Singularity-Free Lagrange-Poincaré Equations on Lie Groups for Vehicle-Manipulator Systems

Borna Monazzah Moghaddam, Robin Chhabra, *Member, IEEE*

***Abstract*—It has been long known that the Euler-Lagrange dynamical equations of fixed-base manipulators with single-degree-of-freedom joints can be formulated on Lie groups following exponential joint parameterizations. Whereas, dynamics of symmetric vehicles can be captured using the Euler-Poincaré equations on Lie groups, with no need to choose any local parameterization. We combine these two geometric approaches to develop a singularity-free Lagrangian formalism for the dynamics of vehicle-manipulator systems. We consider vehicles whose configuration manifolds are Lie sub-groups of the Special Euclidean group, encompassing arbitrary base vehicle motions corresponding to, e.g., ball, planar, or free joints. We revisit the Hamilton-d'Alembert principle for systems on principal bundles to derive the Lagrange-Poincaré equations for vehicle-manipulators with possibly symmetry-breaking external applied wrenches. These equations effectively separate the external (locked-arm system) and internal dynamics (arm's motion) by introducing a block-diagonalized inertia matrix. We then incorporate the exponential parameterization of manipulators to explicitly formulate the reduced dynamics on Lie groups. The resulting equations are in matrix form and can be immediately implemented in simulations and model-based control strategies. The geometrical significance of the proposed formalism is further demonstrated via the step-by-step presentation of a case study.**

***Index Terms*—Vehicle-Manipulator Systems, Lagrange-Poincaré Equations, Singularity-Free Kinematics, Lie Groups**

I. INTRODUCTION

AUTONOMOUS vehicle-manipulator systems (consisting of a robotic arm mounted on a mobile vehicle) are uniquely fit to precise manipulation tasks in harsh environments (see Figure 1). Manipulators provide accurate tracking and manipulation of a target, while the mobility of their base allows them to operate in impassable or distant environments [1]. For example, spacecraft-manipulator systems are currently used for handling payloads via tele-operation [2] and are proposed for on-orbit servicing [3]. Similarly, aerial, underwater, and seaborne manipulators have a demonstrated potential for inspection and servicing [4]–[6]. For the purposes of analysis, mission planning, and control, researchers have been developing alternative dynamic models of vehicle-manipulators as multi-body systems [7], [8], to achieve lucid and rigorous models that can reflect their constraints and capabilities.

The Special Euclidean ($\mathbf{SE}(3)$) group is a matrix Lie group that provides a global representation of the pose of a rigid body [10], [11]. This representation has roots in screw theory,

B. Monazzah Moghaddam is with the Autonomous Space Robotics Laboratory, Carleton University, Ottawa, On, Canada

R. Chhabra is the head of the Autonomous Space Robotics Laboratory, Carleton University, Ottawa, On, Canada

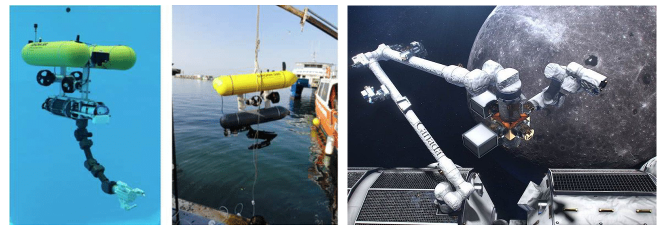

Fig. 1: Examples of vehicle-manipulator systems for underwater recovery missions [9] and in-orbit operations (credit: Canadian Space Agency)

originated by Ball [12] based on the work of Chasles in the 1800s [13]. Using the exponential map of the $\mathbf{SE}(3)$, Brockett pioneered the development of the Product Of Exponentials (POE) formula to describe the forward kinematics of rigid multi-body systems [14]. Having provided its geometric connection to screw theory, Murray presented a Lie group framework for the kinematics, dynamics, and control of fixed-base manipulators in his book [13]. The computational and analytical advantages of this framework was demonstrated by Park *et al.* [15], who integrated the POE into the recursive Newton-Euler and the Euler-Lagrange dynamical equations of multi-body systems [16], [17]. It is argued that the resulting equations reduce the complexity of the model without sacrificing its computational efficiency [18]. The POE has also been incorporated in a variety of robotic applications [19]–[22], e.g. cooperative robotics [18], parameter identification [23]–[25], control [26], design, and motion planning [27]. Chhabra and Emami generalized the POE to include a category of multi-dof joints whose motion can be described via Lie subgroups, using the exponential coordinates of the $\mathbf{SE}(3)$ [28]. Other researchers have also explored screw systems and Lie group presentation to study rigid spatial mechanisms [11] and constrained multi-body chains [29], [30].

General vehicle-manipulators are modelled as moving-base multi-body systems with a multi-dof joint capturing the motion of the base vehicle and a chain of rigid bodies interconnected by single-dof joints forming the manipulator. Dynamic models of vehicle-manipulator systems have been developed based on the recursive Newton-Euler approach [6], [31], [32], Lagrangian formalism [33]–[35] and quasi-Lagrangian formulation [36], [37]. In such models while the exponential map of the $\mathbf{SE}(3)$ can provide a minimal and global parameterization of the single-dof joints [13], [38], [39], the exponential coordinates do not globally parametrize multi-dof vehicles and introduce singularities that are not physically meaningful [40]. The matrix representation however introduces global coordinate-independent parameterization of the multi-dof joints in the system that avoids singularities or the double covering problem of Quaternions [41]–[43]. Since this representation is not necessarily minimal, forming the dynamical equations based on it is not trivial. Duindam *et al.* derived the dynamical equations of systems with general multi-dof joints by introducing a local diffeomorphism between their relative configuration manifold and its tangent space, based on the exponential map of the $\mathbf{SE}(3)$ [44], [45]. This work resulted in the singularity-free Boltzmann-Hamel equations of open-chain multi-body systems with holonomic and non-holonomic constraints [46]. Based on these equations, From *et al.* developed singularity-free formulations for systems with multi-dof joints by introducing minimal quasi-coordinates [47], [48], and applied it to realistic multi-body mechanisms [46], [49] such as under-water [50] and space manipulators [51]. Although being explicit, such formulations for vehicle-manipulator systems do not reveal symmetric properties of the system and often can be complex and difficult to derive.

Lie groups are instrumental to exploit symmetries in multi-body systems, by studying invariance of the energy metric of the system under their actions [52]. Marsden and Ratiu pioneered the utilization of these symmetries in reduction of mechanical systems, including Poisson [53] and symplectic [54], [55] reduction of Hamiltonian systems, and the reduction of Lagrangian systems [56]. By taking advantage of the invariance of systems' Lagrangian on Lie groups, singularity-free Euler-Poincaré equations are formed that can describe vehicles' motion on the $\mathbf{SE}(3)$ or its sub-groups [57], [58]. These equations have been employed to develop controllers for underactuated or fully-actuated systems [59]–[61]. Hamel combined the Euler-Lagrange and Euler-Poincaré equations to propose a more general form of the Lagrangian reduction in a local principal bundle [62], [63], whose work led to defining the notion of quasi-velocities. Inspired by his work, Scheurle and Marsden developed the Lagrangian parallel to the symplectic and Poisson reduction to obtain a set of reduced Euler-Lagrange equations on the tangent bundle of the configuration manifold [64], which was later used for optimal control purposes [65]. Bloch and McClamroch *et al.* built upon this formalism in their works on nonholonomic systems [66], and investigated its integration in and benefits to the controlled mechanical systems [67]. Cendra *et al.* showed that by exploiting symmetries in systems on a Cartesian product of a Lie group and a manifold, Hamilton's principle leads to two sets of equations that combine to form the Lagrange-Poincaré equations [68], [69]. These equations include a set of reduced Euler-Lagrange equations on a conventional phase space e.g., of the manipulator in a vehicle-manipulator, and a set of Euler-Poincaré equations on the Lie group, e.g., representing the vehicle in a vehicle-manipulator [57]. In this formalism the group elements do not appear in the dynamical model [57], and hence the equations are singularity-free in the sense of local parameterization of the Lie group. The reduction approaches developed by or based on the works of Marsden and his collaborators, are often formulated in an abstract language that may conceal physical meanings and may not immediately provide explicit dynamical equations in matrix form. Such properties are yet to be rigorously explored in, e.g., multi-body systems. Further, these approaches, specifically in the case of Lagrange-Poincaré equations, can be extended to handle symmetry-breaking forces, which is neglected in most treatments.

Directly or indirectly, researchers have investigated use of the inherent symmetries of vehicle-manipulators to remove the requirement of parameterizing the base vehicle's motion and to form their singularity-free equations of motion. Incorporating the zero-momentum constraint, Dubowsky and Papadopoulos effectively removed the vehicle's motion on $\mathbf{SE}(3)$ from the Lagrangian dynamical equations and introduced the notion of virtual manipulator to extend the use of fixed-base manipulator models [34], [70]. In a Hamiltonian setup, Chhabra and Emami developed symplectic reduction [71] of holonomic and an extended Chaplygin reduction [72], [73] of nonholonomic vehicle-manipulator systems to remove the vehicle's configuration from the equations of motion. This framework was proved beneficial in the trajectory planning and control of underactuated robotic systems [74], [75]. Taking advantage of the principal bundle structure of their configuration manifold, vehicle-manipulators can be modelled by Lagrange-Poincaré equations. Recently, Mishra *et al.*, without providing enough physical insight, applied these equations from [57], [76] to formulate the equations of motion of a free-floating spacecraft-manipulator [77]. They exploited the block-diagonal property of the inertia matrix in this formalism to set up a hardware-in-the-loop simulation of space manipulators [78] and develop an effective observer for the spacecraft's motion [79].

This article develops a unified model applicable to any vehicle-manipulator system, mathematically modeled as a multi-body system. The multi-body system consists of a multi-dof vehicle and a serial-link manipulator. Our derivations are partially based on the Lagrange-Poincaré equations [57] and on the Hamel equations [64], [80], built upon Lie group treatments of robot kinematics and dynamics in the works of Murray *et al.* [13], Lynch and Park [81], and Bloch [57]. Noteworthy contributions of this paper are summarized as:

1) We provide a singularity-free dynamical model of vehicle-manipulators on Lie groups, and thus, establish a numerically stable model that can be smoothly integrated into model-based nonlinear control logics.
2) Using the Hamilton-d'Alembert principle and the exponential formalism for multi-body systems, we derive the forced Lagrange-Poincaré equations that can handle symmetry-breaking forcing and potential terms.
3) We Provide explicit closed-form matrix equations for inertia, Coriolis and potential terms in the set of reduced dynamical equations on Lie groups, where the measurable quantities of the system are included with respect to the base vehicle.

The provided dynamical formalism effectively separates the representations of the external and internal dynamics, and eliminates the constraint equations for the vehicle. In our treatment, we ensure conveying the physical interpretations of all quantities and geometric structures introduced throughout the paper that are often left abstract in the existing literature.

The paper is structured as follows: We describe the kinematics of relative rigid motions and vehicle-manipulator systems in Section II. The Lagrange-Poincaré treatment of the coupled vehicle-manipulator systems is rigorously developed in Section III, which is accompanied with a step-by-step case study. Section IV concludes the paper with some remarks.

II. Kinematics of Vehicle-Manipulator Systems on Lie Groups

In this section, we provide the kinematics framework that is utilized throughout the article to describe the relative motion of rigid bodies in a moving-base multi-body system.

A. Relative Pose and Velocity

Let us consider two rigid bodies in the multi-body system indexed by i and j, and attach two orthonormal coordinate frames to them. A relative pose g_i^j between Body i and Body j is a map from the coordinate frame attached to Body i to that attached to Body j that is isometric and orientation-preserving. The set of all such relative poses forms a smooth manifold that is called the relative configuration manifold and is denoted by G_i^j. Fixing a relative pose, say $\bar{g}_i^j$, there exists an identification of G_i^j by the space of coordinate transformations of Body i, G_i^i, through left translation, i.e., $G_i^j = \bar{g}_i^j G_i^i$, and one by G_j^j through right translation, i.e., $G_i^j = G_j^j \bar{g}_i^j$. The spaces of coordinate transformations G_i^i and G_j^j are both isomorphic to the Special Euclidean group $\mathbf{SE}(3)$, as Lie groups, and their Lie algebras, respectively denoted by $\mathfrak{g}_i^i$ and $\mathfrak{g}_j^j$, are isomorphic to $\mathbf{se}(3)$ (the Lie algebra of $\mathbf{SE}(3)$). Using the matrix representation of the $\mathbf{SE}(3)$, an element $g_i^j \in G_i^j$ can be represented via

$$g_i^j = \begin{bmatrix} R_i^j & {}^j p_i^j \\ \mathbb{O}_{1\times 3} & 1 \end{bmatrix}. \tag{1}$$

Here, $R_i^j \in \mathbf{SO}(3)$ is a 3×3 matrix that describes the orientation of the coordinate frame attached to Body i relative to that attached to Body j, and ${}^j p_i^j \in \mathbb{R}^3$ is the vector of relative linear position between the origins of the same coordinate frames expressed in the frame of Body j. The symbol $\mathbb{O}$ denotes a zero matrix with appropriate dimensions. For a curve $g_i^j(t) \in G_i^j$, the relative velocity of Body i with respect to Body j, $\dot{g}_i^j(t) = \frac{d}{dt} g_i^j(t)$, can be observed in the coordinate frame of Body i or Body j via left translation ${}^i\hat{V}_i^j = g_i^j(t)^{-1} \dot{g}_i^j(t) \in \mathfrak{g}_i^i$ or right translation ${}^j\hat{V}_i^j = \dot{g}_i^j(t) g_i^j(t)^{-1} \in \mathfrak{g}_j^j$, respectively. The *hat* operator is the vector space isomorphism between $\mathbb{R}^6$ and $\mathbf{se}(3)$, such that,

$$\hat{V} := \begin{bmatrix} \tilde{\omega} & v \\ \mathbb{O}_{1\times 3} & 0 \end{bmatrix}, \tag{2}$$

for every $V = [v^T \quad \omega^T]^T \in \mathbb{R}^6$, where $\tilde{\omega}$ is an element of the Lie algebra of $\mathbf{SO}(3)$, denoted by $\mathbf{so}(3)$, and $v \in \mathbb{R}^3$, corresponding to the angular and linear velocities, respectively. The inverse of the *hat* operator is denoted by *vee*, i.e., $\hat{V}^\vee = V$. The *tilde* operator transforms a vector $\omega \in \mathbb{R}^3$ to a 3×3 skew-symmetric matrix, such that $\tilde{\omega} p = \omega \times p$, for every $p \in \mathbb{R}^3$. Given the curve $g_i^j(t)$, the two relative velocity matrices are calculated as

$$\begin{aligned} {}^i\hat{V}_i^j &:= \begin{bmatrix} (R_i^j)^T \dot{R}_i^j & (R_i^j)^T \, {}^j\dot{p}_i^j \\ \mathbb{O}_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{g}_i^i, \\ {}^j\hat{V}_i^j &:= \begin{bmatrix} \dot{R}_i^j (R_i^j)^T & -\dot{R}_i^j (R_i^j)^T \, {}^j p_i^j + {}^j\dot{p}_i^j \\ \mathbb{O}_{1\times 3} & 0 \end{bmatrix} \in \mathfrak{g}_j^j. \end{aligned} \tag{3}$$

In order to change the coordinate frame of observation, one should use the *Adjoint operator* (for matrix Lie groups):

$${}^j\hat{V}_i^j = g_i^j ({}^i\hat{V}_i^j)(g_i^j)^{-1}. \tag{4}$$

The 6×6 *Adjoint matrix* which transforms the $\mathbb{R}^6$ representations of the relative velocity vectors from the coordinate frame i to frame j is

$$\mathbf{Ad}_{g_i^j} := \begin{bmatrix} R_i^j & ({}^j\tilde{p}_i^j) R_i^j \\ \mathbb{O}_{3\times 3} & R_i^j \end{bmatrix}, \tag{5}$$

such that

$${}^jV_i^j = \mathbf{Ad}_{g_i^j} \, {}^iV_i^j. \tag{6}$$

Based on the Lie bracket of the Lie algebras $\mathfrak{g}_i^i$ and $\mathfrak{g}_j^j$, we can define the *adjoint operator* $\mathbf{ad}_{{}^jV_i^j}(\cdot) := [{}^j\hat{V}_i^j, \hat{(\cdot)}]^\vee$, where in matrix form

$$\mathbf{ad}_{{}^jV_i^j} = \begin{bmatrix} {}^j\tilde{\omega}_i^j & {}^j\tilde{v}_i^j \\ \mathbb{O}_{3\times 3} & {}^j\tilde{\omega}_i^j \end{bmatrix}. \tag{7}$$

B. Constrained Relative Motion

A joint is a mechanism that restricts the relative motion of body i with respect to body j. It defines a restricted relative configuration manifold (by an abuse of naming convention) $Q_i^j \subseteq G_i^j$ that contains the permitted relative poses g_i^j of Body i relative to Body j. In the manipulator, we exclusively use single-dof joints, whose restricted relative configuration manifolds are globally parameterized by the exponential map in the next section. To capture the motion of the base vehicle we use a class of globally parameterized multi-dof joints that are described in the following definition.

