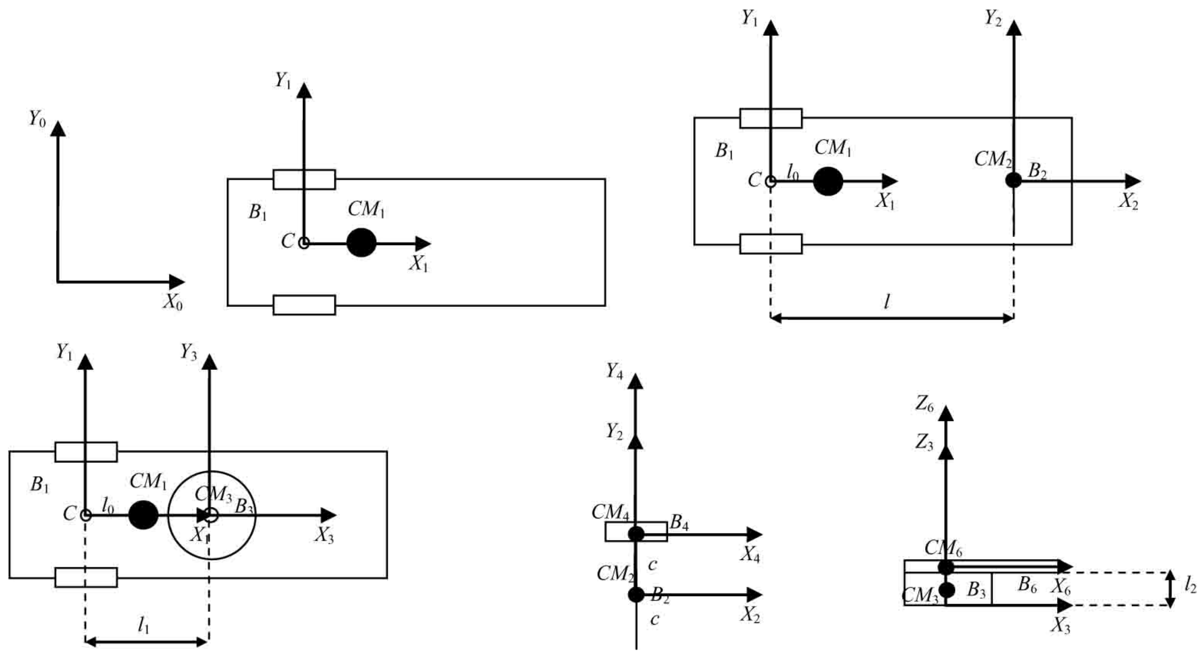

Figure 2: The coordinate frames attached to the bodies of the crane

joints are identified by

$$Q_4^2 = \left\{ r_4^2 = \begin{bmatrix} \cos(\psi_2) & 0 & \sin(\psi_2) & 0 \\ 0 & 1 & 0 & c \\ -\sin(\psi_2) & 0 & \cos(\psi_2) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \,\middle|\, \psi_2 \in \mathbb{S}^1 \right\},$$

$$Q_5^2 = \left\{ r_5^2 = \begin{bmatrix} \cos(\psi_3) & 0 & \sin(\psi_3) & 0 \\ 0 & 1 & 0 & -c \\ -\sin(\psi_3) & 0 & \cos(\psi_3) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \,\middle|\, \psi_3 \in \mathbb{S}^1 \right\},$$

where c is the distance between the steering point and the front wheels. Note that, if we assume that the front wheels are rotating together, then we can substitute the front wheels with a cylinder. Finally, the sixth joint is a one-d.o.f. revolute joint with the Y_6-axis being its axis of revolution. So, we have

$$Q_6^5 = \left\{ r_6^5 = \begin{bmatrix} \cos(\varphi_2) & 0 & \sin(\varphi_2) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi_2) & 0 & \cos(\varphi_2) & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_2 \in \mathbb{S}^1 \right\}.$$

We assume that the initial pose of B_1 with respect to the inertial coordinate frame $r_{1,0}^0$ is the identity element of $SE(3)$. As a result, the initial pose of the centre of mass of B_1 with respect to B_0 is

$$r_{cm,1} = \begin{bmatrix} 1 & 0 & 0 & l_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the second and third body, the initial relative pose with respect to B_1 is

$$r_{2,0}^1 = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_{3,0}^1 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and we have

$$r_{cm,2} = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_{cm,3} = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where we assume that the centre of mass of B_3 is located in the middle of the body. The initial relative pose of B_4 and B_5 with respect to B_2 is

$$r_{i,0}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \pm c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the relative pose of the centre of mass of B_4 and B_5 with respect to the inertial coordinate frame is

$$r_{cm,i} = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & \pm c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $i = 4, 5$ and plus and minus signs refer to B_4 and B_5, respectively. For the sixth body we have

$$r_{6,0}^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_{cm,6} = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where we assume that the centre of mass of this body is at the sixth joint J_6.

Knowing the above specifications of the system, we identify the configuration manifold of the nonholonomic open-chain multi-body system in this case study by $\mathcal{Q} = \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_6$, where

$$\mathcal{Q}_1 = \left\{ q_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & x \\ \sin(\theta) & \cos(\theta) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \,\middle|\, x, y \in \mathbb{R}, \theta \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_2 = \left\{ q_2 = \begin{bmatrix} \cos(\psi_1) & -\sin(\psi_1) & 0 & 2l\sin^2(\psi_1/2) \\ \sin(\psi_1) & \cos(\psi_1) & 0 & -l\sin(\psi_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_1 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_3 = \left\{ q_3 = \begin{bmatrix} \cos(\varphi_1) & -\sin(\varphi_1) & 0 & 2l_1\sin^2(\varphi_1/2) \\ \sin(\varphi_1) & \cos(\varphi_1) & 0 & -l_1\sin(\varphi_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_1 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_4 = \left\{ q_4 = \begin{bmatrix} \cos(\psi_2) & 0 & \sin(\psi_2) & 2l\sin^2(\psi_2/2) \\ 0 & 1 & 0 & 0 \\ -\sin(\psi_2) & 0 & \cos(\psi_2) & l\sin(\psi_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_2 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_5 = \left\{ q_5 = \begin{bmatrix} \cos(\psi_3) & 0 & \sin(\psi_3) & 2l\sin^2(\psi_3/2) \\ 0 & 1 & 0 & 0 \\ -\sin(\psi_3) & 0 & \cos(\psi_3) & l\sin(\psi_3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \psi_3 \in \mathbb{S}^1 \right\},$$

$$\mathcal{Q}_6 = \left\{ q_6 = \begin{bmatrix} \cos(\varphi_2) & 0 & \sin(\varphi_2) & 2l_1\sin^2(\varphi_2/2) - l_2\sin(\varphi_2) \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi_2) & 0 & \cos(\varphi_2) & l_1\sin(\varphi_2) + 2l_2\sin^2(\varphi_2/2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3) \middle| \varphi_2 \in \mathbb{S}^1 \right\}.$$

In order to calculate the kinetic energy for the system under study, we need to first form the function $F\colon \mathcal{Q} \to \mathcal{P} = \overbrace{SE(3) \times \cdots \times SE(3)}^{6-times}$, which determines the pose of the coordinate frames attached to the centres of mass of the bodies with respect to the inertial coordinate frame.

$$F(q_1, \cdots, q_5) = (q_1 r_{cm,1}, q_1 q_2 r_{cm,2}, q_1 q_3 r_{cm,3}, q_1 q_2 q_4 r_{cm,4}, q_1 q_2 q_5 r_{cm,5}, q_1 q_3 q_6 r_{cm,6})$$

Using (3.22), we can calculate the kinetic energy metric for the open-chain multi-body system. In matrix form we have the following equation for the tangent map $(T_{F(q)} L_{F(q)^{-1}})(T_q F)\colon T_q \mathcal{Q} \to Lie(\mathcal{P})$

$$(T_{F(q)} L_{F(q)^{-1}})(T_q F) = \begin{bmatrix} \mathrm{Ad}_{r_{cm,1}^{-1}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{Ad}_{r_{cm,6}^{-1}} \end{bmatrix} \mathcal{J}_q \begin{bmatrix} T_{q_1}(L_{q_1^{-1}} \circ \iota_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_{q_6}(L_{q_6^{-1}} \circ \iota_6) \end{bmatrix},$$

where we have the following equalities, using the introduced joint parameters:

$$\mathcal{J}_q = \begin{bmatrix} id_6 & 0_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} \\ \mathrm{Ad}_{q_2^{-1}} & id_6 & 0_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} \\ \mathrm{Ad}_{q_3^{-1}} & 0_{6\times 6} & id_6 & 0_{6\times 6} & 0_{6\times 6} & 0_{6\times 6} \\ \mathrm{Ad}_{(q_2q_4)^{-1}} & \mathrm{Ad}_{q_4^{-1}} & 0_{6\times 6} & id_6 & 0_{6\times 6} & 0_{6\times 6} \\ \mathrm{Ad}_{(q_2q_5)^{-1}} & \mathrm{Ad}_{q_5^{-1}} & 0_{6\times 6} & 0_{6\times 6} & id_6 & 0_{6\times 6} \\ \mathrm{Ad}_{(q_3q_6)^{-1}} & 0_{6\times 6} & \mathrm{Ad}_{q_6^{-1}} & 0_{6\times 6} & 0_{6\times 6} & id_6 \end{bmatrix},$$

$$T_{q_1}(L_{q_1^{-1}} \circ \iota_1) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$T_{q_2}(L_{q_2^{-1}} \circ \iota_2) = \begin{bmatrix} 0 & -l & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$T_{q_3}(L_{q_3^{-1}} \circ \iota_3) = \begin{bmatrix} 0 & -l_1 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$T_{q_4}(L_{q_4^{-1}} \circ \iota_4) = \begin{bmatrix} 0 & 0 & l & 0 & 1 & 0 \end{bmatrix}^T,$$

$$T_{q_5}(L_{q_5^{-1}} \circ \iota_5) = \begin{bmatrix} 0 & 0 & l & 0 & 1 & 0 \end{bmatrix}^T,$$

$$T_{q_6}(L_{q_6^{-1}} \circ \iota_6) = \begin{bmatrix} -l_2 & 0 & l_1 & 0 & 1 & 0 \end{bmatrix}^T.$$