Definition 1 (Displacement Sub-group). We call a joint displacement sub-group if its restricted relative configuration manifold Q_i^j is a b-dimensional smooth embedded sub-manifold of G_i^j and satisfies the following properties:

1) Q_i^j is mapped to a b-dimensional embedded Lie sub-group of G_i^i through the left translation map, using a fixed $\bar{g}_i^j \in Q_i^j$. By $Q_i^i := (\bar{g}_i^j)^{-1} Q_i^j \subseteq G_i^i$ we denote the Lie sub-group whose Lie sub-algebra is $\mathfrak{q}_i^i \subseteq \mathfrak{g}_i^i$.
2) Configurations of the joint are uniquely described by members of a b-dimensional matrix Lie group H (with the Lie algebra $\mathfrak{h}$) via a Lie group isomorphism $\iota \colon H \to Q_i^i$, inducing the Lie algebra isomorphism $\iota_0 \colon \mathfrak{h}^\vee \to (\mathfrak{q}_i^i)^\vee$. Thus, we have $g_i^j(h) = \bar{g}_i^j \iota(h)$ for a joint configuration $h \in H$. Here, we use the same notation ($\wedge$ and $\vee$) to identify the isomorphism between $\mathbb{R}^b$ and $\mathfrak{h}$.
3) Every joint velocity is described by the vector $\mathcal{V} \in \mathfrak{h}^\vee \cong \mathbb{R}^b$, such that the relative velocities are ${}^iV_i^j = \iota_0(\mathcal{V})$ and ${}^jV_i^j = \mathbf{Ad}_{g_i^j(h)}\big(\iota_0(\mathcal{V})\big)$, for every $g_i^j(h) \in Q_i^j$.

Note that equivalently this category of joints can be defined using the right translation map to introduce the Lie sub-group $Q_j^j := Q_i^j(\bar{g}_i^j)^{-1} \subseteq G_j^j$. The dimension b can take values in the set $\{1, 2, 3, 4, 6\}$ corresponding to different conjugacy classes of displacement sub-groups of $\mathbf{SE}(3)$ [28]. Such a category of joints can describe the configuration of a wide variety of vehicles, e.g., rovers, drones, spacecraft, rail vehicles, etc., whose motion can include multiple directions. In this paper, we do not use any explicit parametrization of H, the joint configuration manifold of the base vehicle.

1) Parametrization of 1-dof Joints: We exclusively consider 1-dof joints for the manipulator. Their restricted relative configuration manifold Q_i^j corresponds to a single admissible direction of relative velocity, where Q_i^j maps to a 1-parameter Lie subgroup of G_i^i (G_j^j) through left (right) translation. The exponential map of the $\mathbf{SE}(3)$ is a local diffeomorphism from an open neighborhood of $\mathbb{O}_{4\times4} \in \mathbf{se}(3)$ to an open neighborhood of $\mathbb{I}_4 \in \mathbf{SE}(3)$, which geometrically corresponds to a simultaneous rotation about and translation along a fixed vector in $\mathbb{R}^3$, where the ratio of translation to rotation is constant. The symbol $\mathbb{I}$ denotes an identity matrix with appropriate dimensions. The right (left) translation via a fixed relative configuration $\bar{g}_i^j \in Q_i^j$ along with the group exponential map of G_j^j (G_i^i), induced by $\mathbf{SE}(3)$, introduces a parametrization of the relative configuration manifold Q_i^j of 1-dof joints through

$$g_i^j(q) = \exp({}^j\hat{\xi}_i^j q)\bar{g}_i^j = e^{{}^j\hat{\xi}_i^j q}\bar{g}_i^j \in Q_i^j, \tag{8}$$

$$g_i^j(q) = \bar{g}_i^j \exp({}^i\hat{\xi}_i^j q) = \bar{g}_i^j e^{{}^i\hat{\xi}_i^j q} \in Q_i^j, \tag{9}$$

where q is the joint parameter, corresponding to the amount of rotation and/or translation. The *twist vector* ${}^j\xi_i^j$ (${}^i\xi_i^j$) corresponds to the axis of the relative screw motion observed from the coordinate frame of Body j (Body i). In this paper, we only work with the right translation map for joint parameterization, i.e., (8). The exponential term gives the required transformation to go from the fixed relative configuration $\bar{g}_i^j$ to any relative configuration of the joint. The twist vector and twist matrix for different 1-dof joints can be calculated from the following equations:

$${}^j\xi_i^j = \begin{bmatrix} {}^j\nu_i^j \\ {}^j\varpi_i^j \end{bmatrix} \quad \& \quad {}^j\hat{\xi}_i^j = \begin{bmatrix} {}^j\tilde{\varpi}_i^j & {}^j\nu_i^j \\ \mathbb{O}_{1\times3} & 0 \end{bmatrix}, \tag{10}$$

where ${}^j\varpi_i^j \in \mathbb{R}^3$ is a unit vector representing the axis of rotation of the joint observed from Body j, which for prismatic joints becomes zero. For prismatic joints ${}^j\nu_i^j \in \mathbb{R}^3$ is a unit vector in the direction of translation observed from Body j, for revolute joints ${}^j\nu_i^j = -{}^j\varpi_i^j \times {}^j\rho_i^j$, and for helical joints ${}^j\nu_i^j = -{}^j\varpi_i^j \times {}^j\rho_i^j + \mathfrak{p}_i^j {}^j\omega_i^j$. Here, $\mathfrak{p}_i^j$ is the pitch of the helical joint, and ${}^j\rho_i^j$ is the position vector of a point on the joint axis observed from Body j, commonly considered to be the position of the joint. The use of Rodrigues' formula provides a means for calculating the exponential term in the parameterization of 1-dof joints [13]. Let $\hat{\xi} \in \mathbf{se}(3)$ such that $\xi = [\nu^T \quad \varpi^T]^T$:

$$\begin{aligned} e^{\hat{\xi}q} &= \begin{bmatrix} \mathbb{I}_3 & \nu q \\ \mathbb{O}_{1\times3} & 1 \end{bmatrix}, \ \|\varpi\| = 0 \\ e^{\hat{\xi}q} &= \begin{bmatrix} e^{\tilde{\varpi}q} & (\mathbb{I}_3 - e^{\tilde{\varpi}q})\tilde{\varpi}\nu + \varpi\varpi^T\nu q \\ \mathbb{O}_{1\times3} & 1 \end{bmatrix}, \ \|\varpi\| = 1 \end{aligned} \tag{11}$$

where the exponential term $e^{\tilde{\varpi}q} \in \mathbf{SO}(3)$ corresponds to the rotation part of the transformation, found from Rodrigues' formula ($\|\varpi\| = 1$) [13]:

$$e^{\tilde{\varpi}q} = \mathbb{I}_3 + \tilde{\varpi}\sin(q) + \tilde{\varpi}^2\big(1 - \cos(q)\big), \tag{12}$$

and the term $(\mathbb{I}_3 - e^{\tilde{\varpi}q})\tilde{\varpi}\nu + \varpi\varpi^T\nu q$ corresponds to the linear translation.

C. Vehicle-Manipulator System Kinematics

A vehicle-manipulator system moving with respect to an inertial frame $\mathcal{I}$ can be modelled as an open-chain multi-body system. To avoid complexity in notation, we only study a single-branch arm, i.e., a serial-link multi-body system. The extension of this work to multi-branch manipulators is straightforward. A serial-link multi-body system is a collection of $n+1$ bodies indexed by $\{0, 1, \cdots, n\}$, where 0 corresponds to the base vehicle and n corresponds to the end-effector, and $n+1$ joints between the consecutive bodies labeled by the succeeding body's index. We attach reference frames to each link at its preceding joint and express its inertial parameters in these local frames. The pose of the base vehicle relative to the inertial frame $\mathcal{I}$ is described by a displacement sub-group joint (Definition 1) with the b-dimensional restricted relative configuration manifold Q_0^I. This manifold is identified via a Lie group H through the isomorphism ι and the left translation by a fixed relative pose $\bar{g}_0^I \in Q_0^I$. We choose the initial relative configuration of the vehicle in a robotic operation as the aforementioned fixed pose $\bar{g}_0^I$. To avoid singularities due to parameterization, we will directly work with members of H when formulating the kinematics and dynamics of the system. Since we only consider 1-dof joints for the manipulator system, the relative configuration manifolds Q_i^{i-1} $(i = 1, \cdots, n)$ of consecutive manipulator links, are parameterized using the exponential map:

$$g_i^{i-1} = \exp\left({}^{i-1}\hat{\xi}_i^{i-1} q_i\right)\bar{g}_i^{i-1} \in Q_i^{i-1} \tag{13}$$

where ${}^{i-1}\hat{\xi}_i^{i-1}$ is the joint axis defined in (10), q_i is the joint parameter, and $\bar{g}_i^{i-1}$ belongs to Q_i^{i-1}. Let $Q_\mathfrak{m}$ denote the n-dimensional configuration manifold of the manipulator. A member of this manifold consists of a set of joint parameters of the manipulator $q_\mathfrak{m} = (q_1, \cdots, q_n) \in Q_\mathfrak{m}$. We also denote the $(b+n)$-dimensional configuration manifold of the vehicle manipulator by $Q := H \times Q_\mathfrak{m}$.

1) Forward Kinematics: Given a base vehicle configuration $h \in H$ and a set of joint angles $q_\mathfrak{m} \in Q_\mathfrak{m}$, the relative pose of each body in the system with respect to a chosen frame is a smooth map on Q based on the cascade of a sequence of relative poses between the intermediate bodies. For the purposes of this study, the reference frame can be the base vehicle's or the inertial frame. The relative pose between Body $i \in \{1, \dots, n\}$ and the inertial frame is obtained as:

$$g_i^I(h, q_{\mathfrak{m}}) = g_0^I(h) g_i^0(q_{\mathfrak{m}}), \tag{14}$$

where the base vehicle's configuration is

$$g_0^I(h) = \bar{g}_0^I \iota(h) \in Q_0^I, \tag{15}$$

and based on (13), the product of exponentials formula [13] leads to

$$g_i^0(q_{\mathfrak{m}}) = e^{\hat{\xi}_1 q_1} \dots e^{\hat{\xi}_i q_i} \bar{g}_i^0. \tag{16}$$

Here, $\bar{g}_0^I$ and $\bar{g}_i^0 = \bar{g}_1^0 \cdots \bar{g}_i^{i-1}$ are respectively fixed poses of the vehicle relative to $\mathcal{I}$ and Body i relative to the vehicle (commonly chosen to be the initial pose). Also,

$$\xi_i := \mathbf{Ad}_{\bar{g}_{i-1}^0} {}^{i-1}\xi_i^{i-1} \in (\mathfrak{g}_0^0)^\vee, \tag{17}$$

represent the manipulator's joints twists expressed in the base vehicle's initial coordinate frame. In order to calculate the kinetic energy of the system, we further require to find the pose of the center of mass of Body i with respect to $\mathcal{I}$, denoted by $g_{cm,i}^I$, as:

$$g_{cm,i}^I = g_i^I \bar{g}_{cm,i}^i = g_0^I e^{\hat{\xi}_1 q_1} ... e^{\hat{\xi}_i q_i} \bar{g}_{cm,i}^0, \tag{18}$$

where $\bar{g}_{cm,i}^i$ and $\bar{g}_{cm,i}^0 = \bar{g}_i^0 \bar{g}_{cm,i}^i$ are the constant pose of the coordinate frame attached to the center of mass of Body i with respect to the corresponding joint coordinate frame and the base vehicle's frame in its initial configuration, respectively. The end-effector forward kinematics is then defined for the tip of the n^{th} link:

$$g_{\mathfrak{e}}^I(h, q_{\mathfrak{m}}) = g_0^I(h) e^{\hat{\xi}_1 q_1} ... e^{\hat{\xi}_n q_n} \bar{g}_n^0 \bar{g}_{\mathfrak{e}}^n. \tag{19}$$

Example 1. Here, the kinematics of a manipulator mounted on a planar vehicle is formulated. The motion of the vehicle can be modeled by a 3-dof planar joint as can be seen in Figure 2. The multi-body system consists of three bodies, a base vehicle and two links connected via 1-dof revolute joints. The centers of mass are located at the middle of each body.