Let us denote the standard basis for $se(3)$ by $\{E_1, \cdots, E_6\}$, such that

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, E_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Further, we define the following metrics on the Lie algebras of copies of $SE(3)$ corresponding to the bodies:

$$(h_i)_e = \begin{bmatrix} m_i id_3 & 0_{3\times 3} \\ 0_{3\times 3} & \begin{bmatrix} j_{x,i} & 0 & 0 \\ 0 & j_{y,i} & 0 \\ 0 & 0 & j_{z,i} \end{bmatrix} \end{bmatrix},$$

where $i = 1, \cdots, 6$, m_i is the mass of B_i, and $(j_{x,i}, j_{y,i}, j_{z,i})$ are the moments of inertia of B_i about the X_i, Y_i and Z_i axes of the coordinate frame attached to the centre of mass of B_i. Note that, we chose this coordinate frame such that its axes coincide with the principal axes of the body B_i. For the body B_i ($i = 2, \cdots, 6$), we assume a symmetric cylindrical shape. The cylinder axis is aligned with the Y_i-axis for $i = 2, 4, 5$, so we have $j_{x,i} = j_{z,i}$. Similarly, for the bodies B_3 and B_6, the cylinder axes are aligned with Z_3 and X_6 axes, and we have the equalities $j_{x,3} = j_{y,3}$ and $j_{y,6} = j_{z,6}$. Also, since the wheels are assumed identical, only the dynamic parameters of B_4 is going to appear in the calculations. Therefore, in the coordinates chosen to identify the configuration manifold (joint parameters), we have the following matrix form for $\mathbb{F}L_q$

$$\mathbb{F}L_q = T_q^*(L_{F(q)^{-1}F}) \begin{bmatrix} (h_1)_e & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (h_6)_e \end{bmatrix} T_q(L_{F(q)^{-1}F}) = \begin{bmatrix} K_{11}(q) & \cdots & K_{16}(q) \\ \vdots & \ddots & \vdots \\ K_{61}(q) & \cdots & K_{66}(q) \end{bmatrix}.$$

Here, we have

$$\begin{aligned}
K_{11}(q) &= \begin{bmatrix} m_{tot} & 0 & -\sin(\theta)(lm_2 + 2lm_4 + l_0 m_1 + l_1 m_3 + l_1 m_6) \\ \star & m_{tot} & \cos(\theta)(lm_2 + 2lm_4 + l_0 m_1 + l_1 m_3 + l_1 m_6) \\ \star & \star & j_{z,tot} \end{bmatrix} \\
K_{21}(q) &= K_{12}(q)^T = \begin{bmatrix} 0 & 0 & 2m_4 c^2 + j_{x,2} + 2J_{x,4} \end{bmatrix}, \\
K_{31}(q) &= K_{13}(q)^T = \begin{bmatrix} 0 & 0 & j_{z,3} + j_{x,6}\sin^2(\varphi_2) + j_{y,6}\cos^2(\varphi_2) \end{bmatrix}, \\
K_{i1}(q) &= K_{1i}(q)^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \forall i = 4, 5, 6 \\
K_{22}(q) &= 2m_4 c^2 + j_{x,2} + 2j_{x,4}, \quad K_{i2} = K_{2i} = 0, \forall i = 3, \cdots, 6 \\
K_{33}(q) &= j_{z,3} + j_{x,6}\sin^2(\varphi_2) + j_{y,6}\cos^2(\varphi_2), \quad K_{i3} = K_{3i} = 0, \forall i = 4, 5, 6 \\
K_{44}(q) &= j_{y,4}, \quad K_{i4} = K_{4i} = 0, \forall i = 5, 6 \\
K_{55}(q) &= j_{y,4}, \quad K_{65} = K_{56} = 0, \quad K_{66}(q) = j_{y,6},
\end{aligned}$$

where

$$\begin{aligned}
m_{tot} &= m_1 + m_2 + m_3 + 2m_4 + m_6, \\
j_{z,tot} &= j_{z,1} + j_{x,2} + j_{z,3} + 2j_{x,4} + j_{x,6} + l^2 m_2 + 2l^2 m_4 + l_0^2 m_1 + l_1^2 m_3 \\
&\quad + l_1^2 m_6 + 2c^2 m_4 - j_{x,6}\cos^2(\varphi_2) + j_{y,6}\cos^2(\varphi_2).
\end{aligned}$$

For $K_{11}(q)$, we did not include the lower diagonal elements, since the matrix is symmetric. The kinetic energy is calculated by

$$K_q(\dot{q}, \dot{q}) = \frac{1}{2}\dot{q}^T \mathbb{F}L_q \dot{q},$$

where, with an abuse of notation, $\dot{q}$ is the vector corresponding to the speed of the joint parameters.