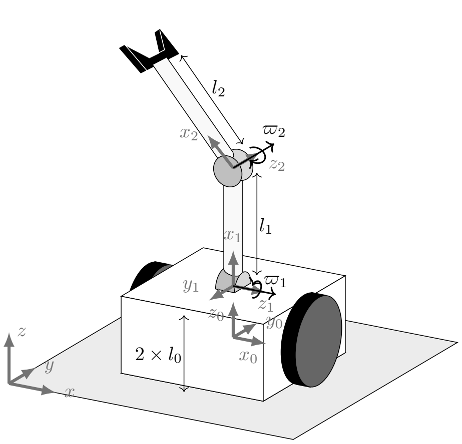

Fig. 2: Sample Rover-Manipulator's Geometric Representation

The fixed initial absolute pose of the vehicle joint which is the same as the pose of its center of mass is

$$\bar{g}_0^I = \bar{g}_{cm,0}^I = \begin{bmatrix} \mathbb{I}_3 & \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T \\ \mathbb{O}_{1\times 3} & 1 \end{bmatrix}. \tag{20}$$

The configuration of the vehicle is well-described by the matrix Lie group $H = \mathbf{SE}(2) \subset \mathbb{R}^{3\times 3}$, such that

$$g_0^I(h) = \bar{g}_0^I \begin{bmatrix} \begin{bmatrix} R_h & \mathbb{O}_{2\times 1} \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} & \begin{bmatrix} p_h \\ 0 \\ 1 \end{bmatrix} \\ \mathbb{O}_{1\times 3} & 1 \end{bmatrix} \tag{21}$$

for every $h = \begin{bmatrix} R_h & p_h \\ \mathbb{O}_{1\times 2} & 1 \end{bmatrix} \in H$ with $R_h \in \mathbf{SO}(2)$ and $p_h \in \mathbb{R}^2$. The fixed relative poses of the first link and its center of mass are

$$\bar{g}_1^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad \bar{g}_{cm,1}^1 = \begin{bmatrix} \mathbb{I}_3 & \begin{bmatrix} \frac{l_1}{2} \\ 0 \\ 0 \end{bmatrix} \\ \mathbb{O}_{1\times 3} & 1 \end{bmatrix}. \tag{22}$$

The first arm link is attached to the vehicle at ${}^0\rho_1^0 = \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T$, and its corresponding revolute joint generates rotation about ${}^0\varpi_1^0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, and ${}^0\nu_1^0 = \begin{bmatrix} 0 & l_0 & 0 \end{bmatrix}^T$, resulting in:

$$\xi_1 = \begin{bmatrix} 0 \\ l_0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e^{\hat{\xi}_1 q_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q_1 & -\sin q_1 & l_0 \sin q_1 \\ 0 & \sin q_1 & \cos q_1 & l_0(1 - \cos q_1) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Therefore, pose of Body 1 relative to the vehicle is $g_1^0(q_1) = e^{\hat{\xi}_1 q_1} \bar{g}_1^0$. The second link is attached to the first at the fixed relative linear position ${}^1\rho_2^1 = \begin{bmatrix} l_1 & 0 & 0 \end{bmatrix}^T$ in the first link's frame, which corresponds to the ${}^0\rho_2^0 = \begin{bmatrix} 0 & 0 & l_{01} \end{bmatrix}^T$,where $l_{01} = l_0 + l_1$. The corresponding revolute joint generates rotation about ${}^0\varpi_2^0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and ${}^0\nu_2^0 = \begin{bmatrix} -l_{01} & 0 & 0 \end{bmatrix}^T$:

$$\xi_2 = \begin{bmatrix} -l_{01} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e^{\hat{\xi}_2 q_2} = \begin{bmatrix} \cos q_2 & 0 & \sin q_2 & -l_{01} \sin q_2 \\ 0 & 1 & 0 & 0 \\ -\sin q_2 & 0 & \cos q_2 & l_{01}(1 - \cos q_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

The fixed relative poses corresponding to the second link are

$$\bar{g}_2^1 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad \bar{g}_{cm,2}^2 = \begin{bmatrix} \mathbb{I}_3 & \begin{bmatrix} \frac{l_2}{2} \\ 0 \\ 0 \end{bmatrix} \\ \mathbb{O}_{1\times 3} & 1 \end{bmatrix}. \tag{25}$$

Therefore, pose of Body 2 relative to the vehicle is $g_2^0(q_1, q_2) = e^{\hat{\xi}_1 q_1} e^{\hat{\xi}_2 q_2} \bar{g}_1^0 \bar{g}_2^1$. Finally, the end-effector is at the tip of the last link and is assumed to be oriented in a manner to match the orientation of the last link,

$$g_{\mathfrak{e}}^2 = \bar{g}_{\mathfrak{e}}^2 = \begin{bmatrix} \mathbb{I}_3 & \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \\ \mathbb{O}_{1\times 3} & 1 \end{bmatrix}. \tag{26}$$

Hence, the end-effector forward kinematics is computed through $g_{\mathfrak{e}}^I(h, q_1, q_2) = g_0^I(h) e^{\hat{\xi}_1 q_1} e^{\hat{\xi}_2 q_2} \bar{g}_1^0 \bar{g}_2^1 \bar{g}_{\mathfrak{e}}^2$.

2) Differential Kinematics: In this section, we formulate the absolute velocity of rigid bodies in a vehicle-manipulator system expressed in the inertial or body frame, using appropriate Jacobian mappings. Let $\mathfrak{h} \cong \mathfrak{q}_0^0 \subseteq \mathfrak{g}_0^0$ be the Lie algebra of $H \cong Q_0^0 := (\bar{g}_0^I)^{-1} Q_0^I \subseteq G_0^0$ with the isomorphism $\iota_0 : \mathfrak{h}^\vee \to (\mathfrak{q}_0^0)^\vee$. We study the differential kinematics of the system based on the left-trivialization of TH, i.e., elements in the form $(h, \mathcal{V}, q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}}) \in H \times \mathfrak{h}^\vee \times TQ_{\mathfrak{m}}$. Note that through ι_0, every element of $\mathfrak{h}$ maps to a body velocity of the vehicle.

Lemma 1. *In matrix form, the velocity of Body i relative to the inertial frame can be expressed in the inertial or Body i coordinate frame as*

$${}^I V_i^I = {}^I J_i^I \begin{bmatrix} \mathcal{V} \\ \dot{q}_{\mathfrak{m}} \end{bmatrix} \quad \& \quad {}^i V_i^I = {}^i J_i^I \begin{bmatrix} \mathcal{V} \\ \dot{q}_{\mathfrak{m}} \end{bmatrix}, \tag{27}$$

where the inertial and body Jacobian matrices are

$${}^I J_i^I(h, q_{\mathfrak{m}}) = \boldsymbol{Ad}_{g_0^I(h)} {}^0 J_i^I \quad \& \quad {}^i J_i^I(q_{\mathfrak{m}}) = \boldsymbol{Ad}_{(g_i^0)^{-1}} {}^0 J_i^I, \tag{28}$$

and the base Jacobian matrix is

$${}^0 J_i^I(q_{\mathfrak{m}}) = \begin{bmatrix} \iota_0 & \xi_1 & \cdots & \boldsymbol{Ad}_{e^{\hat{\xi}_1 q_1} \ldots e^{\hat{\xi}_{i-1} q_{i-1}}} \xi_i & \mathbb{O}_{6\times(n-i)} \end{bmatrix}, \tag{29}$$

where the twists ξ_i ($i = 1, \ldots, n$) have been defined by (17).

Proof . In an open-chain multi-body system, there exists a distinct path between the inertial frame and each body in the chain. The velocity of Body i in the multi-body system relative to the inertial frame is found by adding up all the relative velocities of neighbouring bodies:

$${}^I V_i^I = {}^I V_0^I + {}^I V_1^0 + \cdots + {}^I V_i^{i-1}, \tag{30}$$

where all the individual relative velocities must belong to the common Lie algebra $(\mathfrak{g}_I^I)^\vee$. The relative velocity of Body i with respect to its neighboring Body $i-1$ for all $i \in \{0, 1, \cdots, n\}$ and expressed in the inertial frame is

$${}^I V_i^{i-1} = \mathbf{Ad}_{g_0^I g_{i-1}^0} {}^{i-1} V_i^{i-1}, \tag{31}$$

where the relative velocity ${}^{i-1}\hat{V}_i^{i-1} \in \mathfrak{g}_{i-1}^{i-1}$ corresponding to Joint i can be obtained from (3). Substituting (31) in (30),

$${}^I V_i^I = {}^I V_0^I + \mathbf{Ad}_{g_0^I} {}^0 V_1^0 + \mathbf{Ad}_{g_0^I g_1^0} {}^1 V_2^1 + \cdots + \mathbf{Ad}_{g_0^I g_{i-1}^0} {}^{i-1} V_i^{i-1}. \tag{32}$$

Considering the fact that a vehicle-manipulator's planning and control system often works within the frame of the base vehicle, we express the contributions of the manipulator and the base vehicle in the velocity of Body $i \in \{1, \ldots, n\}$ in the vehicle's coordinate frame:

$${}^I V_i^I = \mathbf{Ad}_{g_0^I} ({}^0 V_0^I + {}^0 V_i^0). \tag{33}$$

The relative poses corresponding to the joints in the manipulator chain are parameterized using (13):

$$\begin{aligned} {}^0 V_i^0 =& {}^0 V_1^0 + \mathbf{Ad}_{e^{\hat{\xi}_1 q_1}} \left(\mathbf{Ad}_{\bar{g}_1^0} {}^1 V_2^1 \right) + \\ & \cdots + \mathbf{Ad}_{e^{\hat{\xi}_1 q_1} \ldots e^{\hat{\xi}_{i-1} q_{i-1}}} \left(\mathbf{Ad}_{\bar{g}_{i-1}^0} {}^{i-1} V_i^{i-1} \right) \\ =& \xi_1 \dot{q}_1 + \cdots + \mathbf{Ad}_{e^{\hat{\xi}_1 q_1} \ldots e^{\hat{\xi}_{i-1} q_{i-1}}} \xi_i \dot{q}_i. \end{aligned} \tag{34}$$

The base's velocity can be identified via a left translation by g_0^I to the Lie algebra of the base

$${}^0 \hat{V}_0^I = (g_0^I)^{-1} \dot{g}_0^I \in \mathfrak{q}_0^0 \subseteq \mathfrak{g}_0^0. \tag{35}$$

As $Q_0^I = \bar{g}_0^I Q_0^0$, each element g_0^I and its derivative can be identified by $g_0^I = \bar{g}_0^I g_0^0$ and $\dot{g}_0^I = \bar{g}_0^I \dot{g}_0^0$ for a unique $g_0^0(t) \in Q_0^0$, which provides

$${}^0 V_0^I = \left((g_0^0)^{-1} \dot{g}_0^0 \right)^\vee \in \iota_0(\mathfrak{h}^\vee) \subseteq (\mathfrak{g}_0^0)^\vee. \tag{36}$$

Given the element $\hat{\mathcal{V}} \in \mathfrak{h}$ corresponding to the base vehicle's body velocity, we have

$${}^0 V_0^I = \iota_0 \mathcal{V}. \tag{37}$$

Substituting (37) and (34) into (33), we find

$${}^I V_i^I(h, \mathcal{V}, q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}}) = \mathbf{Ad}_{g_0^I(h)} (\iota_0 \mathcal{V} + {}^0 V_i^0(q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}})). \tag{38}$$

By appropriately defining ${}^0 J_i^I$ as (29) we arrive at the equations in the statement of the lemma. ■

Since these velocities will be paired with body inertia matrices in calculating the kinetic energy, we have represented the absolute velocity of each body in both the inertial and body coordinate frames. The body velocity ${}^i V_i^I$ is independent of g_0^I; and hence, the kinetic energy of the system will be symmetric with respect to the action of the Lie group associated with the configuration of the base vehicle.

Example 2. We now present the differential kinematics of the described sample vehicle-manipulator in 1. The constant isomorphism corresponding to the base vehicle's velocity is

$$\iota_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

The rover's restricted body velocity is:

$$\mathcal{V} = \begin{bmatrix} {}^0 v_{0,x}^I \\ {}^0 v_{0,y}^I \\ {}^0 \omega_{0,z}^I \end{bmatrix}. \tag{40}$$

To calculate the inertial velocities of all bodies in the system, which are essential for the calculations of the dynamics later on, we must calculate the spatial and body Jacobians of each of the bodies, based on (28) and (29). For this, we first calculate the base-vehicle Jacobians:

$$ {}^{0}J_0^I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \& \quad {}^{0}J_1^I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{41} $$

$$ {}^{0}J_2^I = \begin{bmatrix} 1 & 0 & 0 & 0 & -l_1 - l_0 \cos q_1 \\ 0 & 1 & 0 & l_0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos q_1 \\ 0 & 0 & 1 & 0 & \sin q_1 \end{bmatrix} \tag{42} $$

where we have used the $\mathbf{Ad}_{e^{\hat{\xi}_1 q_1}}$ calculated based on (5):

$$ \mathbf{Ad}_{e^{\hat{\xi}_1 q_1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & l_0(1-\cos q_1) & l_0 \sin q_1 \\ 0 & \cos q_1 & -\sin q_1 & l_0(1-\cos q_1) & 0 & 0 \\ 0 & \sin q_1 & \cos q_1 & -l_0 \sin q_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos q_1 & -\sin q_1 \\ 0 & 0 & 0 & 0 & \sin q_1 & \cos q_1 \end{bmatrix}. \tag{43} $$

We can similarly find $\mathbf{Ad}_{e^{\hat{\xi}_2 q_2}}$ based on (5):

$$ \mathbf{Ad}_{e^{\hat{\xi}_2 q_2}} = \begin{bmatrix} \cos q_2 & 0 & \sin q_2 & 0 & (l_{01})(\cos q_2 - 1) & 0 \\ 0 & 1 & 0 & (l_{01})(\cos q_2 - 1) & 0 & (l_{01})\sin q_2 \\ -\sin q_2 & 0 & \cos q_2 & 0 & -(l_{01})\sin q_2 & 0 \\ 0 & 0 & 0 & \cos q_2 & 0 & \sin q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin q_2 & 0 & \cos q_2 \end{bmatrix}. \tag{44} $$

Then, the spatial Jacobians can be found:

$$ {}^{I}J_0^I = \mathbf{Ad}_{g_0^I} {}^{0}J_0^I \quad \& \quad {}^{I}J_1^I = \mathbf{Ad}_{g_0^I} {}^{0}J_1^I \quad \& \quad {}^{I}J_2^I = \mathbf{Ad}_{g_0^I} {}^{0}J_2^I \tag{45} $$

where, based on (5), we have

$$ \mathbf{Ad}_{g_0^I(h)} = \begin{bmatrix} \begin{bmatrix} R_h & \mathbb{O}_{2\times1} \\ \mathbb{O}_{1\times2} & 1 \end{bmatrix} & \widetilde{\begin{bmatrix} p_h \\ l_0 \end{bmatrix}} \begin{bmatrix} R_h & \mathbb{O}_{2\times1} \\ \mathbb{O}_{1\times2} & 1 \end{bmatrix} \\ \mathbb{O}_{3\times3} & \begin{bmatrix} R_h & \mathbb{O}_{2\times1} \\ \mathbb{O}_{1\times2} & 1 \end{bmatrix} \end{bmatrix}. \tag{46} $$

The inertial velocities of each of the bodies are then found based on (27).

III. Vehicle-Manipulator System Dynamics

In this section, we present an intrinsic formulation for the dynamics of vehicle-manipulator systems evolving on the Cartesian product of the Lie group H representing the base motion and the smooth manifold $Q_{\mathfrak{m}}$ capturing the manipulator's configuration. In the derivation of the equations we use the principal bundle structure of the configuration manifold. The equations are in the form of Lagrange-Poincaré equations combining a reduced set of Euler-Lagrange equations for the arm motion on $Q_{\mathfrak{m}}$ and a set of Euler-Poincaré equations for the vehicle motion on H. This introduces a decomposition of the equations of motion into the singularity-free dynamics of the base vehicle and the manipulator dynamics. The decomposition also allows for studying the effects of internal and external forces and their couplings in an elucidated fashion. The dynamics of the vehicle-manipulator system can be derived through the Hamilton-d'Alembert principle, based on the applied forces to the system and the $\mathcal{L}$agrangian $\mathcal{L}\colon TQ \to \mathbb{R}$ defined as:

$$ \mathcal{L} = K - U, \tag{47} $$

where $K\colon TQ \to \mathbb{R}$ is the total kinetic energy, and $U\colon Q \to \mathbb{R}$ is the potential energy. The kinetic energy of the vehicle-manipulator system is the sum of the kinetic energies of all bodies in the chain. Using absolute velocities in the body coordinate frames:

$$ \begin{aligned} K(h, \dot{h}, q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}}) &= \frac{1}{2} \ll {}^{i}V_i^I, {}^{i}V_i^I \gg \\ &= \sum \frac{1}{2} {}^{i}V_i^{I\,T} ({}^{i}\mathcal{M}_i) {}^{i}V_i^I. \end{aligned} \tag{48} $$

where the body velocity ${}^{i}V_i^I$ is found from (27). The inner product $\ll, \gg$ is defined based on the left-invariant metrics ${}^{i}\mathcal{M}_i\colon \mathfrak{g}_i^i \to (\mathfrak{g}_i^i)^*$, where $(\cdot)^*$ denotes the dual of a vector space, found from Body i's inertia properties in the respective joint coordinate frame:

$$ {}^{i}\mathcal{M}_i = \mathbf{Ad}^{-T}_{g^i_{cm,i}} \begin{bmatrix} m_i \times \mathbb{I}_3 & \mathbb{O}_{3\times3} \\ \mathbb{O}_{3\times3} & \mathfrak{I}_i \end{bmatrix} \mathbf{Ad}^{-1}_{g^i_{cm,i}}. \tag{49} $$

Here, m_i is the mass of Body i and the 3×3 matrix $\mathfrak{I}_i$ is the same body's moment of inertia. The body velocities can be collected in a vector form $\begin{bmatrix} ({}^{0}V_0^I)^T & \cdots & ({}^{n}V_n^I)^T \end{bmatrix}^T$, and each substituted from (27) to find

$$ \begin{bmatrix} {}^{0}V_0^I \\ \vdots \\ {}^{n}V_n^I \end{bmatrix} = \begin{bmatrix} {}^{0}J_0^I(q_{\mathfrak{m}}) \\ {}^{1}J_1^I(q_{\mathfrak{m}}) \\ \vdots \\ {}^{n}J_n^I(q_{\mathfrak{m}}) \end{bmatrix} \begin{bmatrix} \mathcal{V} \\ \dot{q}_{\mathfrak{m}} \end{bmatrix}. \tag{50} $$

We introduce the total Jacobian matrix of the multi-body system $\mathcal{J}$ as

$$ \mathcal{J}(q_{\mathfrak{m}}) := \mathrm{diag}_0^n\{\mathbf{Ad}_{\bar{g}_i^0}\} \begin{bmatrix} {}^{0}J_0^I(q_{\mathfrak{m}}) \\ {}^{1}J_1^I(q_{\mathfrak{m}}) \\ \vdots \\ {}^{n}J_n^I(q_{\mathfrak{m}}) \end{bmatrix}, \tag{51} $$

where the function $\mathrm{diag}_0^n\{\cdot\}$ is the block diagonal matrix of its arguments for $i = 0, \ldots, n$ (note that $\mathbf{Ad}_{\bar{g}_0^0} = \mathbb{I}_{6\times6}$). Therefore,

$$ \begin{bmatrix} {}^{0}V_0^I \\ \vdots \\ {}^{n}V_n^I \end{bmatrix} = \mathrm{diag}_0^n\{\mathbf{Ad}^{-1}_{\bar{g}_i^0}\} \mathcal{J} \begin{bmatrix} \mathcal{V} \\ \dot{q}_{\mathfrak{m}} \end{bmatrix}, \tag{52} $$

and the kinetic energy in (48) becomes

$$ K = \frac{1}{2} \begin{bmatrix} \mathcal{V} \\ \dot{q}_{\mathfrak{m}} \end{bmatrix}^T M(q_{\mathfrak{m}}) \begin{bmatrix} \mathcal{V} \\ \dot{q}_{\mathfrak{m}} \end{bmatrix}, \tag{53} $$

for the generalized mass matrix

$$ M(q_{\mathfrak{m}}) = \mathcal{J}^T (\mathrm{diag}_0^n\{\mathfrak{M}_i\}) \mathcal{J}. \tag{54} $$

Here, $\mathfrak{M}_i$ is the inertial mass matrix of Body i oriented in its initial configuration:

$$ \mathfrak{M}_i := \mathbf{Ad}^{-T}_{\bar{g}_i^0} ({}^{i}\mathcal{M}_i) \mathbf{Ad}^{-1}_{\bar{g}_i^0}. \tag{55} $$

Since the resulting kinetic energy is independent of the pose of the base, we can drop the metric used in the definition of the $\mathcal{L}$agrangian $\mathcal{L}$ to $\mathfrak{h}^\vee \times TQ_\mathfrak{m}$. The potential U can also be dropped to $Q_\mathfrak{m}$ if it is assumed invariant. We discuss inclusion of H-dependent potentials in Remark 1.

Lemma 2. *For a symmetric potential energy independent of base configuration (elements of H), the $\mathcal{L}$agrangian of a vehicle-manipulator system on TQ drops to a reduced $\mathcal{L}$agrangian $\ell^0 \colon \mathfrak{h}^\vee \times TQ_\mathfrak{m} \to \mathbb{R}$:*

$$\ell^0(\mathcal{V}, q_\mathfrak{m}.\dot{q}_\mathfrak{m}) = \frac{1}{2}\begin{bmatrix}\mathcal{V}\\ \dot{q}_\mathfrak{m}\end{bmatrix}^T M(q_\mathfrak{m})\begin{bmatrix}\mathcal{V}\\ \dot{q}_\mathfrak{m}\end{bmatrix} - u, \tag{56}$$

where $M(q_\mathfrak{m})\colon (\mathfrak{h}^\vee \times T_{q_\mathfrak{m}}Q_\mathfrak{m}) \times (\mathfrak{h}^\vee \times T_{q_\mathfrak{m}}Q_\mathfrak{m}) \to \mathbb{R}$ is the reduced kinetic energy metric, given in (54)*, and $u := U(\mathbb{I}, q_\mathfrak{m})\colon Q_\mathfrak{m} \to \mathbb{R}$ is the reduced potential energy.*

Lemma 3. *The total Jacobian defined in* (51) *can be decomposed into a constant matrix Ξ and a configuration-dependent matrix $\mathfrak{L}$:*

$$\mathcal{J}(q_\mathfrak{m}) = \mathfrak{L}(q_\mathfrak{m})\Xi. \tag{57}$$

where

$$\Xi := \begin{bmatrix}\iota_0 & \mathbb{O}_{6\times n}\\ \mathbb{O}_{6n\times b} & diag_1^n\{\xi_i\}\end{bmatrix} = \begin{bmatrix}\iota_0 & \mathbb{O}_{6\times n}\\ \mathbb{O}_{6n\times b} & \Xi_\mathfrak{m}\end{bmatrix}, \tag{58}$$

with each ξ_i defined in (17)*. Further,*

$$\mathfrak{L}(q_\mathfrak{m}) := \begin{bmatrix}\mathbb{I}_6 & \mathbb{O}_{6\times6} & \cdots & \cdots & \mathbb{O}_{6\times6}\\ \mathfrak{Ad}_1^1 & \mathbb{I}_6 & \mathbb{O}_{6\times6} & \cdots & \mathbb{O}_{6\times6}\\ \mathfrak{Ad}_2^1 & \mathfrak{Ad}_2^2 & \mathbb{I}_6 & \cdots & \mathbb{O}_{6\times6}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathfrak{Ad}_n^1 & \mathfrak{Ad}_n^2 & \mathfrak{Ad}_n^3 & \cdots & \mathbb{I}_6\end{bmatrix} \tag{59}$$

where

$$\mathfrak{Ad}_i^j = \boldsymbol{Ad}_{e^{-\xi_i q_i}\cdots e^{-\xi_j q_j}}, \quad i \geq j \in \{1, \cdots, n\}. \tag{60}$$

Proof. By substituting the body and base Jacobians from (28) and (29) into the definition of the total Jacobian in (51), we observe that the resulting $\mathcal{J}$ is equal to the product $\mathfrak{L}(q_\mathfrak{m})\Xi$ as in the statement of the lemma. Note that the block diagonal elements of the matrix $\mathfrak{L}$ have been substituted by the identity matrix, since for every $i \in \{1, \cdots, n\}$ we have the identity:

$$\mathbf{Ad}_{e^{\xi_i q_i}}\xi_i = \xi_i. \tag{61}$$

■

The matrix $\mathfrak{L}$ can be further decomposed into block components that will be used in the definition of the mechanical connection:

$$\mathfrak{L} = \begin{bmatrix}\mathbb{I}_{6\times6} & \mathbb{O}_{6\times6n}\\ \mathfrak{L}_{\mathfrak{m}0} & \mathfrak{L}_\mathfrak{m}\end{bmatrix} \tag{62}$$

where

$$\mathfrak{L}_{\mathfrak{m}0} := \begin{bmatrix}\mathfrak{Ad}_1^1\\ \mathfrak{Ad}_2^1\\ \vdots\\ \mathfrak{Ad}_n^1\end{bmatrix} \quad \& \quad \mathfrak{L}_\mathfrak{m} = \begin{bmatrix}\mathbb{I}_{6\times6} & \mathbb{O}_{6\times6} & \cdots & \mathbb{O}_{6\times6}\\ \mathfrak{Ad}_2^2 & \mathbb{I}_{6\times6} & \cdots & \mathbb{O}_{6\times6}\\ \vdots & \vdots & \vdots & \vdots\\ \mathfrak{Ad}_n^2 & \mathfrak{Ad}_n^3 & \cdots & \mathbb{I}_{6\times6}\end{bmatrix}. \tag{63}$$

It can be observed from the definition of the mass matrix in (51), (54), and (59) that:

$$\ell^0 = \frac{1}{2}\begin{bmatrix}\mathcal{V}\\ \dot{q}_\mathfrak{m}\end{bmatrix}^T \begin{bmatrix}M_0 & M_{0\mathfrak{m}}\\ M_{0\mathfrak{m}}^T & M_\mathfrak{m}\end{bmatrix}\begin{bmatrix}\mathcal{V}\\ \dot{q}_\mathfrak{m}\end{bmatrix} - u, \tag{64}$$

where the block components $M_0 \colon \mathfrak{h}^\vee \to (\mathfrak{h}^\vee)^*$, $M_\mathfrak{m} \colon TQ_\mathfrak{m} \to (\mathfrak{h}^\vee)^*$ and $M_{0\mathfrak{m}} \colon TQ_\mathfrak{m} \to (\mathfrak{h}^\vee)^*$ are given by:

$$M_0 = \iota_0^T(\mathfrak{M}_0 + \mathfrak{L}_{\mathfrak{m}0}^T(\text{diag}_1^n\{\mathfrak{M}_i\})\mathfrak{L}_{\mathfrak{m}0})\iota_0, \tag{65}$$

$$M_{0\mathfrak{m}} = \iota_0^T {\mathfrak{L}_{\mathfrak{m}0}}^T(\text{diag}_1^n\{\mathfrak{M}_i\})\mathfrak{L}_\mathfrak{m}\Xi_\mathfrak{m}, \tag{66}$$

$$M_\mathfrak{m} = \Xi_\mathfrak{m}^T\mathfrak{L}_\mathfrak{m}^T(\text{diag}_1^n\{\mathfrak{M}_i\})\mathfrak{L}_\mathfrak{m}\Xi_\mathfrak{m}. \tag{67}$$

Definition 2. We define the *spatial mechanical connection* [57] $\mathcal{A}_I \colon H \times \mathfrak{h}^\vee \times TQ_\mathfrak{m} \to \mathfrak{h}^\vee$ on the principal bundle $H \times Q_\mathfrak{m} \to Q_\mathfrak{m}$ based on the left translation action of H to be the map that assigns to each $(\mathcal{V}, \dot{q}_\mathfrak{m})$ the corresponding absolute velocity of the locked system expressed in the inertial coordinate frame:

$$\mathcal{A}_I(h, \mathcal{V}, q_\mathfrak{m}, \dot{q}_\mathfrak{m}) := \mathbf{Ad}_h(\mathcal{V} + \mathcal{A}(q_\mathfrak{m})\dot{q}_\mathfrak{m}), \tag{68}$$

with the fiber-wise linear mapping $\mathcal{A} \colon TQ_\mathfrak{m} \to \mathfrak{h}^\vee$ defined as

$$\mathcal{A}(q_\mathfrak{m}) := M_0^{-1}(q_\mathfrak{m})M_{0\mathfrak{m}}(q_\mathfrak{m}). \tag{69}$$

We define the *body mechanical connection* $\mathcal{A}_0 : H \times \mathfrak{h}^\vee \times TQ_\mathfrak{m} \to \mathfrak{h}^\vee$ on the principal bundle $H \times Q_\mathfrak{m} \to Q_\mathfrak{m}$ based on the right action of H as the map that assigns to each $(\mathcal{V}, \dot{q}_\mathfrak{m})$ the corresponding absolute velocity of the locked system expressed in the base vehicle's coordinate frame:

$$\mathcal{A}_0(h, \mathcal{V}, q_\mathfrak{m}, \dot{q}_\mathfrak{m}) := \mathcal{V} + \mathcal{A}(q_\mathfrak{m})\dot{q}_\mathfrak{m}. \tag{70}$$

Note that $\mathcal{A}_0$ is independent of the elements of H. ■

Both mechanical connections are defined on the tangent bundle of the configuration manifold $TQ = TH \times TQ_\mathfrak{m}$, where the component TH has been conveniently left-trivialized. The vertical and horizontal sub-bundles of the connections $\mathcal{A}_I$ and A_0 are the same, and are respectively identified by:

$$\begin{aligned}\mathbf{Ver}_q &:= \{(\Omega_{loc}, 0) | \Omega_{loc} \in \mathfrak{h}^\vee\} \subset \mathfrak{h}^\vee \times T_{q_\mathfrak{m}}Q_\mathfrak{m},\\ \mathbf{Hor}_q &:= \{(\mathcal{V}, \dot{q}_\mathfrak{m}) | \mathcal{V} + \mathcal{A}\dot{q}_\mathfrak{m} = 0\} \subset \mathfrak{h}^\vee \times T_{q_\mathfrak{m}}Q_\mathfrak{m}.\end{aligned} \tag{71}$$

The corresponding vertical and horizontal projection maps are:

$$\begin{aligned}\mathbf{ver}_q(\mathcal{V}, \dot{q}_\mathfrak{m}) &:= (\mathcal{V} + \mathcal{A}\dot{q}_\mathfrak{m}, 0),\\ \mathbf{hor}_q(\mathcal{V}, \dot{q}_\mathfrak{m}) &:= (-\mathcal{A}\dot{q}_\mathfrak{m}, \dot{q}_\mathfrak{m}).\end{aligned} \tag{72}$$

Let

$$\Omega_{loc} := \mathcal{V} + \mathcal{A}\dot{q}_\mathfrak{m} \in \mathfrak{h}^\vee \tag{73}$$

be the absolute velocity of the locked system about its center of mass expressed in the vehicle's coordinate frame. These two projection maps allow the decomposition of motion/dynamics into two complementary directions, conveniently parameterized by Ω_{loc} and $\dot{q}_\mathfrak{m}$.