For this case study, the potential energy of the nonholonomic open-chain multi-body system is constant, and it does not enter the dynamical equation. As a result, the Hamiltonian of the nonholonomic open-chain multi-body system is calculated by

$$H(q, p) = \frac{1}{2}p^T \mathbb{F}L_q^{-1} p,$$

where p is the vector of generalized momenta corresponding to the joint parameters.

The nonholonomic constraints for the multi-body system under study are the non-slipping conditions on the wheels, i.e., B_1, B_4 and B_5. The linearly independent 1-forms corresponding to the constraints are

$$\begin{aligned}
\omega_1^1 &= -\sin(\theta)dx + \cos(\theta)dy,\\
\omega_1^2 &= -\sin(\theta+\psi_1)dx + \cos(\theta+\psi_1)dy + l\cos(\psi_1)d\theta,\\
\omega_1^3 &= \cos(\theta+\psi_1)dx + \sin(\theta+\psi_1)dy + (l\sin(\psi_1) - c)d\theta - cd\psi_1 - bd\psi_2,\\
\omega_1^4 &= \cos(\theta+\psi_1)dx + \sin(\theta+\psi_1)dy + (l\sin(\psi_1) + c)d\theta + cd\psi_1 - bd\psi_3,
\end{aligned}$$

where b is the radius of each wheel. The distribution $\mathcal{D} \subset T\mathcal{Q}$ is the annihilator of these constraint 1-forms, and it is the span of the following vector fields:

$$\begin{aligned}
\Bigg\{ &\frac{\partial}{\partial\psi_1} + \frac{cl\cos(\psi_1)}{l - c\sin(\psi_1)}\left(\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} + \frac{\tan(\psi_1)}{l}\frac{\partial}{\partial\theta} + \frac{2}{b\cos(\psi_1)}\frac{\partial}{\partial\psi_3}\right)\\
&, \frac{\partial}{\partial\psi_2} + \frac{bl\cos(\psi_1)}{l - c\sin(\psi_1)}\left(\cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} + \frac{\tan(\psi_1)}{l}\frac{\partial}{\partial\theta} + \frac{l + c\sin(\psi_1)}{bl\cos(\psi_1)}\frac{\partial}{\partial\psi_3}\right)\\
&, \frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2}\Bigg\}.
\end{aligned}$$

Here in this example, the base of the multi-body system consists of four bodies, B_1, B_2, B_4 and B_5, and its configuration manifold is $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_4 \times \mathcal{Q}_5$. The Hamiltonian of the system H and the distribution $\mathcal{D}$ are invariant under the action of $\mathcal{G} = \mathcal{Q}_1 \times \mathcal{Q}_3 \times \mathcal{Q}_4 \times \mathcal{Q}_5$, which is isomorphic to $SE(2) \times SO(2) \times SO(2) \times SO(2)$ as a group, by left translation. Now, consider the action of $G = \mathcal{Q}_1 \times \mathcal{Q}_5 \subset \mathcal{G}$ as a subgroup of $\mathcal{G}$, which satisfies the dimensional assumption (2.7) for Chaplygin systems. Using the joint parameters, $\forall (x_0, y_0, \theta_0, \psi_{3,0}) \in G$ we have

$$\widehat{\Phi}_{(x_0,y_0,\theta_0,\psi_{3,0})}(q) = (x\cos(\theta_0) - y\sin(\theta_0) + x_0, x\sin(\theta_0) + y\cos(\theta_0) + y_0, \theta + \theta_0, \psi_3 + \psi_{3,0}, \widehat{q}_1, \overline{q}),$$

where $\widehat{q}_1 = (\psi_1, \psi_2)$ and $\overline{q} = (\varphi_1, \varphi_2)$. We have the principal G-bundle $\widehat{\pi}\colon \mathcal{Q} \to \widehat{\mathcal{Q}} = \mathcal{Q}_2 \times \mathcal{Q}_4 \times \mathcal{Q}_3 \times \mathcal{Q}_6$, and using the joint parameters its corresponding principal connection $\widehat{\mathcal{A}}\colon T\mathcal{Q} \to Lie(G)$ is defined by

$$\widehat{\mathcal{A}}_q = \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & y & 0 \\ \sin(\theta) & \cos(\theta) & -x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{\mathrm{Ad}_{\mathfrak{h}}} \left[\overbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{T_{\mathfrak{h}} L_{\mathfrak{h}^{-1}}} \quad \overbrace{\frac{1}{l - c\sin(\psi_1)} \begin{bmatrix} -lc\cos(\psi_1) & -lb\cos(\psi_1) \\ 0 & 0 \\ -c\sin(\psi_1) & -b\sin(\psi_1) \\ -2lc/b & -(l + c\sin(\psi_1)) \end{bmatrix}}^{\widehat{A}_{\widehat{q}_1}} \quad 0_{4\times 2} \right],$$

where $\mathfrak{h} = (x, y, \theta, \psi_3)$ is an element of $\mathcal{Q}_1 \times \mathcal{Q}_5$. And consequently, the horizontal lift map $\widehat{\mathrm{hl}}_q\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to T_q\mathcal{Q}$ is

$$\widehat{\mathrm{hl}}_q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{l - c\sin(\psi_1)} \begin{bmatrix} lc\cos(\psi_1)\cos(\theta) & lb\cos(\psi_1)\cos(\theta) \\ lc\cos(\psi_1)\sin(\theta) & lb\cos(\psi_1)\sin(\theta) \\ c\sin(\psi_1) & b\sin(\psi_1) \\ 2lc/b & l + c\sin(\psi_1) \end{bmatrix} & 0_{4\times 2} \\ id_4 & \end{bmatrix},$$

where in the above formulation, the first matrix in the multiplication is necessary only to match the order of parameters.

Then, we have

$$\mathbb{F}\widehat{L}_{\widehat{q}} = \widehat{\mathrm{hl}}_q^T \mathbb{F}L_q \widehat{\mathrm{hl}}_q = \begin{bmatrix} \widehat{K}_{11}(\widehat{q}) & \cdots & \widehat{K}_{14}(\widehat{q}) \\ \vdots & \ddots & \vdots \\ \widehat{K}_{41}(\widehat{q}) & \cdots & \widehat{K}_{44}(\widehat{q}) \end{bmatrix},$$

where the following equalities hold:

$$\begin{aligned}
\widehat{K}_{11}(\widehat{q}) = \widehat{A}_{\widehat{q}_1}^T &\begin{bmatrix} K_{11}((e_1,\widehat{q})) & K_{15}((e_1,\widehat{q})) \\ K_{51}((e_1,\widehat{q})) & K_{55}((e_1,\widehat{q})) \end{bmatrix} \widehat{A}_{\widehat{q}_1} - \widehat{A}_{\widehat{q}_1}^T \begin{bmatrix} K_{21}((e_1,\widehat{q})) & K_{25}((e_1,\widehat{q})) \\ K_{41}((e_1,\widehat{q})) & K_{45}((e_1,\widehat{q})) \end{bmatrix}^T \\
&- \begin{bmatrix} K_{21}((e_1,\widehat{q})) & K_{25}((e_1,\widehat{q})) \\ K_{41}((e_1,\widehat{q})) & K_{45}((e_1,\widehat{q})) \end{bmatrix} \widehat{A}_{\widehat{q}_1} + \begin{bmatrix} K_{22}((e_1,\widehat{q})) & K_{24}((e_1,\widehat{q})) \\ K_{42}((e_1,\widehat{q})) & K_{44}((e_1,\widehat{q})) \end{bmatrix}, \\
\widehat{K}_{12}(\widehat{q}) = -\widehat{A}_{\widehat{q}_1}^T &\begin{bmatrix} K_{13}((e_1,\widehat{q})) \\ K_{53}((e_1,\widehat{q})) \end{bmatrix} + \begin{bmatrix} K_{23}((e_1,\widehat{q})) \\ K_{43}((e_1,\widehat{q})) \end{bmatrix} = \widehat{K}_{21}(\widehat{q})^T, \\
\widehat{K}_{13}(\widehat{q}) = -\widehat{A}_{\widehat{q}_1}^T &\begin{bmatrix} K_{16}((e_1,\widehat{q})) \\ K_{56}((e_1,\widehat{q})) \end{bmatrix} + \begin{bmatrix} K_{26}((e_1,\widehat{q})) \\ K_{46}((e_1,\widehat{q})) \end{bmatrix} = \widehat{K}_{31}(\widehat{q})^T, \\
\begin{bmatrix} \widehat{K}_{22}(\widehat{q}) & \widehat{K}_{23}(\widehat{q}) \\ \widehat{K}_{32}(\widehat{q}) & \widehat{K}_{33}(\widehat{q}) \end{bmatrix} &= \begin{bmatrix} K_{33}((e_1,\widehat{q})) & K_{36}((e_1,\widehat{q})) \\ K_{63}((e_1,\widehat{q})) & K_{66}((e_1,\widehat{q})) \end{bmatrix}.
\end{aligned}$$