Lemma 4. *By the change of variables $(\mathcal{V}, \dot{q}_\mathfrak{m}) \mapsto (\Omega_{loc}, \dot{q}_\mathfrak{m})$, the $\mathcal{L}$agrangian of a vehicle-manipulator system ℓ^0 in* (64) *transforms into*

$$\ell^{cm} = \frac{1}{2}\begin{bmatrix}\Omega_{loc}\\ \dot{q}_\mathfrak{m}\end{bmatrix}^T \begin{bmatrix}M_0 & \mathbb{O}_{b\times n}\\ \mathbb{O}_{n\times b} & \hat{M}_\mathfrak{m}\end{bmatrix}\begin{bmatrix}\Omega_{loc}\\ \dot{q}_\mathfrak{m}\end{bmatrix} - u \tag{74}$$

with a block-diagonal mass matrix for expressing the kinetic energy of the system. Here, we have defined

$$\hat{M}_{\mathfrak{m}} := M_{\mathfrak{m}} - \mathcal{A}^T M_0 \mathcal{A}. \tag{75}$$

Proof. The proof is a straightforward computation. ■

Based on l^{cm} in (74) and Ω_{loc} in (73), we can now introduce the generalized momentum of the locked system about its center of mass $P \in (\mathfrak{h}^\vee)^*$ represented in the base vehicle's coordinate frame:

$$P = \frac{\partial l^{cm}}{\partial \Omega_{loc}} = M_0 \Omega_{loc}, \tag{76}$$

$$= M_0 \mathcal{V} + M_0 \mathcal{A} \dot{q}_{\mathfrak{m}}. \tag{77}$$

Let $t \mapsto q(t) \in Q$ be a smooth curve. A variation of $q(t)$, where $a \leq t \leq b$, with fixed end points $q_0 = q(a)$ and $q_f = q(b)$ is a smooth map $\beta : [a,b] \times \mathbb{R} \to Q$ that satisfies the conditions $\beta(t,0) = q(t)$, $\beta(a,\epsilon) = q_0$, and $\beta(b,\epsilon) = q_f$. This variation defines the vector field

$$\delta q(t) = (\delta h(t), \delta q_{\mathfrak{m}}(t)) = \frac{\partial \beta(t,\epsilon)}{\partial \epsilon}|_{\epsilon=0} \in T_{q(t)}Q \tag{78}$$

along the curve $q(t)$, such that $\delta q(a) = \delta q(b) = 0$. The operator δ always refers to variation of an entity, i.e., its composition with β and taking the derivative with respect to ϵ on the curve $q(t)$. Let us define the variation

$$\hat{\eta} := h^{-1}\delta h \in \mathfrak{h}, \tag{79}$$

as the left translation of the variation of the curve $h(t) \in H$ to the lie algebra $\mathfrak{h}$, with the conditions

$$\eta(a,\epsilon) = \eta(b,\epsilon) = 0. \tag{80}$$

A generalized force exerted at the coordinate frame attached to Body j consists of a linear force $f_j \in \mathbb{R}^3$ and an angular moment $\tau_j \in \mathbb{R}^3$, where $\begin{bmatrix} f_j^T & \tau_j^T \end{bmatrix}^T \in (\{\mathfrak{g}_j^j\}^\vee)^*$ is called a *body wrench*. The power generated by a wrench can be calculated by the pairing with the body twist of the frame where the wrench is applied $\left\langle \begin{bmatrix} f_j^T & \tau_j^T \end{bmatrix}^T, {}^jV_j^i \right\rangle$.

Lemma 5. *A vehicle-manipulator system with the $\mathcal{L}$agrangian in* (47) *and the applied force $F = (F_0, F_{\mathfrak{m}}) \in T^*Q \cong T^*H \times T^*Q_{\mathfrak{m}}$ satisfies the Hamilton-d'Alembert principle:*

$$\delta \int_a^b \mathcal{L}(q,\dot{q})dt - \int_a^b \langle F, \delta q \rangle \, dt = 0, \tag{81}$$

for variations of the type δq, if and only if ℓ^{cm} in (74) *satisfies the Hamilton-d'Alembert principle [57]:*

$$\delta \int_a^b l^{cm}(\Omega_{loc}, q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}})dt - \int_a^b \big(\langle F_\eta, \eta \rangle + \langle F_{\mathfrak{m}}, \delta q_{\mathfrak{m}} \rangle \big) dt = 0 \tag{82}$$

for a variation of the type:

$$\delta \Omega_{loc} = \dot{\eta} + [\mathcal{V}, \eta] + \mathcal{A}\delta \dot{q}_{\mathfrak{m}} + \delta \mathcal{A} \dot{q}_{\mathfrak{m}}, \tag{83}$$

and

$$F_\eta = T_{\mathbb{I}}^* L_h F_0 \in (\mathfrak{h}^\vee)^*. \tag{84}$$

The mapping $T_{\mathbb{I}}^* L_h \colon T_h^*H \to (\mathfrak{h}^\vee)^*$ is the dual of the tangent map corresponding to the left translation at the identity $\mathbb{I} \in H$. Here, F_η is the applied body wrench at the vehicle's coordinate frame.

Proof . To show the equivalence of the first terms appearing in (81) and (82), we need to compute the variation $\delta\Omega_{loc}$ induced on $\mathfrak{h} \times T_{q_{\mathfrak{m}}} Q_{\mathfrak{m}}$ by the variation δh. This variation can be found from the chain rule, using the definition in (73):

$$\delta \Omega_{loc} = \delta \mathcal{V} + \delta(\mathcal{A}\dot{q}_{\mathfrak{m}}) \tag{85}$$

Knowing that for a curve on H the order of the variation and differentiation operators is interchangeable, i.e. $\delta \dot{h}(t) = \frac{d}{dt}\delta h(t)$, the variation of $\mathcal{V}$ is derived from

$$\begin{aligned}
\delta \hat{\mathcal{V}} &= \delta(h^{-1}\dot{h}) = -(h^{-1}\delta h)(h^{-1}\dot{h}) + (h)^{-1}\frac{d}{dt}\delta h \\
&= -\hat{\eta}\hat{\mathcal{V}} + \dot{\hat{\eta}} - \frac{dh^{-1}}{dt}\delta h \\
&= -\hat{\eta}\hat{\mathcal{V}} + \dot{\hat{\eta}} + (h^{-1}\dot{h})(h^{-1}\delta h) \\
&= -\hat{\eta}\hat{\mathcal{V}} + \hat{\mathcal{V}}\hat{\eta} + \dot{\hat{\eta}} = [\hat{\mathcal{V}}, \hat{\eta}] + \dot{\hat{\eta}} = (\dot{\eta} + \mathbf{ad}_{\mathcal{V}}\eta)^\wedge.
\end{aligned}$$

The $\mathbf{ad}_{\mathcal{V}} : \mathfrak{h} \to \mathfrak{h}$ operator for the Lie algebra $\mathfrak{h} \cong \mathfrak{q}_0^0 \subseteq \mathfrak{g}_0^0$ is defined according to the Lie algebra isomorphism ι_0 and (7). As a result:

$$\delta \Omega_{loc} = \dot{\eta} + \mathbf{ad}_{\mathcal{V}}\eta + \mathcal{A}\delta\dot{q}_{\mathfrak{m}} + \delta\mathcal{A}\dot{q}_{\mathfrak{m}}. \tag{86}$$

To show the equality of the forcing terms appearing in (81) and (82), we start with the virtual work $\langle F, \delta q \rangle$ corresponding to the generalized forces $F = (F_0, F_{\mathfrak{m}})$ with the components F_0 and $F_{\mathfrak{m}}$ collocated with the vehicle and the arm joint velocities, respectively. Therefore,

$$\langle F, \delta q \rangle = \langle F_0 | \delta h \rangle + \langle F_{\mathfrak{m}}, \delta q_{\mathfrak{m}} \rangle. \tag{87}$$

Note that $\langle \cdot | \cdot \rangle$ denotes the pairing between elements of T_h^*H and T_hH. We define the mapping $T_{\mathbb{I}}L_h \colon \mathfrak{h}^\vee \to T_hH$ as the tangent map corresponding to the left translation at the identity $\mathbb{I} \in H$. Hence based on (79), $T_{\mathbb{I}}L_h\eta = \delta h$ and rewriting the virtual work:

$$\langle F_0 | \delta h \rangle = \langle F_0 | T_{\mathbb{I}} L_h \eta \rangle = \langle T_{\mathbb{I}}^* L_h F_0, \eta \rangle = \langle F_\eta, \eta \rangle, \tag{88}$$

This completes the proof. ■

From a practical perspective, we consider three types of forces acting on the system: (i) vehicle wrenches collocated with the base vehicle's body velocity $f_0 \in (\mathfrak{h}^\vee)^*$, (ii) arm forces collocated with the joint velocities $f_{\mathfrak{m}} \in T^*Q_{\mathfrak{m}}$, and (iii) the external wrench acting at the end-effector $f_{\mathfrak{e}} \in (\{\mathfrak{g}_{\mathfrak{e}}^{\mathfrak{e}}\}^\vee)^*$. The equivalent body wrench at the vehicle $f_{\mathfrak{e},0}$ and forces at the joints $f_{\mathfrak{e},\mathfrak{m}}$ due to $f_{\mathfrak{e}}$ can be calculated based on the principle of virtual work:

$$\begin{aligned}
\left\langle f_{\mathfrak{e}}, \left((g_{\mathfrak{e}}^I)^{-1}\delta g_{\mathfrak{e}}^I\right)^\vee \right\rangle &= \left\langle f_{\mathfrak{e}}, {}^{\mathfrak{e}}J_{\mathfrak{e}}^I \begin{bmatrix} \eta \\ \delta q_{\mathfrak{m}} \end{bmatrix} \right\rangle \\
&= \langle f_{\mathfrak{e},0}, \eta \rangle + \langle f_{\mathfrak{e},\mathfrak{m}}, \delta q_{\mathfrak{m}} \rangle,
\end{aligned} \tag{89}$$

where the Jacobian ${}^{\mathfrak{e}}J_{\mathfrak{e}}^I$ is calculated based on (28) and (29). The equivalent wrench and forces are computed as:

$$f_{\mathfrak{e},0} = J_{\mathfrak{e},0}^T f_{\mathfrak{e}} \in (\mathfrak{h}^\vee)^* \tag{90}$$

$$f_{\mathfrak{e},\mathfrak{m}} = J_{\mathfrak{e},\mathfrak{m}}^T f_{\mathfrak{e}} \in T^*Q_{\mathfrak{m}}, \tag{91}$$

where the end-effector Jacobians are:

$$J_{\mathfrak{e},0}(q_{\mathfrak{m}}) \coloneqq \mathbf{Ad}^{-1}_{g_n^0}\mathbf{Ad}^{-1}_{\bar{g}^n_{\mathfrak{e}}}\iota_0, \quad (92)$$

$$J_{\mathfrak{e},\mathfrak{m}}(q_{\mathfrak{m}}) \coloneqq \mathbf{Ad}^{-1}_{g_n^0}\mathbf{Ad}^{-1}_{\bar{g}^n_{\mathfrak{e}}}\left[\xi_1 \cdots \mathbf{Ad}_{e^{\hat{\xi}_1 q_1}\dots e^{\hat{\xi}_{n-1}q_{n-1}}}\xi_n\right]. \quad (93)$$

Thus, the total body wrench collocated with the vehicle velocity (F_η) is found from:

$$F_\eta = f_0 + J^T_{\mathfrak{e},0} f_{\mathfrak{e}} \in (\mathfrak{h}^\vee)^*, \quad (94)$$

and the total force collocated with the joint velocities is:

$$F_{\mathfrak{m}} = f_{\mathfrak{m}} + J^T_{\mathfrak{e},\mathfrak{m}} f_{\mathfrak{e}} \in T^*Q_{\mathfrak{m}}. \quad (95)$$

Theorem 1 (Lagrange-Poincaré Equations for Vehicle–Manipulator Systems). *Given a vehicle-manipulator system with the vehicle configuration $h \in H$, a set of joint angles $q_{\mathfrak{m}} \in Q_{\mathfrak{m}}$, input wrenches $f_0 \in (\mathfrak{h}^\vee)^*$, $f_{\mathfrak{m}} \in T^*Q_{\mathfrak{m}}$, and $f_{\mathfrak{e}} \in (\{\mathfrak{q}^{\mathfrak{e}}_{\mathfrak{e}}\}^\vee)^*$ that are collocated with the vehicle's body velocity, the joint velocities, and the end-effector's body velocity, respectively, and the $\mathcal{L}$agrangian $\mathcal{L}$ in* (47) *that is invariant with respect to the base vehicle's configuration, the singularity-free dynamical equations of the system reads:*

$$-\dot{P} + \boldsymbol{ad}^T_{\mathcal{V}} P = f_0 + J^T_{\mathfrak{e},0} f_{\mathfrak{e}} \quad (96)$$

$$\hat{M}_m \ddot{q}_{\mathfrak{m}} + \hat{C}_m \dot{q}_{\mathfrak{m}} + \hat{N}_m + \frac{\partial u}{\partial q_{\mathfrak{m}}} = f_{\mathfrak{m}} - \mathcal{A}^T f_0 + (J^T_{\mathfrak{e},\mathfrak{m}} - \mathcal{A}^T J^T_{\mathfrak{e},0}) f_{\mathfrak{e}}, \quad (97)$$

$$\mathcal{V} = (h^{-1}\dot{h})^\vee = {M_0}^{-1}(P - \mathcal{A}^T \dot{q}_{\mathfrak{m}}) \quad (98)$$

with the manipulator mass matrix $\hat{M}_{\mathfrak{m}}(q_{\mathfrak{m}})$ defined in (75)*, the locked mass matrix M_0 defined in* (67)*, and*

$$\hat{C}_{\mathfrak{m}}(q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}}) = \sum_{i=1}^n \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_i}\dot{q}_i - \frac{1}{2}\begin{bmatrix} \dot{q}^T_{\mathfrak{m}} \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_1} \\ \dot{q}^T_{\mathfrak{m}} \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_2} \\ \vdots \\ \dot{q}^T_{\mathfrak{m}} \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_n} \end{bmatrix} \quad (99)$$

$$\hat{N}_{\mathfrak{m}}(q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}}, P) = \mathcal{A}^T \boldsymbol{ad}^T_{\mathcal{V}} P - \left(\sum_{i=1}^n (\frac{\partial \mathcal{A}}{\partial q_i})\dot{q}_i\right)^T P + \begin{bmatrix} \dot{q}^T_{\mathfrak{m}} \frac{\partial \mathcal{A}}{\partial q_1}^T \\ \vdots \\ \dot{q}^T_{\mathfrak{m}} \frac{\partial \mathcal{A}}{\partial q_n}^T \end{bmatrix} P + \frac{1}{2}\begin{bmatrix} P^T \frac{\partial M_0^{-1}}{\partial q_1} \\ P^T \frac{\partial M_0^{-1}}{\partial q_2} \\ \vdots \\ P^T \frac{\partial M_0^{-1}}{\partial q_n} \end{bmatrix} P, \quad (100)$$

where $\mathcal{A}$ is found from (69).

Proof. The equations of motion of a vehicle-manipulator system with the $\mathcal{L}$agrangian in (47) between two fixed configurations $q_0 = q(a)$ and $q_f = q(b)$ under the effect of the applied force $F = (F_0, F_m) \in T^*Q$ can be formed using the Hamilton-d'Alembert principle for a variation of the type δq with fixed endpoints, i.e., $\delta q(a) = \delta q(b) = 0$. Based on Lemma 5, the reduced Lagrangian ℓ^{cm} then has to satisfy the Hamilton-d'Alembert principle in (82) for variations defined in (83) and forces defined in (87). Using the chain rule, and recongizing that $l^{cm} = k^{cm}(\Omega_{loc}, q_{\mathfrak{m}}, \dot{q}_{\mathfrak{m}}) - u(q_{\mathfrak{m}})$, the Hamilton-d'Alembert principle in (82) can be re-written as:

$$\int (\underbrace{\left\langle \frac{\partial k^{cm}}{\partial \Omega_{loc}}, \delta\Omega_{loc} \right\rangle}_{(I)} + \underbrace{\left\langle \frac{\partial k^{cm}}{\partial q_{\mathfrak{m}}}, \delta q_{\mathfrak{m}} \right\rangle + \left\langle \frac{\partial k^{cm}}{\partial \dot{q}_{\mathfrak{m}}}, \delta \dot{q}_{\mathfrak{m}} \right\rangle}_{(II)})dt - \int (\underbrace{\langle F_\eta, \eta\rangle}_{(III)} + \underbrace{\langle F_{\mathfrak{m}}, \delta q_{\mathfrak{m}}\rangle + \left\langle \frac{\partial u}{\partial q_{\mathfrak{m}}}, \delta q_{\mathfrak{m}} \right\rangle}_{(IV)})dt = 0, \quad (101)$$

The term (I) in 101 is expanded using the definition of the variation $\delta\Omega_{loc}$ in Lemma 5:

$$(I) = \int (\underbrace{\left\langle \frac{\partial k^{cm}}{\partial \Omega_{loc}}, \dot{\eta} + \mathbf{ad}_{\mathcal{V}}\eta \right\rangle}_{(I.A)} + \underbrace{\left\langle \frac{\partial k^{cm}}{\partial \Omega_{loc}}, \delta(\mathcal{A}\dot{q}_{\mathfrak{m}}) \right\rangle}_{(I.B)})dt. \quad (102)$$

Calculation of the Euler-Poincaré Equation in (96)*:* The term $(I.A)$ in (102) is rewritten using the definition of the momentum P in (76):

$$\begin{aligned} (I.A) &= \int (\langle P, \dot{\eta}\rangle + \langle P, \mathbf{ad}_{\mathcal{V}}\eta\rangle)dt \\ &= \int \Big(-\left\langle \dot{P}, \eta \right\rangle + \left\langle \mathbf{ad}^T_{\mathcal{V}} P, \eta \right\rangle \Big) dt \\ &= \int \left\langle -\dot{P} + \mathbf{ad}^T_{\mathcal{V}} P, \eta \right\rangle dt. \end{aligned} \quad (103)$$

Collecting the terms $(I.A)$ and (III) from (103) and (101) respectively, expanding the forcing term based on (94), and noticing the arbitrariness of variation η provides the Euler-Poincaré formula for the dynamics of the base vehicle as presented in the left hand side of (96).

Calculation of the Mass Matrix $\hat{M}_{\mathfrak{m}}$ in (97)*:* The term (II) in (101) can be expanded via integration by parts:

$$\begin{aligned} (II) &= \int \left\langle \frac{\partial k^{cm}}{\partial q_{\mathfrak{m}}}, \delta q_{\mathfrak{m}} \right\rangle dt - \int \left\langle \frac{d}{dt}\frac{\partial k^{cm}}{\partial \dot{q}_{\mathfrak{m}}}, \delta q_{\mathfrak{m}} \right\rangle dt \\ &= \int \left\langle \frac{\partial k^{cm}}{\partial q_{\mathfrak{m}}} - \frac{d}{dt}\frac{\partial k^{cm}}{\partial \dot{q}_{\mathfrak{m}}}, \delta q_{\mathfrak{m}} \right\rangle dt. \end{aligned} \quad (104)$$

Using the definition of the kinetic energy and l^{cm} in (74), the following chain rule can then be utilized in (104) to find that the mass matrix in (97) is in fact $\hat{M}_{\mathfrak{m}}$:

$$\frac{d}{dt}\frac{\partial k^{cm}}{\partial \dot{q}_{\mathfrak{m}}} = \hat{M}_{\mathfrak{m}}\ddot{q}_{\mathfrak{m}} + \underbrace{\frac{d}{dt}(\hat{M}_{\mathfrak{m}})}_{\hat{C}_1}\dot{q}_{\mathfrak{m}}. \quad (105)$$

Calculation of the Coriolis Matrix $\hat{C}_{\mathfrak{m}}$ in (97)*:* The two parts of the Coriolis matrix come from the expansion of (104). The time-derivative of the mass matrix in (105) can be written in coordinates:

$$\hat{C}_1\dot{q}_{\mathfrak{m}} = \frac{d}{dt}(\hat{M}_{\mathfrak{m}})\dot{q}_{\mathfrak{m}} = (\sum_{i=1}^n \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_i}\dot{q}_i)\dot{q}_{\mathfrak{m}}. \quad (106)$$

Remembering the definition of the kinetic energy in (74) and the definition of the momentum P in (76), the second term in

(104) can be expanded via a chain rule:

$$\frac{\partial k^{cm}}{\partial q_{\mathfrak{m}}} = \underbrace{\frac{1}{2}\begin{bmatrix} \dot{q}_{\mathfrak{m}}^T \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_1} \\ \dot{q}_{\mathfrak{m}}^T \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_2} \\ \vdots \\ \dot{q}_{\mathfrak{m}}^T \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_n} \end{bmatrix}}_{\hat{C}_2} \dot{q}_{\mathfrak{m}} + \underbrace{\frac{1}{2}\begin{bmatrix} P^T \frac{\partial M_0^{-1}}{\partial q_1} \\ P^T \frac{\partial M_0^{-1}}{\partial q_2} \\ \vdots \\ P^T \frac{\partial M_0^{-1}}{\partial q_n} \end{bmatrix}}_{\hat{N}_1} P. \tag{107}$$

Collecting the terms $\hat{C}_1$ and $\hat{C}_2$ in (106) and (107) (the terms that are dependant on $\dot{q}_{\mathfrak{m}}$) completes the calculation of $\hat{C}$ in (99).

Calculation of $\hat{N}_{\mathfrak{m}}$ in (97)*:* The term $(I.B)$ in (102) is expanded after substituting the momentum P from (76):

$$\begin{aligned}(I.B) &= \int \langle P, \mathcal{A}\delta\dot{q}_{\mathfrak{m}} + \delta\mathcal{A}\dot{q}_{\mathfrak{m}}\rangle\, dt \\ &= \int (\mathcal{A}^T \langle P, \delta\dot{q}_{\mathfrak{m}}\rangle + \langle P, \delta\mathcal{A}\dot{q}_{\mathfrak{m}}\rangle) dt \\ &= \int (\underbrace{-\left\langle \mathcal{A}^T\dot{P}, \delta q_{\mathfrak{m}}\right\rangle}_{I.B.1} \underbrace{-\left\langle \dot{\mathcal{A}}^T P, \delta q_{\mathfrak{m}}\right\rangle + \langle P, \delta\mathcal{A}\dot{q}_{\mathfrak{m}}\rangle}_{I.B.2}) dt.\end{aligned} \tag{108}$$

$\dot{P}$ from (96) is susbstituted into the term $(I.B.1)$ in (108):

$$\begin{aligned}(I.B.1) &= -\int \left\langle \mathcal{A}^T(\mathbf{ad}_{\mathcal{V}}^T P - F_\eta), \delta q_{\mathfrak{m}}\right\rangle dt \\ &= \int (\langle \mathcal{A}^T F_\eta, \delta q_{\mathfrak{m}}\rangle + \left\langle \underbrace{-\mathcal{A}^T \mathbf{ad}_{\mathcal{V}}^T P}_{\hat{N}_2}, \delta q_{\mathfrak{m}}\right\rangle) dt.\end{aligned} \tag{109}$$

The term $I.B.2$ in (108) can be expanded as:

$$\begin{aligned}(I.B.2) &= \int (-\left\langle \{\sum_{i=1}^n (\frac{\partial \mathcal{A}}{\partial q_i})\dot{q}_i\}^T P, \delta q_{\mathfrak{m}}\right\rangle \\ &\qquad + \left\langle P, \sum_{i=1}^n (\frac{\partial \mathcal{A}}{\partial q_i}\delta q_i)\dot{q}_{\mathfrak{m}}\right\rangle) dt \\ &= \int (-\left\langle \{\sum_{i=1}^n (\frac{\partial \mathcal{A}}{\partial q_i})\dot{q}_i\}^T P, \delta q_{\mathfrak{m}}\right\rangle \\ &\qquad + \left\langle P, \begin{bmatrix} \frac{\partial \mathcal{A}}{\partial q_1}\dot{q}_m & \cdots & \frac{\partial \mathcal{A}}{\partial q_n}\dot{q}_m \end{bmatrix} \delta q_{\mathfrak{m}}\right\rangle) dt \\ &= \int \left\langle \underbrace{-\{\sum_{i=1}^n (\frac{\partial \mathcal{A}}{\partial q_i})\dot{q}_i\}^T P + \begin{bmatrix} \dot{q}_{\mathfrak{m}}^T \frac{\partial \mathcal{A}}{\partial q_1}^T \\ \vdots \\ \dot{q}_{\mathfrak{m}}^T \frac{\partial \mathcal{A}}{\partial q_n}^T \end{bmatrix} P}_{\hat{N}_3}, \delta q_{\mathfrak{m}}\right\rangle dt.\end{aligned} \tag{110}$$

The $\hat{N}_{\mathfrak{m}}$ matrix as presented in (100) is formed by collecting the terms $\hat{N}_1$, $\hat{N}_2$ and $\hat{N}_3$ from (107), (109) and (110), respectively.

We collect the terms $\hat{M}_{\mathfrak{m}}\ddot{q}_{\mathfrak{m}}$, $\hat{C}_{\mathfrak{m}}\dot{q}_{\mathfrak{m}}$ and $\hat{N}_{\mathfrak{m}}$, the force and potential in the term (IV) of (101), and the force $\mathcal{A}^T F_{\mathfrak{m}}$ in (109). Then, by noticing the arbitrariness of the variation $\delta q_{\mathfrak{m}}$ and expanding the forces F_η and $F_{\mathfrak{m}}$ based on (94) and (95), this results in the internal dynamics equation in (97). ■

Calculation of the closed-form of $\hat{C}_{\mathfrak{m}} \in \mathbb{R}^{n\times n}$ and $\hat{N}_{\mathfrak{m}} \in \mathbb{R}^{n\times 1}$ requires the knowledge of the evolution of the partial derivatives of the locked mass matrix, the manipulator mass matrix and the connection $\mathcal{A}$ with respect to each of the joint parameters which are presented in detail in Section III-A.

Remark 1. *Note that, for simplicity, the preceding formalism has been developed under the assumption of having a symmetric potential function $u(q_{\mathfrak{m}})$ which is independent of $h \in H$. Whereas, one can incorporate a symmetry-breaking potential function $u(h, q_{\mathfrak{m}})$ (e.g. gravity potential for an aerial manipulator or an inclined rover-manipulator) in a similar manner to the treatment of F_η. This potential can not be dropped to $Q_{\mathfrak{m}}$ any more as was done in* (56) *and adds the following term to* (101) *in the proof of* (96) *and* (97)*:*

$$\left\langle \frac{\partial u}{\partial h} | \delta h \right\rangle = \left\langle T_{\mathbb{I}}^* L_h(\frac{\partial u}{\partial h}), \eta \right\rangle =: \langle f_u, \eta\rangle, \tag{111}$$

where $f_u \in (\mathfrak{h}^\vee)^*$. The problem is explicitly computing the partial derivative of u on an abstract manifold. We argue that we can instead calculate f_u, regardless of the choice of any local coordinates for H. We use the fact that the configuration manifold of the vehicle is a matrix Lie group embedded in a (matrix) vector space. Accordingly, the first term in (111)*, which is the directional derivative of the function u along the vector field δh (defined on the curve $h(t) \in H$), can be calculated as:*

$$\left\langle \frac{\partial u}{\partial h} | \delta h \right\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u(h + \epsilon\delta h, q_{\mathfrak{m}}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u(h + h\epsilon\hat{\eta}, q_{\mathfrak{m}}). \tag{112}$$

We choose a basis $\{\hat{\zeta}_i \in \mathfrak{h} | i = 1, \dots, b\}$ for $\mathfrak{h}$ and evaluate each component of f_u in this basis by

$$\eta_i f_{ui} := \eta_i \langle f_{ui}, \zeta_i\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u(h + \eta_i h\epsilon\hat{\zeta}_i, q_{\mathfrak{m}}), \quad i = 1, \dots, b$$

where $\hat{\eta} = \sum_{i=1}^b \eta_i\hat{\zeta}_i$. Knowing that the right hand side is linear in η_i:

$$f_{ui} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u(h + h\epsilon\hat{\zeta}_i, q_{\mathfrak{m}}). \quad i = 1, \dots, b \tag{113}$$

We can now form the vector $f_u \in (\mathfrak{h}^\vee)^*$ in the chosen basis. This wrench appears on the right hand side of (96)*, and the term $\mathcal{A}^T f_u$ shows up on the right hand side of* (97)*.*

A. Closed Form Equations for Configuration-Dependent Matrices

In the singularity-free Lagrange-Poincaré equations in (96) and (97), there are terms involving $\frac{\partial M_0}{\partial q_i}$, $\frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_i}$ and $\frac{\partial \mathcal{A}}{\partial q_i}$. The purpose of this section is to provide closed-form matrix equations for such terms in the dynamical equations, using the exponential parameterization of the manipulator joints. It can be observed from the definitions (65)- (67), (69), and (75) that these partial derivatives can be calculated, knowing the partial derivatives of the matrices $\mathfrak{L}_{\mathfrak{m}0}$ and $\mathfrak{L}_{\mathfrak{m}}$ with respect to the joint coordinates q_i. It is worth noting that we need not calculate the derivatives of Ξ and the mass matrices $\mathfrak{M}_i$, as they only depend on the system's initial configuration, and geometry and physical properties of the rigid bodies.