Here,

$$\widehat{A}_{\widehat{q}_1} = \frac{1}{l - c\sin(\psi_1)} \begin{bmatrix} -lc\cos(\psi_1) & -lb\cos(\psi_1) \\ 0 & 0 \\ -c\sin(\psi_1) & -b\sin(\psi_1) \\ -2lc/b & -(l + c\sin(\psi_1)) \end{bmatrix}.$$

As a result, we can calculate the 2-form $\widehat{\Omega}$ by (2.12)

$$\widehat{\Upsilon}_{12}(\widehat{q},\widehat{p}) = \widehat{p}^T \mathbb{F}\widehat{L}_{\widehat{q}}^{-1} \begin{bmatrix} -\widehat{A}_{\widehat{q}_1}^T \begin{bmatrix} K_{11}((e_1,\widehat{q})) & K_{15}((e_1,\widehat{q})) \\ K_{51}((e_1,\widehat{q})) & K_{55}((e_1,\widehat{q})) \end{bmatrix} + \begin{bmatrix} K_{21}((e_1,\widehat{q})) & K_{25}((e_1,\widehat{q})) \\ K_{41}((e_1,\widehat{q})) & K_{45}((e_1,\widehat{q})) \end{bmatrix} \\ \begin{bmatrix} K_{31}((e_1,\widehat{q})) & K_{35}((e_1,\widehat{q})) \\ K_{61}((e_1,\widehat{q})) & K_{65}((e_1,\widehat{q})) \end{bmatrix} \end{bmatrix}$$
$$\begin{bmatrix} -b(c - l\sin(\psi_1)) \\ 0 \\ -b\cos(\psi_1) \\ -2c\cos(\psi_1) \end{bmatrix} \frac{l}{(l - c\sin(\psi_1))^2},$$
$$\widehat{\Upsilon}_{13} = \widehat{\Upsilon}_{14} = \widehat{\Upsilon}_{23} = \widehat{\Upsilon}_{24} = \widehat{\Upsilon}_{34} = 0,$$

where $\widehat{p}$ is the vector of generalized momenta in the reduced space. Finally, in matrix form we have the following reduced equations of motion for the nonholonomic multi-body system under study:

$$\begin{bmatrix} \dot{\widehat{q}} \\ \dot{\widehat{p}} \end{bmatrix} = \begin{bmatrix} 0_{4\times 4} & id_4 \\ -id_4 & \begin{bmatrix} 0 & \Upsilon(\widehat{q},\widehat{p}) & 0 & 0 \\ -\Upsilon(\widehat{q},\widehat{p}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{\partial \widehat{H}}{\partial \widehat{q}} \\ \frac{\partial \widehat{H}}{\partial \widehat{p}} \end{bmatrix},$$

where $\widehat{H}$ is calculated by (5.30), with $V(q) = 0$.

6.1. Further Reduction of the System

In this subsection we investigate if the system under study demonstrates any conserved quantity due to the action of a bigger symmetry group (bigger than $\mathcal{Q}_1 \times \mathcal{Q}_5$). In this case study, since originally $\mathbb{F}L_q$ is independent of φ_1, the Hamiltonian $\widehat{H}$ is invariant under the action of $\mathcal{N} = \mathcal{Q}_3$ by left translation. Using the joint parameters, for any $\varphi_{1,0}$ we have the action of $\mathcal{N}$ on $T^*\widehat{\mathcal{Q}}$ defined as

$$T^*\breve{\Phi}_{\varphi_{1,0}}(\psi_1,\psi_2,\varphi_1,\varphi_2,\widehat{p}_{\psi_1},\widehat{p}_{\psi_2},\widehat{p}_{\varphi_1},\widehat{p}_{\varphi_2}) = (\psi_1,\psi_2,\varphi_1+\varphi_{1,0},\varphi_2,\widehat{p}_{\psi_1},\widehat{p}_{\psi_2},\widehat{p}_{\varphi_1},\widehat{p}_{\varphi_2}),$$

where $(\psi_1,\psi_2,\varphi_1,\varphi_2,\widehat{p}_{\psi_1},\widehat{p}_{\psi_2},\widehat{p}_{\varphi_1},\widehat{p}_{\varphi_2})$ is a set of coordinates for $T^*\widehat{\mathcal{Q}}$, which can be considered as the reduced space of joint parameters and their corresponding momenta. Also, it is easy to check that the conditions (i) and (ii) on Page 11 are satisfied for this case study. As the result, we have that the momentum map $\breve{\mathbf{M}}_{\widehat{q}}\colon T^*_{\widehat{q}}\widehat{\mathcal{Q}} \to Lie^*(\mathcal{N})$ for the $\mathcal{N}$-action is conserved along the solution curves of the reduced system. Here, the momentum map is defined by

$$\breve{\mathbf{M}}_{\widehat{q}} = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}.$$