Lemma 6 (Calculation of derivatives of $\mathfrak{L}_{\mathfrak{m}0}$ and $\mathfrak{L}_{\mathfrak{m}}$)**.** *Based on* (63)*, the partial derivatives of* $\mathfrak{L}_{\mathfrak{m}0}$ *and* $\mathfrak{L}_{\mathfrak{m}}$ *are calculated by*

$$\frac{\partial \mathfrak{L}_{\mathfrak{m}0}}{\partial q_i} = \begin{bmatrix} \frac{\partial \mathfrak{Ad}_1^1}{\partial q_i} \\ \frac{\partial \mathfrak{Ad}_2^1}{\partial q_i} \\ \vdots \\ \frac{\partial \mathfrak{Ad}_n^1}{\partial q_i} \end{bmatrix} \tag{114}$$

$$\frac{\partial \mathfrak{L}_{\mathfrak{m}}}{\partial q_i} = \begin{bmatrix} \mathbb{I}_{6\times 6} & \mathbb{O}_{6\times 6} & \cdots & \mathbb{O}_{6\times 6} \\ \frac{\partial \mathfrak{Ad}_2^2}{\partial q_i} & \mathbb{I}_{6\times 6} & \cdots & \mathbb{O}_{6\times 6} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \mathfrak{Ad}_n^2}{\partial q_i} & \frac{\partial \mathfrak{Ad}_n^3}{\partial q_i} & \cdots & \mathbb{I}_{6\times 6} \end{bmatrix}, \tag{115}$$

where

$$\frac{\partial \mathfrak{Ad}_j^k}{\partial q_i} = \begin{cases} -\mathfrak{Ad}_j^{i+1} \boldsymbol{ad}_{\xi_i} \mathfrak{Ad}_i^k & j > i \geq k \in \{1, \ldots, n\} \\ -\boldsymbol{ad}_{\xi_i} \mathfrak{Ad}_i^k & j = i \geq k \in \{1, \ldots, n\}. \\ \mathbb{O}_{6\times 6} & \textit{Otherwise} \end{cases} \tag{116}$$

Proof. The proof is straight forward computation presented in Appendix 1. ∎

1) Derivatives of M_0 and $M_{0\mathfrak{m}}$: From (65) and (66), for every $i = 1, \ldots, n$ we have

$$\frac{\partial M_0}{\partial q_i} = \iota_0^T \Big(\frac{\partial \mathfrak{L}_{\mathfrak{m}0}^T}{\partial q_i} (\mathrm{diag}_1^n\{\mathfrak{M}_i\}) \mathfrak{L}_{\mathfrak{m}0}) + \mathfrak{L}_{\mathfrak{m}0}^T (\mathrm{diag}_1^n\{\mathfrak{M}_i\}) \frac{\partial \mathfrak{L}_{\mathfrak{m}0}}{\partial q_i} \Big) \iota_0, \tag{117}$$

and

$$\frac{\partial M_{0\mathfrak{m}}}{\partial q_i} = \iota_0^T \Big(\frac{\partial \mathfrak{L}_{\mathfrak{m}0}^T}{\partial q_i} (\mathrm{diag}_1^n\{\mathfrak{M}_i\}) \mathfrak{L}_{\mathfrak{m}0}) + \mathfrak{L}_{\mathfrak{m}0}^T (\mathrm{diag}_1^n\{\mathfrak{M}_i\}) \frac{\partial \mathfrak{L}_{\mathfrak{m}}}{\partial q_i} \Big) \Xi_{\mathfrak{m}}, \tag{118}$$

where the partial derivatives of $\mathfrak{L}_{\mathfrak{m}0}$ and $\mathfrak{L}_m$ are expressed in (114) and (115).

2) Derivatives of $\hat{M}_{\mathfrak{m}}$ and $\mathcal{A}$: From (75), the derivative of the arm mass matrix with respect to q_i is calculated:

$$\frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_i} = \frac{\partial M_{\mathfrak{m}}}{\partial q_i} - \frac{\partial (\mathcal{A}^T M_0 \mathcal{A})}{\partial q_i}, \tag{119}$$

where

$$\frac{\partial M_{\mathfrak{m}}}{\partial q_i} = \Xi_{\mathfrak{m}}^T \Big(\frac{\partial \mathfrak{L}_{\mathfrak{m}}}{\partial q_i}^T (\mathrm{diag}_1^n\{\mathfrak{M}_i\}) \mathfrak{L}_{\mathfrak{m}} + \mathfrak{L}_{\mathfrak{m}}{}^T (\mathrm{diag}_1^n\{\mathfrak{M}_i\}) \frac{\partial \mathfrak{L}_{\mathfrak{m}}}{\partial q_i} \Big) \Xi_{\mathfrak{m}}, \tag{120}$$

with the partial derivative of $\mathfrak{L}_m$ found from (115), and

$$\frac{\partial (\mathcal{A}^T M_0 \mathcal{A})}{\partial q_i} = \frac{\partial \mathcal{A}^T}{\partial q_i} M_0 \mathcal{A} + \mathcal{A}^T \frac{\partial M_0}{\partial q_i} \mathcal{A} + \mathcal{A}^T M_0 \frac{\partial \mathcal{A}}{\partial q_i}. \tag{121}$$

Here, it is evident that

$$\frac{\partial \mathcal{A}}{\partial q_i} = M_0^{-1} \frac{\partial M_0}{\partial q_i} M_0^{-1} M_{0\mathfrak{m}} + M_0^{-1} \frac{\partial M_{0\mathfrak{m}}}{\partial q_i}, \tag{122}$$

where we used the definition of the principal connection in (69), and $\frac{\partial M_0}{\partial q_i}$ and $\frac{\partial M_{0\mathfrak{m}}}{\partial q_i}$ are calculated in (117) and (118), respectively. The procedure explained in this section, provides the closed form matrix equations for all of the terms appearing in the Lagrange-Poincaré equations, in an algorithmic fashion. This procedure is appropriate for simulation or model-based control purposes.

Example 3. In this section, we derive the governing dynamical equations of the sample two-link rover-manipulator system described in 1, in an algorithmic fashion. The equations include a set of second order nonlinear differential equations for the manipulator and two sets of first order differential equations for the vehicle motion. The knowledge of inertia properties of the bodies, the manipulator configuration $q_{\mathfrak{m}} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$, its velocity $\dot{q}_{\mathfrak{m}} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$, the locked momentum of the system P, and the applied wrenches f_0, f_u, and $f_{\mathfrak{e}}$ are used to derive the dynamics.

First, we set up the constant dynamic properties used to form the inertia matrices M_0 and $\hat{M}_m$. The body inertia matrices of each of the three bodies in the chain relative to their respective joint frames are found from (49):

$${}^0\mathcal{M}_0 = \begin{bmatrix} m_0 \times \mathbb{I}_{3\times 3} & \mathbb{O}_{3\times 3} \\ \mathbb{O}_{3\times 3} & \mathfrak{I}_0 \end{bmatrix} \tag{123}$$

$${}^1\mathcal{M}_1 = \begin{bmatrix} m_1 \times \mathbb{I}_{3\times 3} & \begin{bmatrix} 0 & \frac{m_1 l_1}{2} & 0 \\ -\frac{m_1 l_1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -\frac{m_1 l_1}{2} & 0 \\ \frac{m_1 l_1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \mathfrak{I}_1 - \begin{bmatrix} \frac{1}{2} m_1 l_1^2 & 0 & 0 \\ 0 & \frac{1}{2} m_1 l_1^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \tag{124}$$

$${}^2\mathcal{M}_2 = \begin{bmatrix} m_2 \times \mathbb{I}_{3\times 3} & \begin{bmatrix} 0 & \frac{m_1 l_2}{2} & 0 \\ -\frac{m_1 l_2}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -\frac{m_2 l_2}{2} & 0 \\ \frac{m_2 l_2}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \mathfrak{I}_2 - \begin{bmatrix} \frac{1}{2} m_2 l_2^2 & 0 & 0 \\ 0 & \frac{1}{2} m_2 l_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{125}$$

We find the inertia matrices of each body with respect to the rover's frame $\mathfrak{M}_0$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ based on (55) using the calculated body inertia matrices in (123), (124) and (125), and the adjoint maps $\mathbf{Ad}_{\bar{g}_1^0}$ and $\mathbf{Ad}_{\bar{g}_2^0}$ based on (5):

$$\mathbf{Ad}_{\bar{g}_1^0} = \begin{bmatrix} 0 & 0 & 1 & 0 & l_0 & 0 \\ 0 & -1 & 0 & 0 & 0 & l_0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Ad}_{\bar{g}_2^0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -l_{01} \\ 0 & 0 & 1 & 0 & l_{01} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{126}$$

The twists corresponding to the joints are collected into:

$$\Xi_{\mathfrak{m}} := \begin{bmatrix} 0 & l_0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(l_0+l_1) & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T. \tag{127}$$

Second, we formulate the configuration-dependent matrices M_0, $\hat{M}_{\mathfrak{m}}$ and $\mathcal{A}$. The adjoint maps $\mathfrak{Ad}_1^1$, $\mathfrak{Ad}_2^1$ and $\mathfrak{Ad}_2^2$, defined in (60), are found using the adjoint maps $\mathbf{Ad}_{e^{-\xi_1 q_1}}$ and $\mathbf{Ad}_{e^{-\xi_2 q_2}}$ in (43) and (44):

$$\mathfrak{Ad}_1^1 = \mathbf{Ad}_{e^{-\xi_1 q_1}}, \quad \mathfrak{Ad}_2^2 = \mathbf{Ad}_{e^{-\xi_2 q_2}}, \quad \mathfrak{Ad}_2^1 = \mathfrak{Ad}_2^2 \mathfrak{Ad}_1^1. \tag{128}$$

The matrices $\mathfrak{L}_{\mathfrak{m}0}$ and $\mathfrak{L}_{\mathfrak{m}}$ are calculated based on (63) and (128):

$$\mathfrak{L}_{\mathfrak{m}0} := \begin{bmatrix} \mathfrak{Ad}_1^1 \\ \mathfrak{Ad}_2^1 \end{bmatrix} \quad \& \quad \mathfrak{L}_{\mathfrak{m}} := \begin{bmatrix} \mathbb{I}_{6\times6} & \mathbb{O}_{6\times6} \\ \mathfrak{Ad}_2^2 & \mathbb{I}_{6\times6} \end{bmatrix}. \tag{129}$$

Knowing the inclusion map ι_0 as presented in (39), the inertia matrices observed from the rover's frame $\mathfrak{M}_0$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and the $\mathfrak{L}$ elements calculated in (129), we can find the inertia matrix blocks M_0, $M_{0\mathfrak{m}}$ and $M_{\mathfrak{m}}$ based on (65), (66), and (67). The connection $\mathcal{A}$ can thus be calculated from M_0 and $M_{0\mathfrak{m}}$ based on (69). This allows the calculation of the generalized mass matrix $\hat{M}_{\mathfrak{m}}$ according to (75).

Third, we collect all forces applied to the system, including the potential terms (f_u, $\frac{\partial u}{\partial q_{\mathfrak{m}}}$), the joint torques forming $f_{\mathfrak{m}}$, and the external wrenches applied at the rover and the end-effector, i.e., f_0 and $f_{\mathfrak{e}}$, respectively. We present a constant gravity vector via $\vec{\mathbf{g}} \in \mathbb{R}^4$ with its last component being 0, which is not necessarily in the z_0 direction. Therefore, the potential function of the multi-body system is the sum of the potentials of all three bodies:

$$\begin{aligned} u = \sum_{i=0}^{2} u_i = u_0+u_1+u_2 := m_0\Big\langle \vec{\mathbf{g}}, g_0^I(h)\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T\Big\rangle \\ +m_1\Big\langle \vec{\mathbf{g}}, g_0^I(h)e^{\hat{\xi}_1 q_1}\bar{g}_{cm,1}^I\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T\Big\rangle \\ +m_2\Big\langle \vec{\mathbf{g}}, g_0^I(h)e^{\hat{\xi}_1 q_1}e^{\hat{\xi}_2 q_2}\bar{g}_{cm,2}^I\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T\Big\rangle. \end{aligned} \tag{130}$$

The wrench f_u corresponding to the derivative of the potential function with respect to the base pose is computed from (113) for the standard basis of $\mathbf{se}(2)$:

$$\hat{\zeta}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \hat{\zeta}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \hat{\zeta}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Each component takes the following form:

$$\begin{aligned} f_{ui} = &\frac{d}{d\epsilon}\Big|_{\epsilon=0} u_0(h + h\epsilon\hat{\zeta}_i, q_{\mathfrak{m}}) \\ &+ \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_1(h + h\epsilon\hat{\zeta}_i, q_{\mathfrak{m}}) \\ &+ \frac{d}{d\epsilon}\Big|_{\epsilon=0} u_2(h + h\epsilon\hat{\zeta}_i, q_{\mathfrak{m}}). \quad i = 1,2,3 \end{aligned} \tag{131}$$

Note that the only term dependent on $h \in H$ in u is g_0^I. We compute the term

$$\begin{aligned} \Gamma_1(h) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_0^I(h + \epsilon h\hat{\zeta}_1) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \bar{g}_0^I \begin{bmatrix} \begin{bmatrix} R_h & \mathbb{O}_{2\times1} \\ \mathbb{O}_{1\times2} & 1 \end{bmatrix} & \begin{bmatrix} p_h + \epsilon R_h \begin{bmatrix}1 & 0\end{bmatrix}^T \\ 0 \end{bmatrix} \\ \mathbb{O}_{1\times3} & 1 \end{bmatrix} \\ &= \bar{g}_0^I \begin{bmatrix} \mathbb{O}_{3\times3} & \begin{bmatrix} R_h \begin{bmatrix}1 & 0\end{bmatrix}^T \\ 0 \end{bmatrix} \\ \mathbb{O}_{1\times3} & 0 \end{bmatrix}, \end{aligned}$$

and similarly,

$$\Gamma_2(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_0^I(h + \epsilon h\hat{\zeta}_2) = \bar{g}_0^I \begin{bmatrix} \mathbb{O}_{3\times3} & \begin{bmatrix} R_h \begin{bmatrix}0 & 1\end{bmatrix}^T \\ 0 \end{bmatrix} \\ \mathbb{O}_{1\times3} & 0 \end{bmatrix},$$

$$\Gamma_3(h) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_0^I(h + \epsilon h\hat{\zeta}_3) = \bar{g}_0^I \begin{bmatrix} R_h \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \mathbb{O}_{2\times2} \\ \mathbb{O}_{2\times2} & \mathbb{O}_{2\times2} \end{bmatrix}.$$

Therefore, for $i = 1,2,3$

$$\begin{aligned} f_{ui} =& m_0\Big\langle \vec{\mathbf{g}}, \Gamma_i(h)\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T\Big\rangle \\ &+m_1\Big\langle \vec{\mathbf{g}}, \Gamma_i(h)e^{\hat{\xi}_1 q_1}\bar{g}_{cm,1}^I\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T\Big\rangle \\ &+m_2\Big\langle \vec{\mathbf{g}}, \Gamma_i(h)e^{\hat{\xi}_1 q_1}e^{\hat{\xi}_2 q_2}\bar{g}_{cm,2}^I\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T\Big\rangle. \end{aligned}$$

The derivatives of the potential function with respect to the manipulator joint angles q_1 and q_2 in (97) are calculated as:

$$\frac{\partial u}{\partial q_{\mathfrak{m}}} = \begin{bmatrix} \frac{\partial u}{\partial q_1} \\ \frac{\partial u}{\partial q_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial q_1} + \frac{\partial u_2}{\partial q_1} \\ \frac{\partial u_2}{\partial q_2} \end{bmatrix}, \tag{132}$$

where

$$\begin{aligned} \frac{\partial u_1}{\partial q_1} =& m_1 \left\langle \vec{\mathbf{g}}, g_0^I \xi_1 e^{\xi_1 q_1}\bar{g}_{cm,1}^0 \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T \right\rangle, \\ \frac{\partial u_2}{\partial q_1} =& m_2 \left\langle \vec{\mathbf{g}}, g_0^I \xi_1 e^{\xi_1 q_1} e^{\xi_2 q_2}\bar{g}_{cm,2}^0 \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T \right\rangle, \\ \frac{\partial u_2}{\partial q_2} =& m_2 \left\langle \vec{\mathbf{g}}, g_0^I e^{\xi_1 q_1} \xi_2 e^{\xi_2 q_2}\bar{g}_{cm,2}^0 \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T \right\rangle. \end{aligned} \tag{133}$$

Given $f_{\mathfrak{m}}$, f_0, $f_{\mathfrak{e}}$, f_u, and having calculated $\mathcal{A}$ and $J_{\mathfrak{e},\mathfrak{m}}$ in (69) and (93), we expand the right hand side of the Euler-Poincaré equation (96) and Euler-Lagrange equation (97), based on (94) and (95):

$$\begin{aligned} &F_\eta = f_0 + f_u + J_{\mathfrak{e},0}^T f_{\mathfrak{e}}, \\ &F_{\mathfrak{m}} + \mathcal{A}^T F_\eta = f_{\mathfrak{m}} - \mathcal{A}^T(f_0 + f_u) + (J_{\mathfrak{e},\mathfrak{m}}^T - \mathcal{A}^T J_{\mathfrak{e},0}^T) f_{\mathfrak{e}}. \end{aligned}$$

Last, we find the evolution of the system states $q_{\mathfrak{m}}$, $\dot{q}_{\mathfrak{m}}$, $\mathcal{V}$ and P. Knowing M_0, $\mathcal{A}$ and the momentum P and manipulator configuration $\dot{q}_{\mathfrak{m}}$, the rover's restricted body velocity, $\mathcal{V} = \begin{bmatrix} {}^0v_{0,x}^{I\,T} & {}^0v_{0,y}^{I\,T} & {}^0\omega_{0,z}^{I\,T} \end{bmatrix}^T$ is calculated based on (98):

$$\mathcal{V} = {M_0}^{-1}(P - \mathcal{A}^T \dot{q}_{\mathfrak{m}}). \tag{134}$$

We calculate the rate of change of the momentum $\dot{P}$ using (96):

$$\dot{P} = \mathbf{ad}_{\mathcal{V}}^* P - f_0 - (J_{\mathfrak{e},0})^T f_{\mathfrak{e}} + f_u, \tag{135}$$

where $\mathbf{ad}^*_{\mathcal{V}} = \mathbf{ad}^T_{\mathcal{V}}$ is defined according to (7) as:

$$\mathbf{ad}_{\mathcal{V}} = \begin{bmatrix} 0 & -{}^0\omega^I_{0,z} & {}^0v^I_{0,y} \\ {}^0\omega^I_{0,3} & 0 & -{}^0v^I_{0,z} \\ 0 & 0 & 0 \end{bmatrix}, \tag{136}$$

and the jacobian $J_{\mathfrak{e},0}$ is calculated based on (92) using the adjoint maps $\mathbf{Ad}_{g^0_2}$ and $\mathbf{Ad}_{\bar{g}^2_{\mathfrak{e}}}$ found in 2 and the inclusion map ι_0 in (39):

$$J_{\mathfrak{e},0} = \mathbf{Ad}^{-1}_{g^0_2}\mathbf{Ad}^{-1}_{\bar{g}^2_{\mathfrak{e}}}\iota_0. \tag{137}$$

The derivatives of the $\mathfrak{L}$ matrix can be calculated from (114) and (115):

$$\frac{\partial \mathfrak{L}_{\mathfrak{m}0}}{\partial q_1} = -\begin{bmatrix} \mathbf{ad}_{\xi_1}\mathfrak{A}\mathfrak{d}^1_1 \\ \mathfrak{A}\mathfrak{d}^2_2\mathbf{ad}_{\xi_1}\mathfrak{A}\mathfrak{d}^1_1 \end{bmatrix} \quad \& \quad \frac{\partial \mathfrak{L}_{m0}}{\partial q_2} = -\begin{bmatrix} \mathbb{O}_{6\times 6} \\ \mathbf{ad}_{\xi_2}\mathfrak{A}\mathfrak{d}^1_2 \end{bmatrix} \tag{138}$$

$$\frac{\partial \mathfrak{L}_{\mathfrak{m}}}{\partial q_1} = \mathbb{O}_{12\times 12} \quad \& \quad \frac{\partial \mathfrak{L}_{\mathfrak{m}}}{\partial q_2} = \begin{bmatrix} \mathbb{O}_{6\times 6} & \mathbb{O}_{6\times 6} \\ \mathbf{ad}_{\xi_2}\mathfrak{A}\mathfrak{d}^2_2 & \mathbb{O}_{6\times 6} \end{bmatrix}, \tag{139}$$

where the adjoint maps $\mathbf{ad}_{\xi_1}$ and $\mathbf{ad}_{\xi_2}$ are found based on (7):

$$\mathbf{ad}_{\xi_1} = \begin{bmatrix} {}^0\tilde{\varpi}^0_1 & {}^0\tilde{\nu}^0_1 \\ \mathbb{O}_{3\times 3} & {}^0\tilde{\varpi}^0_1 \end{bmatrix} \quad \& \quad \mathbf{ad}_{\xi_2} = \begin{bmatrix} {}^0\tilde{\varpi}^0_2 & {}^0\tilde{\nu}^0_2 \\ \mathbb{O}_{3\times 3} & {}^0\tilde{\varpi}^0_2 \end{bmatrix}, \tag{140}$$

where ${}^0\varpi^0_1$, ${}^0\varpi^0_2$, ${}^0\nu^0_1$ and ${}^0\nu^0_2$ have been defined in 1. This allows the calculation of $\frac{M_0}{\partial q_1}$, $\frac{M_0}{\partial q_2}$, $\frac{M_{0\mathfrak{m}}}{\partial q_1}$, $\frac{M_{0\mathfrak{m}}}{\partial q_2}$, $\frac{\mathcal{A}}{\partial q_1}$, $\frac{\mathcal{A}}{\partial q_2}$, $\frac{M_{\mathfrak{m}}}{\partial q_1}$ and $\frac{M_m}{\partial q_2}$ from (120), (121), (122), (117) and (118), respectively. Next, we can calculate $\frac{\hat{M}_{\mathfrak{m}}}{\partial q_1}$ and $\frac{\hat{M}_{\mathfrak{m}}}{\partial q_2}$ from (119) based on $\frac{M_0}{\partial q_1}$, $\frac{M_0}{\partial q_2}$, $\frac{\mathcal{A}}{\partial q_1}$, $\frac{\mathcal{A}}{\partial q_2}$, $\frac{M_{\mathfrak{m}}}{\partial q_1}$ and $\frac{M_m}{\partial q_2}$.

Knowing the manipulator configuration $q_{\mathfrak{m}}$ and velocities $\dot{q}_{\mathfrak{m}}$, and the partial derivatives $\frac{M_{\mathfrak{m}}}{\partial q_1}$ and $\frac{M_m}{\partial q_2}$, we can calculate the Coriolis matrix $\hat{C}_{\mathfrak{m}}$ from (99):

$$(\hat{C}_m) = \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_1}\dot{q}_1 + \frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_2}\dot{q}_2 - \frac{1}{2}\begin{bmatrix} \dot{q}^T_{\mathfrak{m}}\frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_1} \\ \dot{q}^T_{\mathfrak{m}}\frac{\partial \hat{M}_{\mathfrak{m}}}{\partial q_2} \end{bmatrix}. \tag{141}$$

Knowing P and $\dot{q}_{\mathfrak{m}}$, and having calculated $\mathcal{A}$, $\frac{\mathcal{A}}{\partial q_1}$, $\frac{\mathcal{A}}{\partial q_2}$, $\frac{M_0}{\partial q_1}$, $\frac{M_0}{\partial q_2}$, and $\mathbf{ad}_{\mathcal{V}}$, we can find the $\hat{N}_{\mathfrak{m}}$ matrix from (100). The evolution of the internal states of the manipulator can thus be calculated from (97):

$$\ddot{q}_{\mathfrak{m}} = \hat{M}^{-1}_m\bigg(f_{\mathfrak{m}} - \mathcal{A}^T f_0 + (J^T_{e,\mathfrak{m}} - \mathcal{A}^T(J_{e,0})^T)f_{\mathfrak{e}} - \hat{C}_m\dot{q}_{\mathfrak{m}}\hat{N}_m - \frac{\partial u}{\partial q_{\mathfrak{m}}} - \mathcal{A}^T f_u\bigg).$$

IV. Conclusion

In this paper, we developed the dynamical equations of a category of vehicle-manipulator systems based on the Hamilton-d'Alembert principle. Due to the inherent symmetry of the kinetic energy, the resulting equations are structured similar to the Lagrange-Poincaré equations. We conclude the paper with a number of remarks regarding the advantages of the proposed formalism.

1) The formalism can capture a wide variety of vehicles whose configuration is described by an embedded Lie sub-group of $\mathbf{SE}(3)$. To keep the validity of the equations at every pose of the vehicle, we avoided parameterizing the vehicle's configuration manifold. Hence, the equations are considered singularity-free and can improve the numerical stability of simulations or provide global model-based laws for controlling vehicle-manipulator systems.
2) The equations were derived in the fashion that can incorporate any forcing functions naturally appearing in robotic operations and can handle symmetry-breaking potential functions, without the requirement to parameterize the vehicle's motion.
3) We adopted the geometric exponential formalism to study the manipulator's motion on Lie groups. This provided us with the ability to present the complete set of differential equations in matrix form.
4) The resulting model is ready to be applied without the need for extensive knowledge of the underlying geometry, since tangible physical explanations of all variables and geometric structures and explicit closed-form equations for dynamic matrices were provided.

Throughout the paper, we were committed to the practical consistency of the model with real-life robots by formulating the kinematic properties with respect to the vehicle and accompanying the derivation with a step-by-step case study.

The effectively separated external and internal dynamics in the proposed formalism makes it suitable for hardware-in-the-loop simulation of moving-base manipulators, which is one of the future directions of our current research. We propose to further extend the model to incorporate the effects of a non-inertial reference frame, e.g, spacecraft-manipulators operating relative to a moving orbital frame. We also put forward the idea to exploit the independence of dynamics from the vehicle's pose to develop singularity-free full-pose output-tracking control laws on Lie groups.

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Appendix

Appendix 1. *The derivative of each adjoint map $\mathfrak{Ad}_j^k$ at the identity with respect to q_i is found via*

$$\frac{\partial}{\partial q_i}\mathfrak{Ad}_j^k(.) = \mathfrak{Ad}_j^{i+1}[\mathfrak{Ad}_i^k, \xi_i], \tag{142}$$

or

$$\frac{\partial}{\partial q_i}\mathfrak{Ad}_j^k(.) = -\mathfrak{Ad}_j^{i+1}\boldsymbol{ad}_{\xi_i}\mathfrak{Ad}_i^k(.), \tag{143}$$

where $\boldsymbol{ad}_{\xi_i}$ is found from (7) *and the terms involving $\mathfrak{Ad}_\bullet^\bullet$ are defined in* (60).

Taking the derivative of $\mathfrak{Ad}_j^k$ in (143) when $j > i \geq k$:

$$\begin{aligned}
\frac{\partial \mathfrak{Ad}_j^k}{\partial q_i} &= \frac{\partial}{\partial q_i}\mathbf{Ad}_{e^{-\xi_j q_j}\dots e^{-\xi_k q_k}}(.)\\
&= \frac{\partial\big(e^{-\xi_j q_j}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_j q_j}\big)^\vee}{\partial q_i}\\
&= \big(e^{-\xi_j q_j}\cdots e^{-\xi_{i+1} q_{i+1}}(-\xi_i)\\
&\qquad e^{-\xi_i q_i}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_j q_j}\\
&\quad + e^{-\xi_j q_j}\cdots e^{-\xi_k q_k}(.)\\
&\qquad e^{\xi_k q_k}\cdots e^{\xi_i q_i}(\xi_i)e^{\xi_{i+1} q_{i+1}}\cdots e^{\xi_j q_j}\big)^\vee\\
&= \big(e^{-\xi_j q_j}\cdots e^{-\xi_{i+1} q_{i+1}}\\
&\qquad \big((-\xi_i)e^{-\xi_i q_i}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_i q_i}\\
&\qquad + e^{-\xi_i q_i}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_i q_i}(\xi_i)\big)\\
&\qquad e^{\xi_{i+1} q_{i+1}}\cdots e^{\xi_j q_j}\big)^\vee\\
&= \mathbf{Ad}_{e^{-\xi_j q_j}\dots e^{-\xi_{i+1} q_{i+1}}}\\
&\qquad \big((-\xi_i)e^{-\xi_i q_i}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_i q_i}\\
&\qquad + e^{-\xi_i q_i}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_i q_i}(\xi_i)\big)^\vee\\
&= \mathbf{Ad}_{e^{-\xi_j q_j}\dots e^{-\xi_{i+1} q_{i+1}}}\\
&\qquad \big((-\xi_i)\mathbf{Ad}_{e^{-\xi_i q_i}\dots e^{-\xi_k q_k}}(.)\\
&\qquad + \mathbf{Ad}_{e^{-\xi_i q_i}\dots e^{-\xi_k q_k}}(.)(\xi_i)\big)^\vee\\
&= \mathbf{Ad}_{e^{-\xi_j q_j}\dots e^{-\xi_{i+1} q_{i+1}}}[\mathbf{Ad}_{e^{-\xi_i q_i}\dots e^{-\xi_k q_k}}(.), \xi_i]\\
&= \mathfrak{Ad}_j^{i+1}[\mathfrak{Ad}_i^k(.), \xi_i].
\end{aligned} \tag{144}$$

By using the definition of $\mathbf{ad}_{\xi_i}$ as

$$\mathbf{ad}_{\xi_i}(.) = [\xi_i, (.)] = -[(.), \xi_i] \tag{145}$$

one can find:

$$\frac{\partial}{\partial q_i}\mathfrak{Ad}_j^k(.) = -\mathfrak{Ad}_j^{i+1}\mathbf{ad}_{\xi_i}\mathfrak{Ad}_i^k(.) \quad j > i \geq k \in \{1, \dots, n\}. \tag{146}$$

On the other hand, when in (143) we have $j = i \geq k$:

$$\begin{aligned}
\frac{\partial \mathfrak{Ad}_j^k}{\partial q_i} &= \frac{\partial}{\partial q_i}\mathbf{Ad}_{e^{-\xi_j q_j}\dots e^{-\xi_k q_k}}(.)\\
&= \frac{\partial\big(e^{-\xi_j q_j}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_j q_j}\big)^\vee}{\partial q_i}\\
&= \big((-\xi_i)e^{-\xi_i q_i}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_j q_j}\\
&\qquad + e^{-\xi_j q_j}\cdots e^{-\xi_k q_k}(.)e^{\xi_k q_k}\cdots e^{\xi_i q_i}(\xi_i)\big)^\vee\\
&= \big((-\xi_i)\mathbf{Ad}_{e^{-\xi_i q_i}\dots e^{-\xi_k q_k}}(.)\\
&\qquad + \mathbf{Ad}_{e^{-\xi_i q_i}\dots e^{-\xi_k q_k}}(.)(\xi_i)\big)^\vee\\
&= [\mathbf{Ad}_{e^{-\xi_i q_i}\dots e^{-\xi_k q_k}}(.), \xi_i]\\
&= [\mathfrak{Ad}_i^k(.), \xi_i]\\
&= -\mathbf{ad}_{\xi_i}\mathfrak{Ad}_i^k(.) \quad j = i \geq k \in \{1, \dots, n\}.
\end{aligned} \tag{147}$$