We have a principal bundle $\breve{\pi}\colon \widehat{\mathcal{Q}} \to \breve{\mathcal{Q}} = \mathcal{Q}_2 \times \mathcal{Q}_4 \times \mathcal{Q}_6$ with the (mechanical) principal connection $\breve{\mathcal{A}}_{\widehat{q}}\colon T_{\widehat{q}}\widehat{\mathcal{Q}} \to Lie(\mathcal{N})$

$$\breve{\mathcal{A}}_{\widehat{q}} = \frac{1}{l - c\sin(\psi_1)}\begin{bmatrix}c\sin(\psi_1) & b\sin(\psi_1) & l - c\sin(\psi_1) & 0\end{bmatrix}.$$

For a regular value of the momentum map $\vartheta \in Lie^*(\mathcal{N})$, the coadjoint isotropy group $\mathcal{N}_\vartheta = \mathcal{N}$, and the level set of the momentum map is

$$\breve{\mathbf{M}}^{-1}(\vartheta) = \left\{ (\psi_1, \psi_2, \varphi_1, \varphi_2, \widehat{p}_{\psi_1}, \widehat{p}_{\psi_2}, \widehat{p}_{\varphi_1}, \widehat{p}_{\varphi_2}) \in T^*\widehat{\mathcal{Q}} \,\middle|\, \widehat{p}_{\varphi_1} = \vartheta \right\} \subset T^*\widehat{\mathcal{Q}}.$$

Also, we have

$$\alpha_\vartheta = \frac{\vartheta}{l - c\sin(\psi_1)}\left(c\sin(\psi_1)d\psi_1 + b\sin(\psi_1)d\psi_2 + (l - c\sin(\psi_1))d\varphi_1\right) \in \Omega^1(\widehat{\mathcal{Q}}),$$

and hence,

$$\Xi_\vartheta = \frac{\vartheta bl\cos(\psi_1)}{(l - c\sin(\psi_1))^2} d\psi_1 \wedge d\psi_2 \in \Omega^2(T^*\breve{\mathcal{Q}}).$$

We then can calculate the map $\breve{\varphi}_\vartheta^{-1}\colon T^*\breve{\mathcal{Q}} \to \breve{\mathbf{M}}^{-1}(\vartheta)/\mathcal{N}$ by

$$\breve{\varphi}_\vartheta^{-1}(\psi_1, \psi_2, \varphi_2, \breve{p}_{\psi_1}, \breve{p}_{\psi_2}, \breve{p}_{\varphi_2}) = (\psi_1, \psi_2, \varphi_2, \breve{p}_{\psi_1} + \frac{\vartheta c\sin(\psi_1)}{l - c\sin(\psi_1)}, \breve{p}_{\psi_2} + \frac{\vartheta b\sin(\psi_1)}{l - c\sin(\psi_1)}, \breve{p}_{\varphi_2}).$$

As a result, we determine the 2-forms $\breve{\Omega}_\vartheta \in \Omega^2(T^*\breve{\mathcal{Q}})$:

$$\breve{\Omega}_\vartheta = -d\breve{p} \wedge d\breve{q} - \breve{\Upsilon}'(\breve{q}, \breve{p})d\psi_1 \wedge d\psi_2,$$

where

$$\breve{\Upsilon}'(\breve{q}, \breve{p}) := \Upsilon(\psi_1, \psi_2, 0, \varphi_2, \breve{p}_{\psi_1} + \frac{\vartheta c\sin(\psi_1)}{l - c\sin(\psi_1)}, \breve{p}_{\psi_2} + \frac{\vartheta b\sin(\psi_1)}{l - c\sin(\psi_1)}, \vartheta, \breve{p}_{\varphi_2}).$$

Finally, we have the reduced equations of motion in $T^*\breve{\mathcal{Q}}$ as:

$$\begin{bmatrix} \dot{\breve{q}} \\ \dot{\breve{p}} \end{bmatrix} = \begin{bmatrix} 0_{3\times 3} & id_3 \\ -id_3 & \begin{bmatrix} 0 & \breve{\Upsilon}'(\breve{q},\breve{p}) + \frac{\vartheta bl\cos(\psi_1)}{(l-c\sin(\psi_1))^2} & 0 \\ -\breve{\Upsilon}'(\breve{q},\breve{p}) - \frac{\vartheta bl\cos(\psi_1)}{(l-c\sin(\psi_1))^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \frac{\partial \breve{H}_\vartheta}{\partial \breve{q}} \\ \frac{\partial \breve{H}_\vartheta}{\partial \breve{p}} \end{bmatrix},$$

where

$$\breve{H}_\vartheta(\breve{q},\breve{p}) = \widehat{H}(\psi_1,\psi_2,0,\varphi_2,\breve{p}_{\psi_1} + \frac{\vartheta c\sin(\psi_1)}{l-c\sin(\psi_1)}, \breve{p}_{\psi_2} + \frac{\vartheta b\sin(\psi_1)}{l-c\sin(\psi_1)}, \vartheta, \breve{p}_{\varphi_2}).$$

7. Conclusions and Future Work

In this paper, we developed a two-stage reduction method to reduce the dynamical equations of a nonholonomic multi-body system. Through this process, we considered more general cases of multi-body systems, where there exist multi-d.o.f. holonomic and nonholonomic displacement subgroups as a class of multi-d.o.f. joints. The relative configuration manifold of this class of joints is diffeomorphic to a subgroup of $SE(3)$. We used the Chaplygin reduction theorem to express Hamilton's equation in the cotangent bundle of a quotient manifold. We found some sufficient conditions, under which the kinetic energy metric is invariant under the action of a subgroup of the configuration manifold. Accordingly, we extended the Chaplygin reduction theorem to a two-stage reduction process for the dynamical equations of open-chain multi-body systems with multi-d.o.f. holonomic and nonholonomic joints. Finally, we derived the reduced dynamical equations in the local coordinates for an example of a two d.o.f. crane mounted on a four-wheel car to illustrate the results of this paper.

The reduction process introduced in this paper can be unified with the reduction of holonomic multi-body systems. The investigation of the similarities and differences of these two reduction methods will be the subject of our future research.

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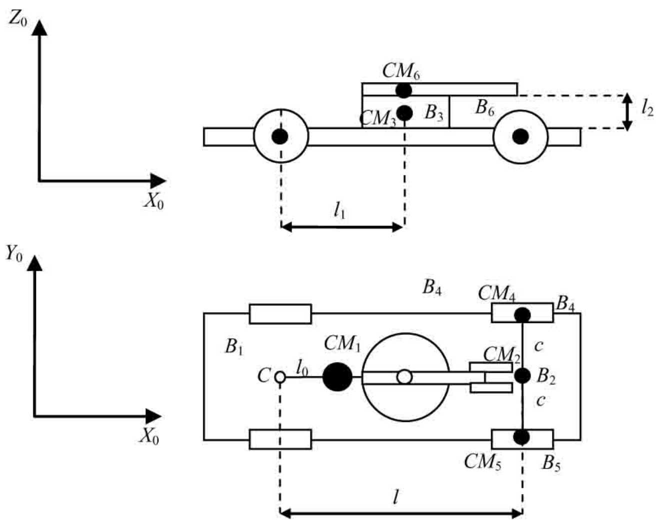
Figure 1: An example of a crane

Figure 2: The coordinate frames attached to the bodies of the crane



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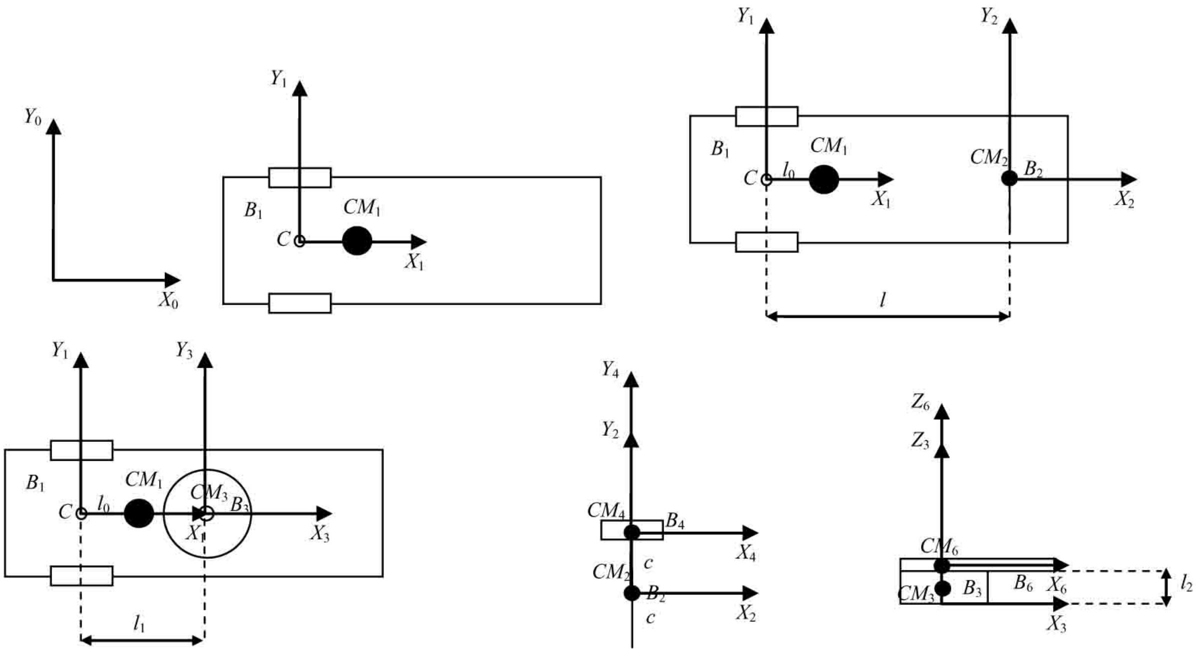

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